1 Time-like, space-like and null

Depending on the sign of the norm, 4-vectors can be time-like, space-like or null (= light-like). This nomenclature makes most sense for spacetime intervals, so we will discuss that first. Afterwards, we will generalize to arbitrary 4-vectors.

1.1 Intervals in Minkowski space

The interval between two events is a 4-vector:

$$\Delta X = (c\Delta t, \Delta x, \Delta y, \Delta z) = (\Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3).$$

The norm of this 4-vector, $\Delta X_\mu \Delta X^\mu$ or $\Delta \overset{\Rightarrow}{X} \star \Delta \overset{\Rightarrow}{X}$, is our familiar invariant interval

$$\Delta X_\mu \Delta X^\mu = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = c^2(\Delta t)^2 - |\Delta\vec{r}|^2$$

in the negative trace metric or $(+,-,-,-)$ metric. Of course, in the other metric the sign of the norm is flipped. Previously in the semester, we also called this quantity Δs^2.

This norm consists of a time contribution and a space contribution. If the time contribution dominates, $c^2(\Delta t)^2 > |\Delta\vec{r}|^2$, we will call the 4-vector time-like. If the spatial contribution dominates, $c^2(\Delta t)^2 < |\Delta\vec{r}|^2$, we will call the 4-vector space-like. When the two contributions are equal, $c^2(\Delta t)^2 = |\Delta\vec{r}|^2$, the norm cancels, and such a 4-vector is known as null or light-like.

Definition: time-like, space-like and light-like intervals

If the negative-trace metric is used, the spacetime interval ΔX is

$$\begin{aligned} \text{time-like} \quad &\text{if} \quad \Delta X_\mu \Delta X^\mu > 0, \\ \text{space-like} \quad &\text{if} \quad \Delta X_\mu \Delta X^\mu < 0, \\ \text{light-like or null} \quad &\text{if} \quad \Delta X_\mu \Delta X^\mu = 0. \end{aligned}$$

- If the interval between two events is **time-like**, then, on a space-time diagram, the line segment joining the points representing the two events has slope **larger** than 1.

  Exercise: Draw lines segments representing intervals on a space-time diagram, and convince yourself that this is true.

  This line could represent the worldline of a physical particle or object (speed $< c$).

- If the interval between two events is **space-like**, then, on a space-time diagram, the line joining them has slope **smaller** than 1.

  Exercise: Draw a spacetime diagram and convince yourself that this is true.

  Such a line cannot represent the worldline of a physical particle or object. (Speed $> c$!) If the line segment is to be interpreted as part of a worldline, it would be the worldline of a tachyon.

- If the interval between two events is **light-like** or **null**, then, on a space-time diagram, the line joining them has slope **equal** to 1.

  Exercise: Draw such an interval on a spacetime diagram.

  Such a line represents the worldline of a photon. This explains the name 'light-like'.

- If an interval is seen to be time-like in one inertial frame, than it is time-like from *any* inertial frame.

  This is because, since Δs^2 is invariant under LT's, its value (and certainly its sign) does NOT depend on which frame one is working in.

  This can also be seen pictorially using spacetime diagrams. A Lorentz transformation squeezes (or un-squeezes) x, ct axes by equal angles, so that the bisection of these axes, representing a photonic worldline, stays the same. Thus a line segment having slope > 1 (speed $< c$) for the x, ct axes (Σ frame) will also have speed $< c$ for the x', ct' axes (Σ' frame). In other words, if a worldline is physical in one inertial frame, it is physical in all inertial frames.

- Similarly, if the interval is space-like (or light-like) in one inertial frame, than it is space-like (or light-like) in *any* inertial frame.

- If the interval between two points is **time-like**:

  - There exists an inertial frame in which the two events are equilocal (happen at the same location).
  - There does NOT exist an inertial frame in which they are simultaneous.

  Graphically (space-time diagrams), you can show this as follows. Since the line joining the two events has slope > 1, you can find a frame Σ' such that the ct' axis is parallel to this line. However, you can't make the x' axis parallel to this line.

Exercise: Try and convince yourself.

- If the interval between two points is **space-like**:
  - There exists an inertial frame in which the two events are simultaneous (happen at the same time).
  - There does NOT exist an inertial frame in which they are equilocal.

  Graphically: The line segment joining the two events has slope < 1. You CANNOT find a frame Σ' such that the ct' axis is parallel to this line. You can however make the x' axis parallel to this line segment.

Exercise: Draw and show.

1.2 Generalize: any type of 4-vectors

The terminology is generalized to any type of 4-vectors. Consider B to be a 4-vector with norm $B_\mu B^\mu$. This could be any type of 4-vector, for example, one of several types of 4-vector that we will meet soon, e.g., a 4-velocity, a 4-force, a 4-potential,.... etc. This 4-vector is

$$\begin{aligned} \text{time-like} \quad &\text{if} \quad B_\mu B^\mu > 0, \\ \text{space-like} \quad &\text{if} \quad B_\mu B^\mu < 0, \\ \text{light-like or null} \quad &\text{if} \quad B_\mu B^\mu = 0, \end{aligned}$$

assuming that the negative-trace metric is used. As before, if the positive-trace metric $(-, +, +, +)$ is used, then this definition is flipped: $B_\mu B^\mu > 0$ indicates space-like and $B_\mu B^\mu < 0$ indicates time-like.

The different classes are associated with different signs of the norm, but this depends on the metric used, and also signs might not be easy to remember. It might be more helpful to remember that a 4-vector is time-/space-like if its temporal/spatial part dominates.

Definition: time-like, space-like and light-like 4-vectors

If $B^\mu = (B^0, \vec{B})$, then the 4-vector is

$$\begin{aligned} \text{time-like} \quad &\text{if} \quad \left(B^0\right)^2 > \left|\vec{B}\right|^2, \\ \text{space-like} \quad &\text{if} \quad \left(B^0\right)^2 < \left|\vec{B}\right|^2, \\ \text{light-like or null} \quad &\text{if} \quad \left(B^0\right)^2 = \left|\vec{B}\right|^2. \end{aligned}$$

This specification does not depend on the choice of metric.

Remember that the the norm of any 4-vector is Lorentz-invariant; hence the sign of the norm does not depend on the frame of measurement. Therefore, if a 4-vector is time-like in one inertial frame, it has to be time-like in any inertial frame.