This is a problem bank for practice - not due for submission.

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1. Let Σ and Σ' be inertial frames. Frame Σ' moves at velocity v with respect to Σ, in the common positive x, x' direction. Measurements of events in the two frames, denoted respectively by (x, y, z, t) and (x', y', z', t'), are related by the standard Lorentz boost

$$x' = \gamma_v(x - vt) \; ; \quad y' = y \; ; \quad z' = z \; ; \quad t' = \gamma_v(t - vx/c^2) \; ;$$

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

(a) A particle has velocity $\vec{u}' = (0, 0, c)$ relative to Σ'.
Find the velocity of the particle relative to Σ.
Explain how your result is consistent with the constancy of the speed of light.

(b) A particle has velocity $(c/2, c/2, 0)$ relative to Σ.
Write down the four-velocity of the particle in the Σ frame.
Find the four-velocity of the particle in the Σ' frame, using the Lorentz tranformation for the four-velocity.
Find the velocity and speed of the particle relative to the Σ' frame. Use these to write the four-velocity in frame Σ'.
Compare the two results.

(c) Two particles of mass m are observed in Σ to move in opposite directions with speed v, one in the positive x and one in the negative x direction.
In the frame Σ, find the individual four-momenta of the two particles, and the total four-momentum of this system.
Calculate the velocities of the two particles relative to frame Σ'.
In the frame Σ', find the individual four-momenta of the two particles, and the total four-momentum of this system.
Show that the total four-momentum calculated in Σ' can be obtained by the Lorentz transformation from that calculated in Σ.
Show that the total four-momentum calculated in frame Σ' has the same norm as that calculated in frame Σ.

(d) A photon leaves the origin of Σ at the time $t = 0$ in a direction which forms an angle of $45°$ with the x-axis. What is the angle with the x'-axis, as observed in Σ'?

(e) A meter stick is at rest in frame Σ. It lies in the xy plane and makes angle $\pi/3$ ($= 60°$) with the x-axis. Using the length contraction formula, find the angle that the stick makes with the x' axis, as observed from the Σ' frame. Also find the length of the stick as observed from the Σ' frame.

(f) Relative to the Σ' frame, an object has speed u' and it travels at an angle θ' with respect to the x' axis. (You can take the motion to be in the $x'y'$ plane.) Find the angle that the trajectory relative to the Σ frame makes with the x axis.

(g) A photon has velocity $\vec{u}' = (\frac{3}{5}c, 0, \frac{4}{5}c)$ relative to Σ'. Find the velocity of the photon relative to Σ. Explain how your result is consistent with the constancy of the speed of light.

(h) A photon has velocity $\vec{u} = (\frac{1}{3}c, \frac{2}{3}c, \frac{2}{3}c)$ relative to Σ. Find the velocity of the photon relative to Σ'. Explain how your result is consistent with the constancy of the speed of light.

(i) Let $T^{\mu\nu}$ be an antisymmetric Minkowski tensor. Write down the transformation equations for its components, i.e., write down expressions for each nonzero $(T')^{\alpha\beta}$ in terms of the nonzero components of $T^{\mu\nu}$.

Hint 1: Each component of a 4-tensor can be thought of as the product of two 4-vectors.

Hint 2: Because antisymmetric, $T^{\mu\nu}$ has only six independent components.

(j) How do densities transform? (A container, at rest relative to frame Σ, carries ρ molecules per unit volume. Viewed from Σ', how many molecules per unit volume does it carry?

Hint: First figure out how the volume transforms.

2. Show that the set of pure Lorentz boosts in the same direction form a group.

   Show that the set of all pure Lorentz boosts in all directions do not form a group. Which group property is not satisfied?

3. A particle of mass m is subjected to a constant force F in the x direction. The particle starts from rest at the origin at time $t = 0$. Find the velocity at time t. Find the position at time t.

4. Collision/decay problems.

   (a) A neutral pion of rest mass m and (relativistic) momentum $p = \frac{3}{4}mc$ decays into two photons, One of the photons is emitted in the same direction as the original pion, and the other in the opposite direction. Find the energy of each photon.

   (b) A particle of mass m whose total energy is twice its rest energy collides with an identical particle at rest. If they stick together, what is the mass of the resulting composite particle? What is its velocity?

   (c) A photon with energy E collides with a stationary mass m. They combine to form one particle. What is the mass of this particle? What is its speed?

   (d) A stationary particle decays into two particles of masses m_1 and m_2, having speeds v_1 and v_2 respectively. Show that

$$m_1^2 \left(\gamma(v_1)^2 - 1\right) \;=\; m_2^2 \left(\gamma(v_2)^2 - 1\right)$$

   (e) A particle of rest mass M breaks up into three particles of rest masses m_1, m_2 and m_3. Show that

$$M \geq m_1 + m_2 + m_3$$

   Hint: there is a frame in which this is almost obvious.

   (f) Particle A, while at rest in the lab, decays into particle B and a photon. The rest masses of A and B are m_A and m_B respectively.

   Show that, after the decay, the particle B and the photon each have momentum of magnitude

$$\frac{m_A^2 - m_B^2}{2m_A}c$$

   as measured in the lab frame.

5. In the lab frame, two identical particles (rest mass M) approach each other, each with energ E. Show that, in the frame of one of the particles, the other has energy

$$E' \;=\; \frac{2E^2}{Mc^2} - Mc^2$$

   Hint: There are several ways to show this. You could calculate the total 4-momentum in both frames and use the fact that the norm-squared of the 4-momentum is invariant.

6. A starship traveling at $v = 0.75c$ past the Earth shoots a missile at a speed of $0.5c$ relative to the ship. Find the speed of the missile relative to the Earth if the rocket is fired (a) straight ahead; (b) straight backwards; (c) straight sideways (in the starship's frame of reference).

7. A Lorentz transformation is given by the matrix

$$L = \begin{pmatrix} \frac{5}{4} & -\frac{3}{4}\cos\theta & -\frac{3}{4}\sin\theta & 0 \\ -\frac{3}{4} & \frac{5}{4}\cos\theta & \frac{5}{4}\sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   (a) Verify that L is indeed a Lorentz transformation.

   (b) Find out whether L is proper and whether L is orthochronous.

   (c) Explain why L cannot represent a pure boost.

   (d) If this transformation relates events measured in inertial frames Σ and Σ', find the relative speed between the two frames.
   Hint: you could first find the coordinates of the origin of one of the frames at some time in the other frame.

8. Relative to the ground (G), train A with proper length $2L$ moves rightward at speed v, while train B with proper length $3L$ moves leftward also at speed v.

   We are interested in the time it takes for the trains to pass each other, and so we consider two events: the fronts of the two trains coinciding (event E_1), and the backs of the two trains coinciding (event E_2).

   Find the time interval between the two events as observed in the ground frame G, in the train A frame, and in the train frame B. Find the spatial distance between the two events, according to observers in each of these frames. (Some of these calculations are tricky, and might be messy. Maybe use $c = 1$ units.)

   Calculate the 'invariant interval' in each of these frames, and show that this combination is indeed frame-independent.

9. A tachyon is a hypothetical particle that travels faster than light.

   (a) Draw the world-line of a tachyon on a space-time diagram.

   (b) Show geometrically (by drawing the axes of another frame) that there exists a frame in which the tachyon moves backward in time.

10. The intertial coordinate systems Σ and $\tilde{\Sigma}$ are related by a standard Lorentz transformation (relative velocity v in the common x or $\tilde{x}$ direction). In the second frame, two photons travel along the $\tilde{x}$ axis separated by a distance $\tilde{D}$. Draw the situation in a neat spacetime diagram where both frames are represented, the worldlines of the photons are shown, and the distance between them as measured in the two frames are shown separately.

    Prove geometrically from your drawing that the distance between the photons measured in the Σ system is $\tilde{D}\sqrt{(c-v)(c+v)}$.

11. Show that the time reversal operator and the spatial reflection operator

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

    are members of the Lorentz group, if viewed as transformation matrices.

    Are these transformations orthochronous?

12. Using the definition of Lorentz transformations ($\Lambda^T g \Lambda = g$), find the form of an infinitesimal boost in the x direction.

    Hence write down the generator of $SO(1,1)$.

    Using the orthogonality condition for rotation matrices, find the form of a 2×2 matrix describing an infinitesimal rotation of the coordinate system in the x-y plane. In other words, find the generator for $SO(2)$.

13. Using $c = 1$ units, the covariant components of the electromagnetic field tensor are

$$F_{\mu\nu} \equiv \begin{Bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{Bmatrix}.$$

Show that the equation

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

encodes the two homogeneous Maxwell equations.
($\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{E} = -\frac{\partial \vec{E}}{\partial t}$)

Express the other two Maxwell equations (those with source terms) in terms of $F_{\mu\nu}$ and the covariant current 4-vector J_μ.

14. The 4-velocity and 4-acceleration of a particle are given by

$$V \;=\; (\gamma_v c,\; \gamma_v \vec{v})\;, \qquad A \;=\; \left(\frac{\gamma_v^4}{c} v\dot{v},\; \frac{\gamma_v^4}{c^2} v\dot{v}\vec{v} + \gamma_v^2 \vec{a} \right)\;,$$

where $\vec{v}$ is the 3-velocity, $v = \sqrt{\vec{v}\cdot\vec{v}}$ is the speed, and $\dot{v} = \dfrac{dv}{dt}$ is NOT the 3-acceleration $\vec{a}$. From problem set 1, we know $v\dot{v} = \vec{v}\cdot\vec{a}$.

(a) Show that the 4-acceleration is always space-like.

(b) Write down the 4-velocity in the particle's rest frame, i.e., in the frame where the particle is instantaneously at rest.

(c) Write down the 4-acceleration in the particle's instantaneous rest frame, i.e., in the frame where the particle is at rest at that instant. You may have to introduce the 3-acceleration as measured in the particle's rest frame; let's call this $\vec{a_0}$. This is called the proper acceleration.

(d) Show that $V{\star}V$ and $V{\star}A$ have the same values in the instantaneous rest frame as in the original frame.

(e) Using the invariance of the norm-squared of the 4-acceleration, express the proper acceleration magnitude $|\vec{a_0}|$ in terms of $\vec{v}$ and $\vec{a}$. Comment on the case $\vec{v} \perp \vec{a}$.

(f) Derive the given expression for the 4-acceleration starting from the given expression for the 4-velocity, using $A^\mu = \dfrac{d}{d\tau} V^\mu$, where τ is the proper time of the body.

(g) Consider one-dimensional motion: both $\vec{v}$ and $\vec{a}$ are in the x-direction. Show that

$$A \;=\; \left(\gamma_v^4 (v/c) a,\; \gamma_v^4 a,\; 0,\; 0\right) \;=\; \gamma_v^4 \left(\frac{v}{c} a,\; a,\; 0,\; 0\right)$$

15. **Poincaré transformations.** If Λ is a 4×4 matrix representing a Lorentz transformation, then transformations of the type

$$x' = \Lambda\, x + a$$

are known as Poincaré transformations. Here x and x' are column vectors 4×1 column vectors (4-vectors) representing spacetime coordinates of events as seen from two-different frames, and a is a 4×1 column vector. In other words, a Poincaré transformation is a combination of a Lorentz transformation plus a possible shift of the space and time coordinates. We will denote this transformation as (Λ, a).

(a) Show that the result of two Poincaré transformations (Λ_1, a_1) and (Λ_2, a_2), applied successively, is the Poincaré transformation

$$(\Lambda_2\, \Lambda_1, \Lambda_2\, a_1 + a_2)$$

(b) Find out whether the spacetime interval between two events, $c^2(\Delta t)^2 - (\Delta\vec{r})^2$, is invariant under Poincaré transformations, where $\Delta\vec{r}$ is the spatial three-vector displacement between the two events.

(c) Find out whether the quantity $c^2t^2 - \vec{r}^2$, for an event $(\vec{r}, t)$, is invariant under Poincaré transformations.

(d) Is the application of successive Poincaré transformations associative?

(e) Is the application of successive Poincaré transformations commutative?

(f) Show that the set of all Poincaré transformations form a group.

16. Measured in one inertial frame, events A and B have spatial coordinates

$$(x_A,\, y_A,\, z_A) = (4L,\, -6L,\, 0)\;, \qquad (x_B,\, y_B,\, z_B) = (7L,\, -2L,\, 0)\;,$$

and temporal coordinates

$$t_A = 2L/c\;, \qquad t_A = 12L/c\;,$$

where L is a positive constant.

Calculate the invariant interval between the events. Is this interval timelike, spacelike, or null?

Explain whether there exists a different inertial frame in which the two events occur simultaneously.

17. Inertial frame S' moves at velocity v with respect to another inertial frame S, in the common x direction. A particle has velocity in the common x direction. It's speed is u in the S frame and u' in the S' frame.

    (a) Write down the four-velocity of the particle in the S frame and in the S' frame.

    (b) Since the four-velocity is a four-vector, it should transform according to a Lorentz transformation. Write down the transformations between the four-vector components.

    (c) Use the transformation equations to obtain u' in terms of u.

    (d) Now write down the inverse transformations, and use these to obtain u in terms of u'.
    Comment: You should obtain the velocity addition formula through this procedure.

18. Two masses M move at speed v, one to the east ($+x$ direction) and one to the west ($-x$ direction).

    (a) Write down the four-momenta of each of the particles.

    (b) Write down the total four-momentum in the system.

    (c) Consider the frame of the east-moving particle. Write down the four-momenta of each of the particles, as seen from this frame.

    (d) Write down the total four-momentum of the system, relative to this frame.

    (e) Show that total four-momentum has the same norm in the two frames.

19. Relative to frame Σ, frame Σ' has velocity v in the common z, z' direction. (Note z, not x.)

    (a) A photon has velocity $\vec{u}' = (\frac{3}{5}c, \frac{4}{5}c, 0)$ relative to Σ'. Find the velocity of the photon relative to Σ. Explain how your result is consistent with the constancy of the speed of light.

    (b) A photon has velocity $\vec{u}' = (\frac{3}{5}c, 0, \frac{4}{5}c)$ relative to Σ'. Find the velocity of the photon relative to Σ. Explain how your result is consistent with the constancy of the speed of light.

20. Is the Lorentz group abelian? Explain why or why not.

21. A physicist arrested for running a red light claims that, because of the forward motion of his car, the red light from the traffic signal was Doppler-shifted so that it appeared green. Estimate how fast the physicist would have to be driving for this to be possible. You can take the wavelengths of red and green light to be 650 nm and 510 nm, respectively.

22. Vehicles A, C, and B are traveling along the same straight lane. Vehicle B, traveling at speed $3c/5$, is chased by vehicle C, which in turn is chased by vehicle A. Vehicle A has speed $4c/5$.

    The pilot of vehicle C sees A and B approaching her from two sides with equal speed. What is the speed of vehicle C with respect to the lane?

23. Twin paradox: Caoimhe travels from earth to a distant planet (distance L from earth) and promptly returns; both legs of the journey are at speed v. Her twin Darragh remains on earth.

    Draw a spacetime diagram where Darragh's worldline is vertical, i.e., the time and space axis of Darragh's frame are perpendicular. (i.e., draw the spacetime diagram from Darragh's frame.) Show the wordlines of both twins.

    Draw a spacetime diagram from the frame of Caoimhe's outgoing journey. Show the wordlines of both twins.

    Draw a spacetime diagram from the frame of Caoimhe's return journey. Show the wordlines of both twins.

    In each case, indicate the velocities (inverse of slope) of each worldline segment, relative to that frame.

24. Consider the two Lorentz transformations:

    Λ_A: a pure Lorentz boost of speed c in the positive y direction.

    Λ_B: a pure rotation around the z axis by angle θ.
    (Either clockwise or counterclockwise - you can choose.)

    (a) Write down the 4×4 matrices representing the two transformations.

    (b) Find the transformation obtained by first applying the rotation and then applying the boost.

    (c) Are the two transformations commutative?

25. For any two objects a and b, show that the inner product of their four velocities is

$$V_A \star V_B \;=\; -\,c^2\gamma(v_{\mathrm{rel}})$$

where the v_{rel} is the relative speed of either of the objects with respect to the other.

26. A particle of type A traveling at speed $5c/6$ in an accelerator decays into two identical particles of type B (through the reaction $A \to B + B$). The masses of the two species are related as $m_B = \frac{2}{5}m_A$. After the reaction, the two final particles travel along the same track as the original particle.

    (a) Find the speed of either B particle in the rest frame of the original A particle.

    (b) Using the velocity addition formulae, use your result for the speeds to find the velocities of the final particles in the accelerator frame.

    (c) If the original particle was moving instead at speed $c/6$, what would the velocities of the final particles be in the accelerator frame?

27. A Lorentz transformation is given by the matrix

$$L \;=\; \begin{pmatrix} \frac{25}{12} & -\frac{9}{20} & -\frac{3}{5} & -\frac{5}{3} \\ -\frac{5}{4} & \frac{3}{4} & 1 & 1 \\ 0 & -\frac{4}{5} & \frac{3}{5} & 0 \\ -\frac{4}{3} & 0 & 0 & \frac{5}{3} \end{pmatrix}$$

    (a) Verify that L is indeed a Lorentz transformation.

    (b) Find out whether L is proper and whether L is orthochronous.

    (c) Explain why L cannot represent a pure boost.

    (d) If this transformation relates events measured in inertial frames Σ and Σ', find the relative speed between the two frames.
    Hint: you could first find the coordinates of the origin of one of the frames at some time in the other frame.

28. When he returns his rented rocket after one week's cruising in the galaxy, Mr. Spock is shocked to be billed for three weeks' rental. Assuming that he traveled straight out and straight back, always at the same speed, how fast was he traveling?

29. Use the fact that rapidities are additive to derive the relativistic formula for addition of velocities along the same direction.

    In other words, if frame S' has speed $v_1 = c\tanh\phi_1$ relative to frame S, and frame S'' has speed $v_2 = c\tanh\phi_2$ relative to frame S' (in the same direction), use the fact that the rapidity of S'' relative to S is $(\phi_i + \phi_2)$, to find the speed of S'' relative to S.

    You may need to derive an identity concerning the hyperbolic tangent of the sume of two quantities. You can derive such an identity using the relations

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$$
$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

    and the definition $\tanh z = \sinh z / \cosh z$.

30. A mass m moves at speed u. It decays into three photons, one of which travels in the forward direction, and the other two of which move at angles of $\frac{\pi}{3} = 120°$ relative to the original direction of motion. What are the energies of these three photons?

31. A particle of mass m starts at rest. You push on it with a constant force F. How much time t does it take for the mass to move a distance d? (Both t and d here are measured in the lab frame.)

    What is the energy of the particle when it has been moved by a distance d?

32. A train of proper length $12d$ moves at speed $4c/5$. How much time does it take to pass a person standing on the ground, as measured by that person?

    Solve this by working in the frame of the person, and then again by working in the frame of the train.