Due on Monday, May 3rd.

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1. Collision/decay problems.

   (a) [**5 pts**] Two photons have the same energy E_γ. They collide at an angle θ and create a single particle. Calculate the mass M of the final particle in terms of E_γ and θ.

   (b) [**5 pts**] Two balls of equal mass (m each) approach each other with equal but opposite velocities of magnitude v. Their collision is perfectly inelastic, so they stick together and form a single body of mass M. What is the velocity of the final body and what is its mass M?

   Find the mass M in the specific cases of $v = 0.01c$, $v = 0.5c$, and $v = 0.9c$.

2. The four-vector A^μ is found to be timelike in one inertial frame.

   (a) [**2 pts**] Write an inequality expressing the fact that A^μ is timelike.

   (b) [**3 pts**] Explain why A^μ has to be timelike in any inertial frame.

3. [**SELF**] The mass m and the charge q of a particle are four-scalars. Explain why the combination

   $$(m, q, m, q)$$

   is not a four-vector.

4. Electromagnetism.

   (a) [**5 pts**] Show that the tensor equation $\partial_\mu J^\mu = 0$ is equivalent to the continuity equation of electromagnetism. Here J^μ is the current density 4-vector.

   (b) [**5 pts**] The two inhomogeneous Maxwell's equations can be expressed as the tensor equation for the field tensor $F^{\mu\nu}$:

   $$\partial_\mu F^{\mu\alpha} \;=\; J^\alpha$$

   Derive the continuity equation from this equation, using the fact that the field tensor is antisymmetric.

Note: Depending on the definition conventions for $F^{\mu\nu}$ and J^{μ}, Maxwell's inhomogeneous equations may also appear in textbooks in the forms

$$\partial_\mu F^{\mu\alpha} = -J^\alpha \qquad \text{or} \qquad \partial_\mu F^{\mu\alpha} = \mu_0 J^\alpha \qquad \text{or} \qquad \partial_\mu F^{\mu\alpha} = -\mu_0 J^\alpha$$

but this will not matter for the problem above.

5. Minkowski tensors.

   (a) [**6 pts**] Consider the Minkowski tensors

   $$D^{\mu\nu}\,, \qquad D^{\mu\nu}B_\sigma\,, \qquad D^{\mu\sigma}B_\sigma\,, \qquad D^{\mu\sigma}D_{\mu\sigma}\,, \qquad C^\mu B_\mu\,, \qquad C^\mu B_\sigma\,.$$

   Explain the rank of each tensor.
   Which of these, if any, are scalars?

   (b) [**3 pts**] Show how the tensor $T^{\alpha\beta}_{\gamma}$ transforms under the Lorentz transformation Λ.

   (c) [**4 pts**] A Minkowski tensor $U_{\mu\nu}$ is said to be symmetric if $U_{\mu\nu} = U_{\nu\mu}$. If a tensor is symmetric in one inertial frame, show that it is symmetric in all inertial frames.

6. **Poincaré transformations.**

   If Λ is a 4×4 matrix representing a Lorentz transformation, then transformations of the type

   $$x' \;=\; \Lambda\, x + a$$

   are known as Poincaré transformations. Here x and x' are 4×1 column vectors (4-vectors) representing spacetime coordinates of events as seen from two-different frames, and a is a 4×1 column vector. In other words, a Poincaré transformation is a combination of a Lorentz transformation plus a possible shift of the space and time coordinates. We will denote this transformation as (Λ, a).

   (a) [**5 pts**] Show that the result of two Poincaré transformations (Λ_1, a_1) and (Λ_2, a_2), applied successively, is the Poincaré transformation

   $$(\Lambda_2\, \Lambda_1,\;\; \Lambda_2\, a_1 + a_2)$$

   (b) [**3 pts**] Are Poincaré transformations commutative?

   (c) [**4 pts**] Are Poincaré transformations associative?

   (d) [**SELF**] Does the set of all Poincaré transformations form a group?