1 The Rotation group

1.1 2D rotations

Consider all possible rotations around the z axis. These transform the planar coordinates (x, y) according to transformation matrices of the form

$$\mathcal{R}_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

with the real number θ representing the angle of rotation. These matrices are orthogonal and have unit determinant. The set of these matrices form a group under group multiplications. (Equivalently, the set of rotations around the z axis form a group of transformations.) This group is known as $SO(2)$.

Sometimes the rotation group is defined to include reflections. This means the set of *all* orthogonal 2×2 matrices, not only those with determinant $= +1$. Imagine reflecting around the y axis, a transformation represented by $P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. (Please check!) If we consider all matrices of the form

$$P\mathcal{R}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

then these represent a reflection plus a rotation within the x-y plane. These matrices have determinant -1 and are also orthogonal.

The larger set containing all orthogonal matrices — those with positive determinant and those with negative determinant — is called $O(2)$. The S is missing in this name — S stands for 'special', meaning determinant exactly $+1$.

The group $O(2)$ has two 'disjoint' pieces — one containing matrices with determinant $+1$, known as $SO(2)$, and the other containing the rest of the matrices, whcih have determinant -1. This piece of $O(2)$ does not have a name.

The unit matrix is contained within $SO(2)$. All elements of $SO(2)$ are continuously connected to the identity element (unit matrix), as you can tune θ continously to zero to turn $\mathcal{R}$ into the identity matrix.

The negative-determinant part of $O(2)$ does not contain the unit matrix, and these matrices cannot be continuously deformed into the unit matrix. To obtain the unit matrix from $P\mathcal{R}$, one has to tune θ but also multiply by P, i.e., reflect, which is a violent discontinuous operation. This part of $O(2)$ does not form a group under matrix multiplication, as it misses an identity element.

$SO(2)$ is a simpler group than those we will meet later. The multiplication of two matrices corresponding to rotation angles θ_1 and θ_2 results in another $SO(2)$ matrix of rotation angle $\theta_1 + \theta_2$. So simple. Because addition of angles is a commutative operation, multiplication of $SO(2)$ matrices must be commutative as well. This is an abelian group.

1.2 3D rotations

The set of all 3D rotations is known as $SO(3)$. This is a non-abelian group. It is the set of orthogonal 3×3 matrices whose determinant is $+1$.

If reflections are included, we obtain the larger group $O(3)$, which is the set of *all* orthogonal 3×3 matrices. Just as we had in the two-dimensional case, this means including matrices with determinant -1. The identity matrix belongs to $SO(3)$ and is continuously connected to the members of $SO(3)$, but is not continuously connected to negative-determinant members of $O(3)$.

The group $O(3)$ can be defined as the set of 3×3 matrices which, acting on 3-vectors (x_1, x_2, x_3), preserves the norm(-squared)

$$\vec{x} \cdot \vec{x} = (x_1)^2 + (x_2)^2 + (x_3)^2 .$$

Equivalently, it is the set of 3×3 matrices which are orthogonal.

1.3 n-dimensional rotations

Clearly, this is straightforwardly generalized: $O(n)$ is the set of $n \times n$ matrices which, when acting on n-vectors $(x_1, x_2, \ldots, x_n)$, preserves the quantity

$$(x_1)^2 + (x_2)^2 + \ldots (x_n)^2 \;=\; \sum_{j=1}^{n} (x_j)^2 .$$

Restricting to those elements which have positive determinant ($= +1$), we get the subgroup $SO(n)$.

2 Lorentz group

The Lorentz group is introduced and discussed in a series of statements and observations below. Hopefully you will find that you already know some of these things.

- A fundamental property of Lorentz transformations is the invariance of $x_\mu x^\mu = c^2t^2 - x^2 - y^2 - z^2$. An equivalent statement is that Lorentz transformation matrices satisfy

$$\Lambda^{\mathrm{T}} g \Lambda = g\,. \tag{1}$$

  Formally, any transformation Λ satisfying Eq. (5) is a Lorentz transformation.

- The set of matrices satisfying Eq. (5) is called the Lorentz group.

  One can show that this set of matrices obey all four aspects of the mathematical definition of a group, under the operation of matrix multiplication.

- Looks like we just defined the Lorentz group as a group of matrices. Physically, of course, the Lorentz group is the group of transformations represented by these matrices. The group operation (matrix multiplication) corresponds to successive application of transformations.

  For example, if Λ_1 and Λ_2 are matrices representing two Lorentz transformations, then the matrix $\Lambda_1\Lambda_2$ represents the following transformation: apply Λ_2 first, and then apply Λ_1. (Note the order.)

- The metric g in Eq. (5) could be either

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{or} \quad g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

  The defining equation (5) is not affected by this choice, since the two g's differ only by a sign.

  The negative signature metric is preferred in particle physics and quantum field theory, while the positive signature metric is preferred in general relativity. We've mostly used the first, in this semester.

- The Lorentz group, as defined, includes also reflections of time and reflections of space. These are not very physical transformations. If we omit these, we obtain the set of PROPER ORTHOCHRONOUS transformations, or the set of physical Lorentz transformations. This restricted set itself forms a group, known as the restricted Lorentz group.

- The sign of the time coordinate gets flipped by an LT if $\Lambda^0{}_0$ is negative.

  A Lorentz transformation that retains the sign of the time coordinate is called orthochronous.

  We will show later that $\left|\Lambda^0{}_0\right| \geq 1$, i.e., values of $\Lambda^0{}_0$ between -1 and $+1$ are excluded.

  Thus a Lorentz transformation Λ is

$$\begin{array}{rll} \text{orthochronous} & \text{if} & \Lambda^0{}_0 > 1, \\ \text{non-orthochronous} & \text{if} & \Lambda^0{}_0 < -1. \end{array}$$

  I hope you like the word 'orthochronous'; you can impress your friends and relatives with it. To the best of knowledge, the word has only this meaning and does not mean anything else in other contexts. (Sigh of relief.)

- **Proper and improper:**

  You can show from the definition (5) that the determinant-squared of an LT is 1, so that $\det \Lambda$ is either $+1$ or -1. The LT is proper if $\det \Lambda = 1$ and improper if $\det \Lambda = -1$.

  This terminology is the same as that used for 3×3 rotation matrices $\mathcal{R}$. If $\det \mathcal{R} = -1$, the matrix represents a reflection in addition to a rotation.

  In the case of LT's, $\det \Lambda = -1$ can mean either spatial reflection or temporal reversal. But not both: A Lorentz transformation that involves both time reversal and spatial reflection will have $\det \Lambda = +1$ and hence is 'proper'. Of course, you would probably not regard this transformation as being physical, despite the name proper.

- The matrix

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  represents the operation of time reversal, as you can see by applying it to a spacetime coordinate (ct, x, y, z). It is a valid Lorentz transformation according to the definition (5). (Please check!)

  The matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

  represents the operation of spatial reflection, also known as a PARITY TRANSFORMATION. This is also a valid Lorentz transformation.

Note that the matrices T and P look formally identical to the metric tensor g or its negative. This does not have any physical meaning, as g does not represent a transformation.

- The unit matrix is a member of the Lorentz group defined by Eq. (5) and hence represents a valid Lorentz transformation.

  What transformation does the unit matrix represent? The do-nothing transformation, of course. It's the transformation that takes you from the spacetime coordinates measured from frame Σ to the spacetime coordinates measured from the same frame Σ.

  The unit matrix is a proper and orthochronous LT. (Please check.)

- Members of the restricted Lorentz group, i.e., the proper orthochronous Lorentz transformations, are connected continuously to the unity matrix. Those LT's which involve flipped temporal or spatial coordinates are NOT continuously connected to the unit matrix.

  Thus, the Lorentz group can be broken into four disjoint pieces, illustrated in the table:

	$\Lambda^0{}_0 > 1$ ORTHOCHRONOUS	$\Lambda^0{}_0 < 1$ NON-ORTHOCHRONOUS
$\det \Lambda = +1$ PROPER	$\Lambda^{(p.o.)}$	$TP\Lambda^{(p.o.)}$
$\det \Lambda = -1$ IMPROPER	$P\Lambda^{(p.o.)}$	$T\Lambda^{(p.o.)}$

  We can't list all the elements in each of the four blocks; so we have labeled them with representative matrices. Here $\Lambda^{(p.o.)}$ is an arbitrary proper orthochronous Lorentz transformation, i.e., a representative of the set of physical Lorentz transformations. The top left block represents all transformations continuously connected to this one, i.e., the whole set of physical Lorentz transformations. The lower left block represents all transformations continuously connected to $P\Lambda^{(p.o.)}$, which is the matrix obtained by reflecting the spatial components of $\Lambda^{(p.o.)}$. You should be able to guess the definitions of the other two blocks.

The identity matrix belongs to the top left segment of the Lorentz group. Thus, the other blocks in the table cannot form groups by themselves — they lack the identity element.

- The Lorentz group is represented as $O(3,1)$ or $O(1,3)$. The numbers show that one of the components is treated specially, i.e., that one of the diagonal elements of the 4×4 metric tensor has opposite sign.

  The subset of the Lorentz group that is *proper* is also a group.

  > Exercise: Show that this subset satisfies closure, i.e., that the product of any two LT matrices having determinant $+1$ is also an LT matrix having determinant $+1$.

  The proper Lorentz group consists of the two upper blocks in the table above. This group is called $SO(3,1)$. The 'S' stands for 'special', meaning positive determinant.

  If we further restrict to transformations that are both proper and orthochronous, we obtain the restricted Lorentz group. This is the top left block in the table. This group is represented by the very fancy name $SO^+(3,1)$. The superscript $+$ indicates that time is moving forward, in the physically meaningful direction.

  Most people would think of $SO^+(3,1)$ as the class of physical transformations. Sometimes, when people say 'Lorentz transformations', they might mean only this class of transformations, leaving out most of the full Lorentz group. As always, you have to figure out from the context what is meant.

  I have also seen the notation $SO(3,1)^\uparrow$ or $SO^\uparrow(3,1)$ used for the restricted (physical) Lorentz group, i.e., using a $\uparrow$ instead of a $+$ symbol as superscript.

- Showing that $\left|\Lambda^0{}_0\right| \geq 1$

  We've claimed this inequality previously; let's prove it.

  The defining relation (5) can be written in tensor-index notation as

$$\Lambda^\mu{}_\alpha \, \Lambda^\nu{}_\beta \, g_{\mu\nu} = g_{\alpha\beta}$$

Let's focus on the values $\alpha = \beta = 0$:

$$\Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{0}g_{\mu\nu} = g_{00}$$

$$\implies \left(\Lambda^{0}{}_{0}\right)^2 - \sum_{i=1}^{3}\left(\Lambda^{i}{}_{0}\right)^2 = 1$$

$$\implies \left(\Lambda^{0}{}_{0}\right)^2 = 1 + \sum_{i=1}^{3}\left(\Lambda^{i}{}_{0}\right)^2 \geq 1$$

$$\implies \left|\Lambda^{0}{}_{0}\right| \geq 1$$

3 Boosts and rotations

The restricted Lorentz group contains BOOSTS and ROTATIONS and combinations of the two.

- Boost matrices are SYMMETRIC.
- Two successive boosts result in a pure boost only if they are in the same direction. For example, consider a boost in the x direction followed by another boost in the x direction.

$$\begin{pmatrix} \gamma_{v_2} & -\gamma_{v_2}\left(\frac{v_2}{c}\right) & 0 & 0 \\ -\gamma_{v_2}\left(\frac{v_2}{c}\right) & \gamma_{v_2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_{v_1} & -\gamma_{v_1}\left(\frac{v_1}{c}\right) & 0 & 0 \\ -\gamma_{v_1}\left(\frac{v_1}{c}\right) & \gamma_{v_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{w} & -\gamma_{w}\left(\frac{w}{c}\right) & 0 & 0 \\ -\gamma_{w}\left(\frac{w}{c}\right) & \gamma_{w} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{with} \quad w = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}$$

  You know this already, but if you don't remember, you should try multiplying and showing this. Physically, this means considering a transformation from frame Σ to frame Σ' (relative speed v_1), and then from Σ' to $\tilde{\Sigma}$ (relative speed v_2), when all three are in standard configuration, i.e., relative motion in the common x, x', $\tilde{x}$ direction. The net transformation obtained by matrix multiplication is the transformation from Σ to $\tilde{\Sigma}$. The relative speed between frames Σ and $\tilde{\Sigma}$ is of course not $v_1 + v_2$ but rather $(v_1 + v_2)/(1 + v_1 v_2/c^2)$.

- However, when we apply successively boosts in different directions, we do not obtain a pure boost. For example,

$$\begin{pmatrix} \gamma_{v_2} & 0 & -\gamma_{v_2}\left(\frac{v_2}{c}\right) & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_{v_2}\left(\frac{v_2}{c}\right) & 0 & \gamma_{v_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_{v_1} & -\gamma_{v_1}\left(\frac{v_1}{c}\right) & 0 & 0 \\ -\gamma_{v_1}\left(\frac{v_1}{c}\right) & \gamma_{v_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

turns out to be a non-symmetric matrix. Thus a boost applied in the x direction, followed by a boost applied in the y direction, does not result in a pure boost.

This shows that Lorentz boosts do not form a group by themselves. Rotations are needed to complete the group, i.e., to make the set of transformations satisfy closure.

- Pure rotations satisfy the $\Lambda^{\mathrm{T}} g \Lambda = g$ condition, i.e., belong to the Lorentz group.

  If $\mathcal{R}$ is a 3×3 rotation matrix, then the 4×4 matrix

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \mathcal{R} & \\ 0 & & & \end{pmatrix} \tag{2}$$

  satisfies $L^{\mathrm{T}} g L = g$, because the spatial part of this equation is simply $\mathcal{R}^{\mathrm{T}}(-I)\mathcal{R} = -I$, i.e., $\mathcal{R}^{\mathrm{T}}\mathcal{R} = I$, which we know is true because rotation matrices are orthogonal.

- Pure rotations form a complete group by themselves. The product of any two rotation matrices is a rotation matrix, i.e., successive application of two rotations is itself a rotation.

  This is in contrast to boosts, which do not satisfy group completion by themselves.

- Symbolically:

$$\text{Restricted Lorentz group} = \text{boosts} + \text{rotations}$$

  and

$$\text{Lorentz group} = \text{boosts} + \text{rotations} + T + P$$

- Decomposition theorem: any orthochronous, proper Lorentz transformation can be decomposed into a pure boost and a pure rotation, in either order.

  In other words, if Λ satisfies Eq. (5), and is proper+orthochronous, then we can write it as

$$\Lambda = B^{(1)} L_R^{(1)} \qquad \text{and also as} \qquad \Lambda = L_R^{(2)} B^{(2)},$$

  where $B^{(1),(2)}$ are pure boost matrices and $L_R^{(1),(2)}$ are rotation matrices of the form of Eq. (6). The labels (1) and (2) indicate that the order of decomposition in general produce different boost and rotation matrices.

- **Wigner rotation and Thomas precession.**

  We have learned that successive application of two non-collinear boosts results in a LT that is not a pure boost — instead, it is a composition of a boost and a rotation. This rotation is called Wigner rotation, Thomas rotation, or Thomas-Wigner rotation.

  (Named after Eugene Wigner and Llewellyn Thomas.)

  Imagine an object in high-speed rotation around an orbit, such as an electron around an atomic nucleus. The rotation might be seen as a sequence of many non-collinear boosts, after a period returning the electron to its original velocity. Each combination of boosts in different directions creates a rotation of frame, which can combine into a net rotation per orbit. This results in the spin axis of the electron rotating in space, i.e, a **precession** of the internal angular momentum. This phenomenon is known as Thomas precession.

4 Collinear boosts, Commutativity

If we consider only collinear boosts, say boosts in the x direction, then we get a closed subset of the Lorentz group which itself is a group. The subset satisfies closure because, if you successively boost in the x direction, the result is a net boost in the x direction.

In much of this semester, we've focused on this subgroup, Inertial frames in what we called "standard configuration" are connected by transformations which are members of this group.

This subgroup can be described as the set of 2×2 matrices Λ which satisfy $\Lambda^{\mathrm{T}} g \Lambda = g$. While this formally looks like Eq. (5), here the matrices are 2×2 (not 4×4), and $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

This subgroup is known as $O(1,1)$. If we restrict to proper boosts, we obtain the smaller group $SO(1,1)$. If we also leave out the time-flipping matrices, we get $SO^{+}(1,1)$.

Relativistic addition in the same direction is a commutative operation; hence successive standard boosts commute. Thus $O(1,1)$ and $SO^{+}(1,1)$ are commutative (abelian) groups.

This is in contrast to $O(1,3)$ and $SO(1,3)$ and $SO^{+}(1,3)$. Boosts in different directions do not commute. A boost generally does not commute with a rotation. Even rotations do not commute if they have different rotation axes. $SO^{+}(1,3)$ is very definitely a non-abelian group.

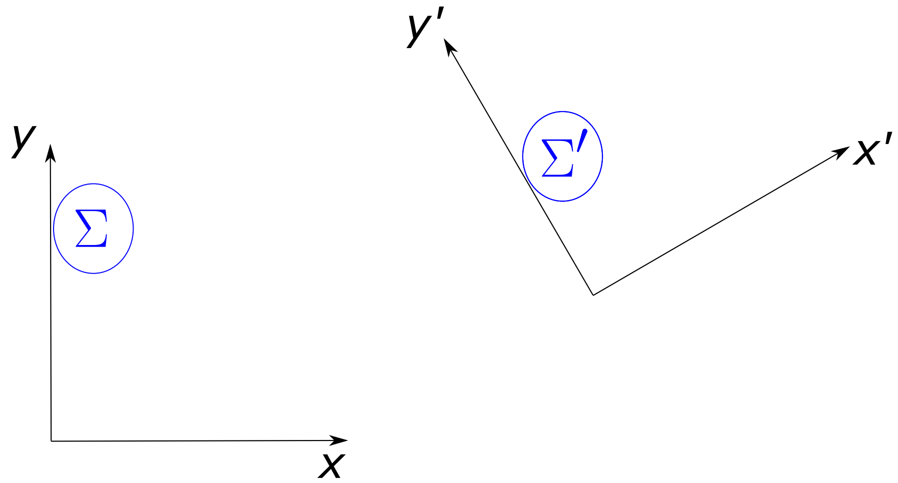

Figure 1: The frame Σ' is obtained by both rotating and translating Σ. For simplicity of drawing, in this example the rotation is around the z axis and the translation is in the x-y plane.

5 Poincaré group

5.1 Euclidean background

In 3D Euclidean physics, you know that rotations by themselves form a group: $SO(2)$ for rotations around a fixed axis, and $SO(3)$ for arbitrary rotations in 3-space. If we include parity flipping (reflections), we have the larger groups $O(2)$ and $O(3)$.

Now, it turns out that rotations + translations together also form a group. This is known as the Euclidean group, because it is the set of all transformations of Euclidean space which preserve the Euclidean distance. The n-dimensional Euclidean group is often written as $E(n)$. It contains the rotation group $O(n)$ as a subset, i.e., as a subgroup.

We consider the combination of a rotation and a translation of the reference frame, for example as shown in Figure 3. Such a transformation changes the coordinantes $\vec{x}$ of a point to $\vec{x'}$ as follows:

$$\vec{x'} = \mathcal{R}\vec{x} + \vec{A} \tag{3}$$

where $\mathcal{R}$ is a 3×3 rotation matrix and $\vec{A}$ is a 3-displacement vector.

Eq. (7) is linear but alas not homogeneous, i.e., it is not expressed as a matrix-vector multiplication alone, but also needs an additive term, $\vec{A}$. In other words, such a transformation is not described by a 3×3 matrix alone. Instead, we need the pair $(\mathcal{R}, \vec{A})$ to describe such a general transformation.

Exercise: In Figure 3 or on a similar diagram, identify the distances A_x and A_y. Which of them are negative?
This can be tricky. Remember that the rotation is applied before the translation. Also remember that we are concerned with passive transformations — not how the frames change, but how the coordinates relative to these frames change.

Is it clear that the set of transformations $(\mathcal{R}, \vec{A})$ is a group? Let's examine the conditions required for a group.

- **Closure:** It seems physically clear that applying two such transformations would produce a net transformation that is a combination of a rotation and a translation. To prove this, apply $(\mathcal{R}_1, \vec{A}_1)$ and $(\mathcal{R}_2, \vec{A}_2)$ successively to a displacement vector $\vec{x}$, and show that the net result can be put in the form $(\mathcal{R}_3, \vec{A}_3)$, for some $(\mathcal{R}_3, \vec{A}_3)$.

  Exercise: Show. Express $\mathcal{R}_3$ and $\vec{A}_3$ in terms of $\mathcal{R}_1$, $\vec{A}_1$, $\mathcal{R}_2$, $\vec{A}_2$.

  If you want, we could invent some notation for this. For example, we could write
  $$(\mathcal{R}_3, \vec{A}_3) \;=\; (\mathcal{R}_2, \vec{A}_2) \odot (\mathcal{R}_1, \vec{A}_1)$$
  to mean that applying $(\mathcal{R}_1, \vec{A}_1)$ and then $(\mathcal{R}_2, \vec{A}_2)$ produces the net transformation $(\mathcal{R}_3, \vec{A}_3)$. This is not standard notation. You can use something else other than $\odot$, or just place transformations one after another without any symbol between them.

- **Identity:** You should be able to identify the combination of $\mathcal{R}$ and $\vec{A}$ which keep the coordinates unchanged. (Exercise!)

  Physically, this is the do-nothing transformation or non-transformation.

- **Inverse:** Geometrically, it's clear that if you take Σ to Σ' by rotating and then translating, you should be able to get back to Σ by rotating and translating.

  Exercise: Given a transformation $(\mathcal{R}_1, \vec{A}_1)$, find the transformation which, combined with this, gives the identity transformation.

- **Associativity:** This needs some work. In previous cases that we have met, the transformations (group elements) could be expressed as matrices, so that combining transformations meant matrix multiplication. Matrix multiplication is known to be associative, so combining the transformations was automatically associative.

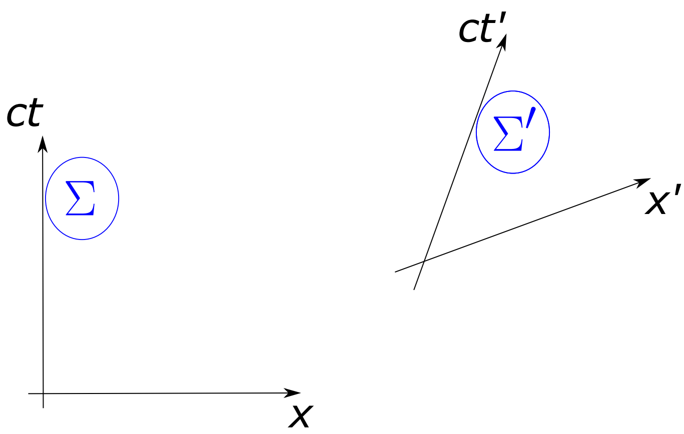

Figure 2: The frame Σ' is obtained from Σ by a Lorentz transformation plus a spacetime translation.

Now, however, the transformation is not represented by a matrix! So you have to show that

$$\left[(\mathcal{R}_3, \vec{A}_3) \odot (\mathcal{R}_2, \vec{A}_2)\right] \odot (\mathcal{R}_1, \vec{A}_1) = (\mathcal{R}_3, \vec{A}_3) \odot \left[(\mathcal{R}_2, \vec{A}_2) \odot (\mathcal{R}_1, \vec{A}_1)\right]$$

using what you have previously worked out for the composition of two groups.

Exercise! Please show associativity.

5.2 Poincaré transformations

In Minkowsi space, a transformation of spacetime coordinates of the form

$$\tilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + a^\nu \tag{4}$$

is known as a Poincaré transformation. This involves a Lorentz transformation Λ and a shift/displacement of space-time coordinates.

When studying Lorentz transformations, we have taken care to insist that the origins of the inertial frames involved all coincide at the common zero time. This guaranteed that the Lorentz transformation is a homogeneous transformation.

In contrast, a Poincaré transformation can include a shift of the spacetime origin. If the transformation is from inertial frame Σ to frame $\tilde{\Sigma}$, then the spacetime origin of the first frame, $(ct, x, y, z) = (0, 0, 0, 0)$, does NOT coincide with the spacetime origin of the second frame, $(c\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) = (0, 0, 0, 0)$.

A Poincaré transformation might contain a rotation or a boost or any combination, plus a shift of spatial origin or/and the zero of time. It is a rich set of transformations! In Figure 4, we have represented using a spacetime diagram a particular Poincaré transformation, namely: one involving an LT that is a pure boost in the x direction, and a shift only in the time and x coordinates. The group considered in the previous subsection (Euclidean rotations + spatial translations) and members of the Lorentz group, are all also included in Poincare transformations.

As Eq. (8) is not homogeneous (although linear), it cannot be expressed by a 4×4 matrix alone. We need a 4×4 LT matrix plus a spacetime 4-vector to express the transformation, e.g. (Λ, a).

Do the set of Poincaré transformations form a group? As in the previous subsection, this is not obvious to see; in particular associativity has to be checked properly. The calculations are formally quite similar to the ones in the previous subsection, for the group containing 3D rotations and spatial translations. The difference is that the transformations we are now considering act on members of 4D Minkowski space. It turns out that all four group properties are satisfied. The Poincaré group is in fact of fundamental importance in our understanding of nature.

6 Infinitesimal transformations and generators

Early in the semester, we considered infinitesimal boosts in a single direction, and hence found the form of a $SO(1,1)$ matrix, i.e., the form of a standard boost.

Writing $\Lambda = I + \epsilon K$, where the matrices are 2×2, we determined the form of K from the condition $\Lambda^{\mathrm{T}} g \Lambda = g$. This was enough to infer the form of finite matrices. The matrix K is the *generator* of $SO(1,1)$.

It is understandable that $SO(1,1)$ has a single generator, as it is a one-parameter family of matrices, e.g., $\begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$, parametrized by the family ϕ.

Similarly, you could consider $SO(2)$, the set of 2D rotations $\mathcal{R}_2$, which satisfy $\mathcal{R}_2^{\mathrm{T}}\mathcal{R}_2 = I$. Defining an infinitesimal rotation $\mathcal{R}_2 = I + \epsilon J$, you could determine the form of J. Then J would be the generator of $SO(2)$.

Now for larger groups. The group of 3D rotations, $SO(3)$, has three generators. We could call them J_1, J_2, J_3. This sounds reasonable, as 3D rotations require three parameters to describe them.

The Lorentz group, $SO^+(3,1)$, contains both rotations and boosts in any spatial direction. Unsurprisingly, it has six generators, three representing infinitesimal boosts (K_1, K_2, K_3) and three representing infinitesimal rotations (J_1, J_2, J_3).

Continuous groups are often characterized by looking at the commutation relations between the generators. Examining the commutation relations teach us a lot about the structure of the group.