1 Some four-vectors, old and new

We will now introduce a few four-vectors. A few of these will be related to a moving object or particle (4-analogs of velocity, acceleration, etc). To prepare for defining these, we will first review our understanding proper time (Subsection 1.1), with particular reference to a moving object.

Once we have identified a 4-vector, we can also identify a corresponding Lorentz scalar or invariant: its norm. These invariants sometimes have easy interpretations, but not always.

1.1 Preparation: proper time and time derivatives

We will construct several 4-vectors describing the kinematics or dynamics of a moving body, such as the 4-vector analogs of ordinary displacement $\vec{r}$, velocity $\vec{u}$, acceleration $\vec{a}$, momentum $\vec{p}$, etc.

In ordinary mechanics, derivatives with respect to time (rates of change) are important: the velocity is the temporal derivative of displacement, the acceleration is the temporal derivative of velocity, etc. In nonrelativistic mechanics, time is invariant (frame-independent). However, when we consider corresponding derivatives in relativistic mechanics, time will have to be specified relative to a particular frame.

It is useful to consider the time as measured from an inertial frame that is, at the moment of consideration, attached to the object, i.e., relative to which the object is at rest. (If the object has nonzero acceleration, the inertial frame cannot remain attached to it, i.e., the object cannot remain at rest relative to the frame. However, at any point of the history of the object, one can identify an inertial frame relative to which the object is at rest.) This is known as the **rest frame** of the object.

If two nearby events occur at the location of the object, then the time interval between them as measured in the rest frame of the object is the *proper* time interval.

If the object moves with velocity $\vec{u}$ relative to the laboratory frame, then the proper time interval between two events on the object is

$$c d\tau = \sqrt{c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}.$$

This is the time interval measured from a frame relative to which the object is at rest. Relative to the lab frame, (dx, dy, dy) is the displacement of the object in time dt, thus $u = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}/dt$. Therefore

$$c d\tau = \sqrt{c^2(dt)^2 - u^2(dt)^2} = c dt\sqrt{1 - u^2/c^2} \qquad \Longrightarrow \qquad \boxed{d\tau = \frac{dt}{\gamma_u}}$$

If the motion of the object is uniform, one could write this in terms of finite differences:

$$\Delta\tau = \frac{\Delta t}{\gamma_u}$$

1.2 Space-time or 4-displacement

Space-time is the 4-vector we have been dealing with from early on. A space-time 4-vector is the coordinate of an event

$$(ct, x, y, z)$$

or an 'interval' between events:

$$(c\Delta t, \Delta x, \Delta y, \Delta z) \qquad \text{or} \qquad (cdt, dx, dy, dz).$$

Each event is a point in Minkowski space, thus a spacetime 4-vector is a 'displacement' in Minkowski space. It is the 4-vector analog of the displacement between spatial points in ordinary 3-space. So we could call it a 4-displacement.

The norm of every 4-vector is a scalar, i.e., a Lorentz invariant. Consider the 4-displacement $\Delta\overset{\Rightarrow}{x} = (c\Delta t, \Delta x, \Delta y, \Delta z)$ representing the interval between two events. Its norm is

$$\left(\Delta\overset{\Rightarrow}{x}\right) \star \left(\Delta\overset{\Rightarrow}{x}\right) = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

which is the Lorentz-invariant interval between the events, as we established very early in the semester.

If the interval $\Delta\overset{\Rightarrow}{x}$ is time-like, i.e., if the norm is positive, then the norm could be written as $c^2(\Delta\tau)^2$, where τ is the proper time interval between the events. If the norm is negative (the interval is space-like), then there is no frame in which the two events are equilocal, so it would not make sense to talk about "proper time".

1.2.1 Notation blues

Above, I have used the notation $\overset{\Rightarrow}{x} = (ct, x, y, z) = (ct, \vec{r})$ for the space-time coordinate 4-vector. Remember the warning that the double-vector notation is not standard, and only invented for these notes.

More conventional notation would be to simply use x for the spacetime coordinate. Then you might write

$$x = (ct, x, y, z).$$

This has the unpleasant feature that the same symbol x means different things on left and right sides of the equation.

It is also common to refer to the spacetime coordinate as x^μ. You might write

$$x^\mu = (ct, x, y, z).$$

You might find this also a bit strange: does x^μ refer to a particular component of the 4-vector depending on the value of μ, or does it represent the complete 4-vector? You have to get used to x^μ sometimes representing the μ-th element of the 4-vector, and sometimes the 4-vector itself.

There is no standard notation that is completely satisfactory. You will encounter both of these mildly inconsistent manners of describing 4-vectors. We will gradually wean ourselves off the double-vector notation and move to these more standard notations, and hope that from the context you can always figure out which symbol refers to what.

1.3 4-velocity

In non-relativistic mechanics, the velocity of an object is obtained by taking the derivative of displacement: $\vec{u} = \frac{d\vec{r}}{dt}$.

Since the displacement (position) vector now appears as components of a 4-vector (spacetime), we could think of defining a velocity 4-vector as the temporal derivative of the spacetime coordinate:

$$\vec{\vec{U}} \stackrel{?}{=} \frac{d}{dt}(ct, \vec{r}) = (c, \vec{u}).$$

One can show (through a somewhat painful calculation) that this combination does not transform like a 4-vector. More easily, one observes that $c^2 - |\vec{u}|^2$ is not Lorentz invariant and hence is not the norm of a 4-vector.

To see what went wrong, we write this is

$$\frac{d}{dt}(ct, \vec{r}) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(c\Delta t, \Delta\vec{r})$$

Here $\Delta\vec{r}$ is the displacement of the particle in time interval Δt, as seen from some frame which is *not* the rest frame of the particle. We know that Δt is not a Lorentz scalar. Since we are dividing a 4-vector by a number which is not a scalar, we cannot expect to find a 4-vector. The solution is to divide by a scalar (Lorentz invariant) quantity, i.e., to replace Δt by the proper time interval $\Delta\tau = \Delta t/\gamma_u$. Thus we define the 4-velocity as

$$\vec{\vec{U}} = \frac{d}{d\tau}(ct, \vec{r}) = \gamma_u \frac{d}{dt}(ct, \vec{r}) = \gamma_u(c, \vec{u})$$

You can show (going through a cumbersome calculation) that this object trasforms under Lorentz transformations as a 4-vector should. We also immediately see that the norm is $\gamma_u^2 c^2 - \gamma_u^2 u^2 = c^2$, which we know to be Lorentz invariant. Thus we have successfully constructed a 4-vector.

Definition: 4-velocity

If an object has 3-velocity $\vec{u} = (u_x, u_y, u_z)$ and speed $u = |\vec{u}|$, then its 4-velocity is

$$\gamma_u(c, \vec{u}) = (\gamma_u c, \gamma_u \vec{u}) = \left(\gamma_u c, \gamma_u u_x, \gamma_u u_y, \gamma_u u_z\right)$$

- The 4-velocity of any particle with nonzero mass is *time-like*, i.e., has positive norm in the $(+, -, -, -)$ metric. Exercise: Show!

- For a photon (massless particle), the speed is $u = c$ so that γ_u is infinite. The definition of 4-velocity unfortunately doesn't make much sense in this case.

  Why does the definition of the 4-velocity not work for a photon? The definition relied on the notion of proper time: the time measured in a frame relative to which the object is at rest. But proper time is not defined for a photon, because there is no frame relative to which a photon is at rest. So, no surprise that the definition only applies to massive objects.

- Using a capital U for 4-velocities seems to be common notation.

  Our definition of the 4-vector could be written as

$$U^\mu = \gamma_u(c, \vec{u}) = \left(\gamma_u c, \gamma_u u_x, \gamma_u u_y, \gamma_u u_z\right)$$

  with the understanding that U^μ on the left represents the whole 4-vector and not one component of the 4-velocity. We will be gradually shifting toward this type of notation and stop using the non-standard thick-vector notation.

  Some texts also use **U** or $\vec{U}$ to represent the 4-vector. I would advise against this. No need to encourage confusion between 4-vectors and 3-vectors!

- For a particle/object at rest, the 4-velocity is simple: $(c, 0, 0, 0)$. Exercise: Show!

  In other words, the 4-velocity of any massive object in the frame moving with the object is $(c, 0, 0, 0)$

- The fact that a 4-velocity is a 4-vector (transforms according to Lorentz transformations) can be used to derive the equations of relativistic 3-vector addition. Consider a particle having 4-velocity $U = \gamma_u(c, u_x, u_y, u_z)$

relative to Σ and $U' = \gamma_{u'}(c, u'_x, u'_y, u'_z)$ relative to Σ'. If Σ and Σ' are related to the standard boost, then

$$\begin{pmatrix} U'^0 \\ U'^1 \\ U'^2 \\ U'^3 \end{pmatrix} = \begin{pmatrix} \gamma_v & -\gamma_v (v/c) & 0 & 0 \\ -\gamma_v (v/c) & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U^0 \\ U^1 \\ U^2 \\ U^3 \end{pmatrix}$$

These equations (and their inverted form) can then be used to express the components of $\vec{u}$ in terms of the components of $\vec{u}'$, and vice versa.

Exercise! This is hopefully irresistible.

- Adding two 4-velocities does not physically mean much, in fact, the result might not even be a valid 4-velocity.

Exercise: Take two relatively simple 4-velocities and add them. (E.g., the 4-velocities of a particle at rest and a particle moving along the x direction.) Find out if the resulting 4-vector has the form of a 4-velocity.

1.4 4-momentum

We have already encountered this 4-vector:

Definition: 4-momentum

An object with energy E and 3-momentum $\vec{p} = (p_x, p_y, p_z)$ has 4-momentum $(E/c, \vec{p})$.

- For a massive particle of mass m moving with 3-velocity $\vec{u}$, we know that $E = \gamma_u mc^2$ and $\vec{p} = \gamma_u m\vec{u}$. The 4-momentum is

$$P^\mu = (\gamma_u mc, \gamma_u m\vec{u}) = \gamma_u m(c, \vec{u}) = \gamma_u m(c, u_x, u_y, u_z).$$

We notice that the right side is m times the 4-velocity! Thus

$$P^\mu = mU^\mu \quad \text{or} \quad \vec{\vec{P}} = m\vec{\vec{U}} \qquad \begin{cases} \text{for objects with} \\ \text{nonzero mass.} \end{cases}$$

This is similar to the familiar 3-vector relation $\vec{p} = m\vec{v}$, which is very pleasant. Of course, you cannot in general explect 4-vectors to obey the same relations among themselves as the corresponding 3-vectors do.

- We have previously seen that the combination $(\gamma_u mc, \gamma_u m\vec{u})$ transforms like a 4-vector under Lorentz transformations. In fact, this is how we motivated the notion of 4-vectors.

  > Exercise: Actually, we worked out the transformation properties only for the case where the object moves in the same direction as the boost. ($\vec{u}$ was in the common x, x' direction, for our standard boost.) Can you work out the transformations when the object 3-velocity is in an arbitrary direction?

- For a massless particle, i.e., a photon, we know how to express the momentum and energy in terms of its frequency f or its wavelength λ:

$$E = hf = \frac{hc}{\lambda} \quad \text{and} \quad \vec{p} = \frac{hf}{c}\hat{n} = \frac{h}{\lambda}\hat{n}$$

  where $\hat{n}$ is a unit vector in the direction of propagation of the photon. Hence the 4-momentum is

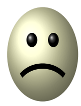
$$P^\mu = \left(\frac{hf}{c}, \frac{hf}{c}\hat{n}\right) = \left(\frac{h}{\lambda}, \frac{h}{\lambda}\hat{n}\right) .$$

  For example, if the photon were traveling in the z direction, its 4-momentum would be $(h/\lambda, 0, 0, h/\lambda)$.

- The 4-momentum of a photon also transforms under LT's as a 4-vector should, although we did not show this explicitly. To show this, we need to use our knowledge of the relativistic Doppler shift.

  > Exercise: Consider inertial frame $\tilde{\Sigma}$ related to Σ by a standard boost. A photon moving in the common x, x' direction has frequency $\tilde{f}$ as measured from $\tilde{\Sigma}$. Find the frequency measured from Σ. Hence show that the 4-momentum of the photon, as measured from $\tilde{\Sigma}$ and Σ, are related by the standard Lorentz transformation for this boost.

- The expression $P^\mu = \text{mass} \times U^\mu$ does not make sense for photons. This is because 4-velocity is not defined for photons, and the mass is zero.



- For a massive particle, the norm of the 4-momentum is $P_\mu P^\mu = m^2c^2$ (if you didn't previously: Show!). This is Lorentz invariant, of course. Also, since this is positive, the 4-momentum of particles with nonzero mass is *time-like*.

  For a mass-less particle, the norm is $P_\mu P^\mu = 0$, also an invariant. The 4-momentum in this case is *null* or *light-like*.

1.5 4-acceleration

We constructed the 4-velocity from the 4-displacement by differentiating with respect to the proper time. Similarly, the 4-acceleration is obtained by differentiating the 4-velocity with respect to the proper time:

Definition: 4-acceleration

$$A^\mu = \frac{dU^\mu}{d\tau}$$

- If the three-velocity of the particle/object is $\vec{u}$, then

$$A^\mu = \frac{d}{d\tau}(\gamma_u c, \gamma_u \vec{u}) = \gamma_u \frac{d}{dt}(\gamma_u c, \gamma_u \vec{u}) = \gamma_u(\dot{\gamma}_u c,\ \dot{\gamma}_u \vec{u} + \gamma_u \vec{a}).$$

  Here the dot represents a derivative with respect to the coordinate time t (and **not** with respect to the proper time τ). Also, $\vec{a} = \frac{d\vec{u}}{dt}$ is the 3-acceleration.

- Early in the semester, we derived the expressions

$$\dot{\gamma}_u = \frac{1}{c^2}\gamma_u^3\,(\vec{u}\cdot\vec{a}) = \frac{1}{c^2}\gamma_u^3 u\dot{u}$$

  for the time derivative of the Lorentz factor. Here, $u = |\vec{u}|$ is the speed and $\dot{u}$ is the time derivative of the speed, which is **not** the acceleration. So the 4-acceleration can be written as

$$A^\mu = \left(\frac{1}{c}\gamma_u^4 \vec{u}\cdot\vec{a},\ \frac{1}{c^2}\gamma_u^4\,(\vec{u}\cdot\vec{a})\,\vec{u} + \gamma_u^2\vec{a}\right)$$

  or

$$A^\mu = \left(\frac{1}{c}\gamma_u^4 u\dot{u},\ \frac{1}{c^2}\gamma_u^4 u\dot{u}\vec{u} + \gamma_u^2\vec{a}\right).$$

  Admittedly, neither of these expressions are very pretty, and not worth memorizing.

- Consider the frame of the object, i.e., the inertial frame in which the object is (at this instant) at rest. This frame has to be re-defined at every instant — if we kept a frame attached to an accelerating objject, that would not be an intertial frame.

  In its instantaneous frame, the object has zero velocity and zero speed. The 4-acceleration becomes $A^\mu = (0, \vec{a}_0)$ (Show!), where $\vec{a}_0$ is the acceleration measured in the rest frame of the object, i.e., the acceleration experienced by the object itself. This is called the **proper acceleration**.

- Like other 4-vectors, the norm $A_\mu A^\mu$ is an invariant. There is a simple expression only in the frame where the object is (instantaneously) at rest: $A_\mu A^\mu = -\vec{a}_0 \cdot \vec{a}_0$. The norm of the 4-acceleration is thus the negative square of the proper acceleration.

- The 4-acceleration of a particle happens to be always 'orthogonal' to its 4-velocity.

  Exercise: Show that $\boxed{U_\mu A^\mu = 0}$, or $\boxed{\overset{\Rightarrow}{U} \star \overset{\Rightarrow}{A} = 0}$, using the expressions derived for them, for a particle having 3-velocity $\vec{v}$ and 3-acceleration $\vec{a}$.

  Since inner products are Lorentz invariant, if $U_\mu A^\mu$ is zero in one inertial frame, then it vanishes in any inertial frame.

  Exercise: Write down U^μ and A^μ in the rest frame of the particle. Use these to show that $U_\mu A^\mu = 0$.

1.6 4-force

We generalize the defintion of the 3-force, $\vec{F} = \dot{\vec{p}}$, in the by-now familiar way. The 4-force on a particle/object is

Definition: 4-force
$$\boxed{K^\mu = \frac{dP^\mu}{d\tau}}$$

where P^μ of is the 4-momentum of the particle/object.

- **Notation:** Why K^μ instead of, say, F^μ or $\overset{\Rightarrow}{F}$?

  Because we will encounter an object called $F^{\mu\nu}$ (with two indices) in electromagnetism, and it would be confusing to have two quantities called F. So the notation K^μ is common. It probably stems from the German word for force, *Kraft*.

  If you are not dealing with electromagnetism, using F^μ is fine of course.

- Since $P^\mu = mU^\mu$ and $A^\mu = \frac{d}{d\tau}U^\mu$, we get for objects which are not losing/gaining mass
  $$K^\mu = mA^\mu$$
  
  The 4-force is mass times the 4-acceleration. Nice! Even though the relativistic 3-force and 3-acceleration refuse to obey the familiar equation, the 4-force and 4-acceleration do obey a relativistic version.

- Noting $\frac{d}{d\tau}P^\mu = \gamma_u \frac{d}{dt}(E/c, \vec{p})$, we get
  $$K^\mu = \gamma_u \left(\frac{1}{c}\frac{dE}{dt}, \vec{F}\right)$$
  The 3-vector part of the 4-force is γ_u times the 3-force.

- Using similar calculations as previously done for the 4-acceleration (or just using $K^\mu = mA^\mu$), one obtains the not-pretty expressions

$$K^\mu = \left(\frac{m}{c}\gamma_u^4 \vec{u}\cdot\vec{a},\ \frac{m}{c^2}\gamma_u^4\left(\vec{u}\cdot\vec{a}\right)\vec{u} + m\gamma_u^2\vec{a}\right)$$

and

$$K^\mu = \left(\frac{m}{c}\gamma_u^4 u\dot{u},\ \frac{m}{c^2}\gamma_u^4 u\dot{u}\vec{u} + m\gamma_u^2\vec{a}\right).$$

- If the velocity at some instant is in the x direction ($v = v_x$ and $v_y = v_z = 0$), then

$$K^\mu = \left(\frac{m}{c}\gamma_u^4 u_x a_x, \gamma_u^4 m a_x, \gamma_u^2 m a_y, \gamma_u^2 m a_z\right).$$

Exercise: Show. This corresponds to the 3-force being

$$\vec{F} = \left(\gamma_u^3 m a_x, \gamma_u m a_y, \gamma_u m a_z\right).$$

1.7 4-potential in electromagnetism

If you have taken a semester on electromagnetism or electrodynamics, you might know that the electric field $\vec{E}$ and the magnetic field $\vec{B}$ can be expressed in terms of two potentials: the scalar potential ϕ and the vector potential $\vec{A}$:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

(In an introductory electromagnetism module, it might be more common to use $V(\vec{r}, t)$ instead of $\phi(\vec{r}, t)$. We will use ϕ.)

The scalar and vector potential combine together to form a 4-vector, which we could call the 4-potential:

$$A^\mu = \left(\frac{\phi}{c}, \vec{A} \right) .$$

You might also remember (or learn soon) that the physical fields $\vec{E}$ and $\vec{B}$ remain unchanged under *gauge transformations*:

$$\phi \to \phi + \frac{\partial f}{\partial t}, \qquad \vec{A} \to \vec{A} - \vec{\nabla} f ,$$

where $f(\vec{r}, t)$ is a scalar function. In 4-vector language, this has the concise form

$$A^\mu \to A^\mu + \partial^\mu f .$$

It's almost special relativity was custom-built for electromagnetism! (Remember: it was indeed.) You might have guessed that ∂^μ, the 4-gradient, is the 4-vector operator

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right) .$$

Notice the minus sign on the spatial derivatives. Where did those come from? This will hopefully become clearer when we introduce index notation.

What about the electric and magnetic fields themselves? It seems likely that $\vec{E}$ and $\vec{B}$ should also be bundled into objects that transform nicely under Lorentz transformations? They are, but these objects are not 4-vectors, rather, they are 4-tensors which are more complicated. We will postpone discussion of the electromagnetic field tensor (and their connection to the 4-potential) until after we've mastered index notation. At that point we will formulate all the basic laws of electrodynamics in terms of 4-vectors and 4-tensors.

The invariant associated with the 4-potential is the quantity

$$\frac{\phi^2}{c^2} - |\vec{A}|^2$$

I have no idea what this means or why this particular quantity is invariant under Lorentz transformations.

Exercise: The quantity $\phi^2/c^2 - |\vec{A}|^2$ is Lorentz invariant. Is it also gauge-invariant?

1.8 Density-current, or 4-current-density

In electromagnetism, you would have met the charge density ρ (the charge per unit volume) and the current density $\vec{J}$, a 3-vector with direction pointing towards the flow of charge and magnitude giving the current through a unit perpendicular area. These two quantities together form a 4-vector

Definition: 4-current $\boxed{J^\mu = (c\rho, \vec{J})}$ or $\boxed{J^\mu = \mu_0(c\rho, \vec{J})}$

The first definition is more common. The second definition (including the magnetic permeability of SI units) arguably makes future formulae easier if we are using SI units.

As you know, electromagnetism has the annoyance of "choice of units." In special relativity, choices of metric and other conventions are a plague. Taken together, when one studies electromagnetism in relativistic notation, a nightmare of differing notation can be expected: comparing different texts can be very difficult. Using $c = 1$ and another set of electromagnetism units (the Heaviside-Lorentz units instead of SI units) might make things easier. We will get through unit issues somehow.

- **Transformation under standard boost:**

  We will now show that the combination $(c\rho, \vec{J})$ indeed transforms like a Lorentz 4-vector. We focus on the case of the standard boost with the charge having velocity component only in the common x, x' direction. Generalization to arbitrary directions of $\vec{J}$ and arbitrary Lorentz transformations is of course possible, but cumbersome.

  Consider a blob of charge moving with velocity $\vec{u}$, $\vec{u}'$ relative to Σ, Σ' frames. Mercifully, these velocities only have x, x' components. In the rest frame of the charge (call it Σ_0), the charge density is ρ_0. Observed from Σ or Σ', the charge density will be different (ρ or ρ'). Then the current densities observed from the Σ and Σ' are respectively $\vec{J} = \rho\vec{u}$ and $\vec{J}' = \rho'\vec{u}'$. We need only consider the x components:

$$J_x = \rho u = \rho u_x, \qquad J'_x = \rho' u' = \rho' u'_x \,.$$

As observed from the Σ frame, the size (length) of the charge-carrying region in the x direction is contracted with respect to the rest frame by a factor γ_u. The contraction factor is γ_u because u is the relative speed between Σ and the rest frame Σ_0. Since the volume of the charged region is observed from Σ to be smaller, the density $= \dfrac{\text{charge}}{\text{volume}}$ is observed to be larger: $\rho = \rho_0\gamma_u$. Similarly, the charge density is observed from Σ' to be $\rho' = \rho_o\gamma'_u$.

We can now work out how (ρ, J_x) are related to (ρ', J'_x), by using the velocity transformation equation:

$$u = \frac{u'+v}{1+u'v/c^2}, \qquad \gamma_u = \gamma_{u'}\gamma_v\left(1+u'v/c^2\right). \tag{1}$$

We obtain

$$\rho = \gamma_u\rho_0 = \gamma_{u'}\gamma_v\left(1+u'v/c^2\right)\rho_0$$
$$= \gamma_v\left(\rho_0\gamma_{u'} + \rho_0\gamma_{u'}u'\frac{v}{c^2}\right) = \gamma_v\left(\rho' + \frac{v}{c^2}J'_x\right)$$

Similarly, you can show that $J_x = \gamma_v\left(J'_x + v\rho'\right)$. Exercise: Show.

Thus, we have shown that $(c\rho, J_x)$ transforms like (ct, x) for a standard Lorentz boost.

This result can be extended, of course, to show that $(c\rho, \vec{J})$ transforms like $(ct, \vec{r})$ for an arbitrary Lorentz transformation.

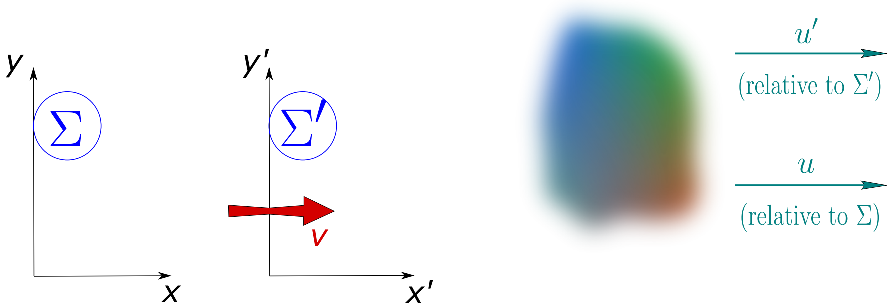


Figure 1: A charge distribution (cloud of charge) moves in the common x, x' direction. It has speed u' relative to Σ' and speed u relative to Σ. Its charge density is ρ_o in its rest frame, ρ when measured from the Σ frame, and ρ' when measured from Σ'.

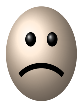
- In electrodynamics, the conservation of charge is expressed through the beautiful continuity equation: $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$. In index notation, this is expressed in the (even more beautiful) form:

  $$\boxed{\text{continuity equation}} \qquad \partial_\mu J^\mu = 0 \quad \text{or} \quad \partial^\mu J_\mu = 0.$$

  We will revisit this after we've learned index notation.

- If we consider mass density (also just known as density) and mass current density, than exactly the same derivation holds. Mass density and mass current density also form a 4-vector.

- At the end of an electrodynamics class, you study the conservation of electromagnetic energy: the energy density and energy current density (Poynting vector) together obey a continuity equation. One wonders: Should the energy density and energy current density also form a Minkowski 4-vector?

  The answer is unfortunately no. This is because the energy itself is not invariant under Lorentz transformations, in contrast to mass or charge. So, the derivation above breaks down. If we try a calculation analogous the one above, the energy density does not get just a factor of γ_u (unlike charge density or mass density) when transforming from Σ_0 to Σ; its transformation is more complicated. As a result we cannot construct a valid 4-vector out of energy-density and energy-current.

1.9 Frequency-wavevector, or 4-wavevector

1.10 4-gradient