1 These notes

This set of notes introduces the expressions for relativistic momentum and energy, and works through a collision example. The relativistic expressions for force are then introduced. Transformation of energy and momentum under a boost is considered.

2 Dynamics, as opposed to kinematics

In most contexts, dynamics simply means time-dependent phenomena. However, when dynamics is contrasted with kinematics, the two words have the following meaning.

Kinematics is the study of motion (trajectories, velocities and accelerations), without regard to the forces that cause the motion, hence without reference to masses, forces, work or energy. In contrast, dynamics is the study of motion which takes into account masses, forces, work and energy.

This distinction is not very strict and somewhat arbitrary, but has been often used as an organizing principle in non-relativistic mechanics as well as in relativistic mechanics.

3 Momentum and Energy in Relativity

In non-relativistic classical mechanics, the momentum and the kinetic energy of a body with mass m are given by

$$\vec{p} = m\vec{v} \tag{1}$$

$$K = \frac{1}{2}mv^2 \tag{2}$$

We will consider the relativistic analogs of these concepts. In non-relativistic mechanics you also learn about the potential energy due to position in a gravitational or electrostatic field. We are still some way off from describing fields relativistically, so this will not a play role in our considerations in this chapter.

Note that momentum and energy are frame-dependent. For example, in the frame moving along with the particle, any particle has zero momentum and zero kinetic energy. Should this frame-dependence bother us? No, we only expect the *laws of physics* to have the same form in all frames — *particular quantities* are not expected to be frame-independent. This is true in the Galilean worldview as well as in the relativistic theory we are slowly building up to.

What are the *laws of physics* associated with these quantities? The momentum of a system is conserved if there are no external forces on the system — this is a consequence of Newton's laws. Does this principle depend on the frame in which momentum is measured? We can check that, with the definition of momentum above, the conservation of momentum in a physical process holds in all inertial frames if it holds in one, provided that the Galilean transformation holds. However, with the Lorentz transformation — which fundamentally alters the transformation of velocities, this is no longer true. Conservation of momentum, formulated with the definition $\vec{p} = m\vec{v}$, is not invariant under Lorentz transformations. So, we need to redefine momentum in a way that the conservation of momentum becomes a frame-independent physical law.

The same holds for energy — the total energy of an isolated system is conserved. Again, the equation we have above for kinetic energy turns out to be inadequate — if the energy so defined is conserved in one frame, it is conserved in other frames according to the Galilean transformation, but not according to the Lorentz transformation.

> Momentum conservation is a special case of Newton's second law (force = rate of change of momentum), applied to the entire system in the case of zero force.

It turns out that momentum and energy conservation are consistent with the Lorentz transformations if we define relativistic momentum and energy of a particle to be

$$\vec{p} = \gamma_v m\vec{v} \tag{3}$$

$$E = \gamma_v mc^2 \tag{4}$$

We can show that these redefinitions allow the laws of momentum conservation and energy conservation to be Lorentz-invariant. First, we discuss some consequences.

The first equation generalizes the definition of momentum, and does not look too different from the non-relativistic definition. At small speeds, the factor γ_v is close to unity, so that the non-relativistic definition of momentum ($\vec{p} = m\vec{v}$) is recovered.

The energy equation however looks quite different from the non-relativistic kinetic energy definition. We will see that it does reproduce the non-relativistic kinetic energy ($\frac{1}{2}mv^2$) at small speeds, but only up to a constant. This calls for some re-interpretation, which we do in the next subsection. The equation will look familiar to most readers because it is similar to the popularly known equation $E = mc^2$, except that it has a γ_v factor. This connection to the popular equation needs some context, to be elaborated in the following subsections.

By elimnating v (and hence γ_v) from the above equations (exercise!), one can directly relate the relativitic momentum and energy of a particle:

$$E^2 = p^2c^2 + m^2c^4 \tag{5}$$

Equations (3), (4), (5) are the most important equations of relativistic dynamics.

3.1 Rest energy and kinetic energy

A remarkable aspect of the energy definition $E = \gamma_v mc^2$ is that it does not vanish at zero speed, i.e. a particle of mass m has energy

$$E = \gamma_0 mc^2 = mc^2$$

when it is at rest. For obvious reasons, this is called the rest energy of a particle. It is the energy that an object has purely due to its mass. This has no analog in non-relativistic physics.

The kinetic energy is the energy due to motion, i.e., obtained by subtracting the rest energy from the energy of the particle at motion.

$$K = \text{energy} - (\text{rest energy}) = \gamma_v mc^2 - mc^2$$

We have seen (earlier in the semester) using the binomial series that $\gamma_v - 1 \approx \frac{1}{2}(v/c)^2$ for small v; therefore the non-relativisitic expression for the kinetic energy is recovered at small speeds:

$$K = (\gamma_v - 1)mc^2 \approx \frac{1}{2}mv^2$$

We can of course use the binomial series further to generate next-oder approximations, i.e., relativistic corrections to the kinetic energy:

$$K = (\gamma_v - 1)mc^2 \approx \frac{1}{2}mv^2 + \frac{3}{8}\left(\frac{v}{c}\right)^2 mv^2 + \frac{5}{16}\left(\frac{v}{c}\right)^4 mv^2 + \cdots$$

Exercise: Derive the next term in this series.

3.2 "$c = 1$" units

A bane of relativity calculations is the appearance of many factors of different powers of c.

We could imagine choosing our units such that c has the value of 1. Then, many of the equations look less messy. For example, the three important equations of relativistic dynamics, (3), (4), (5), become

$$\vec{p} = \gamma_v m\vec{v}, \quad E = \gamma_v m, \quad E^2 = p^2 + m^2.$$

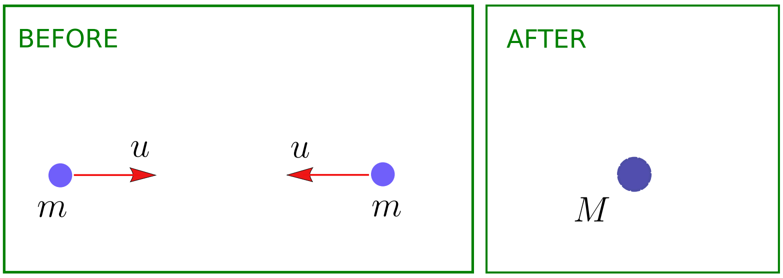

Figure 1: A collision of two particles with the same mass m, resulting in a single final particle at rest, of mass M. Should M be equal to $2m$?

Using so-called "$c = 1$" units has the advantage that many calculations become less messy and hence less error-prone. A disadvantage is that you partially lose the ability to do dimensional analysis. For example, speed/velocity becomes unit-less. As a result, energy, momentum and mass all have the same dimensions!

In more advanced texts, "$c = 1$" units are standard. If in a couple of years you are learning quantum field theory, you certainly will not want to carry along factors of c in calculations that are already very complicated. (In fact, you will also want to set $\hbar = 1$.)

In this text, I will mostly retain factors of c. This will make some calculations look more painful than strictly necessary. In those cases, you are encouraged to carry out the same calculation first using $c = 1$ units, and then again after reinstating the factors of c.

In many introductory texts on special relativity, it is common to define $\beta = v/c$. When one uses "$c = 1$" units, this becomes unnecessary, as β becomes equal to v.

4 A collision

To gain some appreciation of the unusual concept of 'rest energy', we will consider a collision process in which two equal-mass elemenatary particles, each of mass μ, collide and merge to form a single particle of mass M.

Conservation of relativistic energy gives

$$\gamma_v \mu c^2 + \gamma_v \mu c^2 = \gamma_0 M c^2 \qquad \Longrightarrow \qquad M = 2\gamma_v \mu > 2\mu$$

The mass is not conserved in this process! The kinetic energy of the two incoming particles is converted to rest energy during the fusion process. Since rest energy is proportional to mass, the mass in the system increases. We can check that the gain in rest energy (gain in mass times c^2) is equal

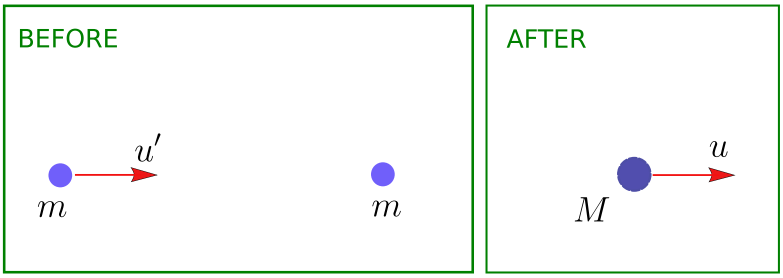

Figure 2: The same fusion process as in the previous figure, but now seen from an frame in which one of the initial particles is at rest.

to the initial kinetic energy:

$$(\text{mass gain})c^2 = (2\gamma_v\mu - 2\mu)c^2 = 2(\gamma_v - 1)\mu c^2$$

The lesson from this calculation is that relativistic energy consists of rest energy and kinetic energy, and they can be converted to each other when multiple particles interact. This is reminiscent of kinetic and potential energy in non-relativistic mechanics, which can also be converted to each other.

How is the fusion phenomenon above described in non-relativistic mechanics? The total mass is conserved in Newtonian mechanics, thus $M = 2\mu$. Where then does the non-relativistic kinetic energy $2\mu c^2$ vanish? In the Newtonian picture, such a process can be possible through the production of heat. Heat is a name given to the internal molecular energy of objects which we do not want to (or are unable to) treat in detail. It may useful when dealing with macroscopic situations, to hide microscopic details into concepts like heat or friction. In developing relativistic principles, we will try to confine ourselves to situtations where we do not need to resort to such fudge-concepts. In other words, we will concentrate on energy-conserving situations.

4.1 The collision from another frame

It is instructive to also analyze this collision using momentum conservation. The collision obviously conserves momentum in the frame that it has been described (let's call it the 'laboratory frame'.) The lab frame can also be referred to as the frame of the final mass M, since the final mass is at rest relative to the lab. Conservation of relativistic momentum in the lab frame is

$$\gamma_v mv - \gamma_v mv = 0,$$

i.e., $0 = 0$, which we knew already.

We can learn a bit more by considering momentum conservation in the frame of one of the incoming particles, say the particle that is left-moving in the lab frame. (Let's call the frame of this incoming particle the $\tilde{\Sigma}$ frame.) In this frame, the collision is between a particle at rest and a particle of speed

$$w = \frac{v+v}{1+v\cdot v/c^2} = \frac{2v}{1+v^2/c^2} = \frac{2vc^2}{c^2+v^2}$$

This quantity — the speed of the right-moving particle relative to the $\tilde{\Sigma}$ frame — is obtaied by using the longitudinal velocity addition formula. (The speed is v relative to the lab frame and the lab frame itself is moving leftward at speed v relative to $\tilde{\Sigma}$; hence we need to relativistically add to equal velocities.)

In the $\tilde{\Sigma}$ frame, after the collision the resulting particle of mass M moves rightward with speed v. Momentum conservation gives us

$$\gamma_w \mu w + 0 = \gamma_v M v$$

The term γ_w is a bit clumsy, but not too difficult to express in terms of v:

$$\gamma_w w = \frac{1}{\sqrt{1-\left(\frac{2vc}{c^2+v^2}\right)^2}} \frac{2vc^2}{c^2+v^2} = \frac{c^2+v^2}{c^2-v^2}\frac{2vc^2}{c^2+v^2} = \frac{2vc^2}{c^2-v^2} = 2\gamma_v^2 v$$

Using this, the momentum conservation equation ($\gamma_w \mu w = \gamma_v M v$) yields immediately $M = 2\mu\gamma_v$. This is the same result we obtained by considering conservation of relativistic energy in the lab frame.

Note: we have made the assumption that the mass of each object is the same relative to each frame. E.g., the masses of the initial particles are μ, whether we are working in the lab frame of the $\tilde{\Sigma}$ frame. This is a fundamental assumption of relativistic dynamics.

We could also consider energy conservation in the $\tilde{\Sigma}$ frame. This gives

$$\gamma_w \mu c^2 + \gamma_0 \mu c^2 = \gamma_v M c^2 \qquad \Longrightarrow \qquad (\gamma_w + 1)\mu = \gamma_v M$$

Now $\gamma_w + 1 = \frac{c^2+v^2}{c^2-v^2} + 1 = \frac{2c^2}{c^2-v^2} = 2\gamma_v^2$. Therefore we again obtain $M = 2\mu\gamma_v$.

The reasoning for the relativistic energy expression $E = \gamma_v mc^2$ is that conservation of energy becomes frame-independent with this form. We have just demonstrated that this form indeed gives consistent energy conservation in different frames. The collision analysis above thus justifies the relativistic energy formula.

4.2 Non-relativistic momentum

It is interesting to ask how non-relativistic momentum would fare in dealing with this situation. The lab frame situation is consistent with non-relativistic momentum conservation ($mv - mv = 0$). In the $\tilde{\Sigma}$ frame, according to Galilean velocity addition the rightmoving incoming particle has velocity $2v$ (instead of the relativistic $\frac{2v}{1+v^2/c^2}$). Momentum conservation then gives $\mu(2v) + 0 = Mv$, so that $M = 2\mu$. This is comforting as we do not expect kinetic energy to be converted to mass in non-relativistic physics. The non-relativistic momentum formula is thus consistent with Galilean transformations, not with Lorentz transformations.

5 Momentum & energy without mass — Photons

In non-relativistic mechanics, an object with zero mass also has zero momentum and zero kinetic energy:

$$p = (0)v = 0\,; \qquad E = \frac{1}{2}(0)v^2 = 0\,.$$

Since such a hypothetical object carries no momentum or energy, it has no effect on the world. Thus an object with zero mass is... nothing. It is not sensible to consider such objects.

But relativity changes the definition of momentum and energy. Considering $p = \gamma_v mv$, we remember that the Lorentz factor can in fact diverge, when $v \to c$. Thus, using physicists' intuition that infinity times zero could be a finite quantity, we see that a zero-mass particle could actually carry momentum in the relativistic world, provided that it travels at the speed of electromagnetic waves, c.

From $E = \gamma_v mc^2$, we see using the same intuition that a massless particle could also carry energy provided γ_v diverges. Indeed, using the relation $E^2 = p^2c^2 + m^2c^4$, we see that for $m = 0$, the momentum and energy of a massless particle are related simply: $E = pc$.

> In a math class, you are not supposed to make poorly defined statements like "infinity times zero". Fortunately, we are not in a math class, so we will allow ourselves physicists' freedom of using such intuition.

Thus, zero-mass particles could carry energy and momentum, i.e., could have physical meaning, as long as they travel at the speed of light. This means that zero-mass particles could have nontrivial interactions with other particles (or each other), e.g., exchanging momentum or energy. This is in sharp contrast to non-relativistic physics!

What type of particle travels at speed c? We have to appeal to the quantum mechanical idea that light has dual nature — it behaves both like waves and like particles. The particles of electromagnetic waves are called photons. Photons are massless particles which nevertheless carry momentum and energy because they always travel at speed c. In early quantum mechanics — to explain blackbody radiation and the photoelectric effect —

it was postulated that light of frequency f comes in packets or discrete particles which carry energy

$$E = hf$$

and hence momentum

$$p = \frac{E}{c} = \frac{hf}{c}.$$

You might have learned this in a 'modern physics' class, or at the end of an Introductory Physics semester, and some of you might have had a full quantum mechanics semester. We will use this bit of quantum idea to include photons (or other massless particles) into our treatment of relativity. Admittedly, this feels a bit inconsistent, as in the rest of our treatment we are building ideas from the ground up, relying as much as possible only on Einstein's two postulates and basic algebra.

The expressions for energy and momentum can also be written in terms of the wavelength $\lambda = c/f$ of the wave. Thus we have for a massless particle

$$\boxed{E = hf = \frac{hc}{\lambda}} \qquad \boxed{p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}} \qquad \begin{cases}\text{for \textbf{massless} particles,}\\ \text{e.g., photons}\end{cases}$$

5.1 Massive and massless particles from different frames

In Figure 3 we summarize some of what we've learned about observing speeds and frequencies in different frames. Observer B moves rightward relative to observer A with relative speed v. We could of course attach frames Σ and $\tilde{\Sigma}$ to the two observers, but this will not be important.

A observes a particle with mass m (a massive particle) moving with speed u, and a photon (a massless particle) having frequency f moving with speed c. How are these observed by B? For simplicity, we will consider both to be in the same direction as the motion of B relative to A.

In physics, **massive** often just means something with a nonzero mass. Unlike its non-technical usage, the word here does not imply a remarkably large mass, just a nonzero mass.

From the frame of B, the particle with finite mass is observed to have the same mass m, but a different speed

$$\tilde{u} = \frac{u - v}{1 - uv/c^2}.$$

On the other hand the photon is observed to have the same speed c, but (due to the relativistic Doppler effect) a different frequency:

$$\tilde{f} = f\sqrt{\frac{c - v}{c + v}} = f\sqrt{\frac{1 - \beta}{1 + \beta}} \qquad \text{where} \quad \beta = v/c.$$

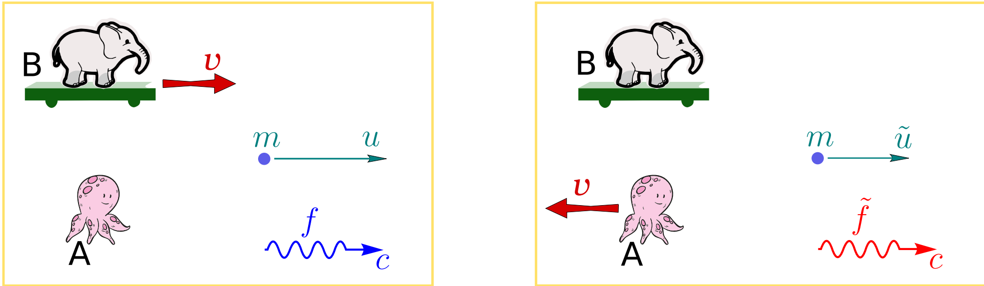

Figure 3: Our smart observers are A.Octopus and B.Elephant. Left: Perspective of A. In this frame, B moves rightward with speed v. Right: Perspective of B. From this frame, A is seen to move leftward with speed v. How do A and B perceive the speed of an object, and the frequency of a light pulse?

For the directions of motion shown in Figure 3, the frequency observed by B is smaller ($\tilde{f} < f$), i.e., the color of light is red-shifted.

The mass of a particle and the speed of light appear the same from different frames — they are Lorentz invariants, also known as Lorentz scalars.

We have simplified this discussion by considering only longitudinal motion of the particle and the photon. As an exercise , you should consider transverse motion and remind yourself of the transformation equations for speed and for Doppler shift.

6 Force

In non-relativistic mechanics, we think of force as either

$$\vec{F} = \frac{d\vec{p}}{dt} = \dot{\vec{p}} \qquad \text{or} \qquad \vec{F} = m\vec{a}$$

The first form is Newton's second law, and the second follows from the first if $\vec{p} = m\vec{v}$.

(There are a few tricky non-relativistic situations for which one can't blindly use $\vec{F} = m\vec{a}$, for example, if the mass of the object is changing. Think about the problem of a rocket propelled by expelling gas, so that the rocket is losing mass as it accelerates forward. However, for situations involving a single object, generally the two forms are equivalent.)

We've learned that, in relativistic mechanics, momentum is no longer $\vec{p} = m\vec{v}$. Thus, the beloved equation $\vec{F} = m\vec{a}$, one of the best-known equations of physics, needs to be corrected!

We have one form of correction for the case of the force acting along the same line as the velocity (linear or 1D motion): in this case the force is $F = \gamma_v^3 ma$, where v is the instantaneous speed of the object. (Derived later.) For transverse forces, i.e., when the force is perpendicular to the instantaneous velocity, the corrected expression is $F = \gamma_v ma$. For example, if the instantaneous velocity is in the x-direction, then the force is

$$\vec{F} = \left(\gamma_v^3 ma_x, \gamma_v ma_y, \gamma_v ma_z\right) \qquad \text{if} \quad \vec{v} = (v, 0, 0).$$

This is admittedly unpleasnt — $\vec{F}$ is not proportional to $\vec{a} = (a_x, a_y, a_z)$. We do not have a single vector equation. This points to a deeper issue: the non-relativistic concept of vectors misses some of the essence of relativity. Later on, we will tackle this issue by introducing a new type of vector, with four components instead of three.

Notice above that we had to refer to the 'instantaneous' speed and velocity. Since a force is being applied to the object, its velocity will change with time.

Let's now derive the results declared above.

6.1 Linear motion: force in same direction as velocity

First consider the force acting parallel or anti-parallel to the velocity, so that the motion is constrained to a straight line (one-dimensional motion). We can then interpret F, v, and a as single-component objects; the other two components of the vectors are zero. Since there is only one dimension, $a = \dfrac{dv}{dt} = \dot{v}$. (If we had multiple dimensions relevant, more care would have been required, as $\dot{v}$ could mean the derivative of speed which is not the acceleration in general. As only one dimension is relevant now, the velocity component v is the speed up to a sign. Using the same symbol v both for velocity component and for speed is a slight abuse of notation, but hopefully won't cause confusion.)

Thus

$$\begin{aligned}\frac{dp}{dt} = \frac{d}{dt}(\gamma_v m v) &= m\left(\frac{d}{dt}(\gamma_v)v + \gamma_v \frac{dv}{dt}\right) \\ &= m\left(\left(\frac{\gamma_v^3 v a}{c^2}\right)v + \gamma_v a\right) \\ &= m\gamma_v a\left(\gamma_v^2 \frac{v^2}{c^2} + 1\right) = m\gamma_v a\left(\gamma_v^2\right)\end{aligned}$$

leading to $F = \gamma_v^3 m a$.

Exercise: You should have previously shown that $\dot{\gamma} = \gamma^3 v\dot{v}/c^2$. If not, please work it out! Also: check that $\gamma^2\beta^2 + 1 = \gamma^2$.

6.2 General directions

Including transverse forces requires a more involved (2D or 3D) version of the above (1D) calculation.

Assuming the instantaneous velocity to be in the x direction, we consider a force having some direction in the xy plane $\vec{F} = (F_x, F_y)$. This force can be obtained by differentiating

$$\vec{p} = \left(\frac{mv_x}{\sqrt{1 - \frac{v_x^2}{c^2} - \frac{v_y^2}{c^2}}}, \frac{mv_y}{\sqrt{1 - \frac{v_x^2}{c^2} - \frac{v_y^2}{c^2}}}\right)$$

and then setting $v_y = 0$. (However, $\frac{dv_y}{dt} = a_y$ is not zero.)

I will omit the steps of this calculation. The factors of c are quite tedious to carry along, so I encourage you to do this after setting $c = 1$. The result is $F_x = \gamma_v^3 m a_x$ and $F_y = \gamma_v m a_y$. An identical calcuation gives $F_z = \gamma_v m a_z$.

Exercise!

6.3 Energy and force

Applying force on an object entails doing work on it, so that its energy increases. In the one-dimensional case, $dE = Fdx$, so that one expects $F = \frac{dE}{dx}$. Is this consistent with our new relativistic expressions for energy and force?

Differentiating

$$E = \gamma_v mc^2$$

with respect to x, one finds

$$\frac{dE}{dx} = \frac{dE/dt}{dx/dt} = \frac{1}{v}\frac{d}{dt}(\gamma_v mc^2) = \frac{mc^2}{v}\frac{d\gamma_v}{dt} = \frac{mc^2}{v}\frac{\gamma_v^3 va}{c^2} = \gamma_v^3 ma = F$$

(Of course, you should check that this works also with the non-relativistic expressions $E = \frac{1}{2}mv^2$ and $F = ma$.)

In summary, the non-relativistic expressions $\vec{F} = \frac{d\vec{p}}{dt}$ and $F = \frac{dE}{dx}$ still hold in relativistic dynamics, but the most famous expression, $\vec{F} = m\vec{a}$, does not.

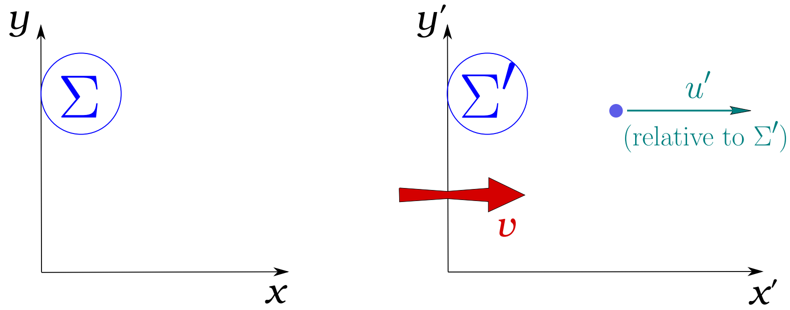

Figure 4: A particle has speed u' relative to Σ', in the common x, x' direction. It's energy and momentum are E and $\vec{p} = (p, 0, 0)$ relative to Σ; and E' and $\vec{p'} = (p', 0, 0)$ relative to Σ'. How are (E, p) related to (E', p')?

7 Lorentz tranformation of energy and momentum

Consider inertial frames Σ, Σ' in standard configuration, with relative speed v. An object has velocity $\vec{u} = (u, 0, 0)$ relative to Σ and $\vec{u'} = (u', 0, 0)$ relative to Σ'. We have learned previously how to relate u to u', and hence also γ_u to $\gamma_{u'}$:

$$u = \frac{u' + v}{1 + u'v/c^2}, \qquad \gamma_u = \gamma_{u'}\gamma_v \left(1 + u'v/c^2\right). \tag{6}$$

The energy and momentum of the particle are E and $\vec{p} = (p, 0, 0)$ relative to Σ, where $E = \gamma_u mc^2$ and $p = \gamma_u mu$. Similarly, the energy E' and momentum $\vec{p'} = (p', 0, 0)$ relative to Σ' are $E' = \gamma_{u'} mc^2$ and $p' = \gamma_{u'} mu'$. Using Eq. (6), it is now straightforward to find out how How E and p is related to E' and p:

$$\begin{aligned} E = \gamma_u mc^2 &= \gamma_{u'}\gamma_v \left(1 + u'v/c^2\right) mc^2 \\ &= \gamma_v \left(\gamma_{u'} mc^2 + v\gamma_{u'} mu'\right) = \gamma_v \left(E' + vp'\right) \end{aligned}$$

and

$$\begin{aligned} p = \gamma_u mu &= \gamma_{u'}\gamma_v \left(1 + u'v/c^2\right) m\frac{u' + v}{1 + u'v/c^2} \\ &= \gamma_{u'}\gamma_v \left(mu' + mv\right) = \gamma_v \left(\gamma_{u'} mu' + \frac{v}{c^2}\gamma_{u'} mc^2\right) = \gamma_v \left(p' + \frac{v}{c^2}E'\right) \end{aligned}$$

The transformation equations for energy and momentum are thus

$$E = \gamma_v \left(E' + vp'_x\right) \qquad p_x = \gamma_v \left(p'_x + \frac{v}{c^2}E'\right)$$

The inverse transformations can be calculated similarly to be

$$E' = \gamma_v (E - v p_x) \qquad p'_x = \gamma_v \left(p_x - \frac{v}{c^2} E \right)$$

These transformation equations for (E, p_x) look very much like the transformation equations for (t, x), except for factors of c. This similarity can be made more explicit by writing the transformation equations for $(E/c, p_x)$ as a matrix equation:

$$\begin{pmatrix} E/c \\ p_x \end{pmatrix} = \begin{pmatrix} \gamma_v & \gamma_v \left(\frac{v}{c}\right) \\ \gamma_v \left(\frac{v}{c}\right) & \gamma_v \end{pmatrix} \begin{pmatrix} E'/c \\ p'_x \end{pmatrix}$$

or

$$\begin{pmatrix} E'/c \\ p'_x \end{pmatrix} = \begin{pmatrix} \gamma_v & -\gamma_v \left(\frac{v}{c}\right) \\ -\gamma_v \left(\frac{v}{c}\right) & \gamma_v \end{pmatrix} \begin{pmatrix} E/c \\ p_x \end{pmatrix}.$$

Thus, under the standard boost, the combination $(E/c, p_x)$ transforms exactly the same way as the combination (ct, x).

By a similar but tedious calculation, one can show that the transverse (y and z) components of the momentum are unchanged under the boost: $p'_y = p_y$ and $p'_z = p_z$. Thus $(E/c, \vec{p}) = (E/c, p_x, p_y, p_z)$ has the same transformation properties as a spacetime coordinate, $(ct, \vec{r}) = (ct, x, y, z)$ under the standard boost. In fact, one can show that the combination $(E/c, \vec{p})$ transforms like $(ct, \vec{r})$ under *any* Lorentz transformation.

7.1 Introducing 4-vectors

Clearly, this cannot be a coincidence. In fact, we will find that there are various combinations of physical quantities which transform the same way. Such objects are known as 4-vectors.

Common notation for the spacetime 4-vector is

$$x = (x^0, x^1, x^2, x^3) \qquad \text{with} \quad x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z.$$

Similarly for the energy-momentum 4-vector

$$p = (p^0, p^1, p^2, p^3) \qquad \text{with} \quad p^0 = E/c,\ p^1 = p_x,\ p^2 = p_y,\ p^3 = p_z.$$

The energy-momentum 4-vector is more commonly known as the four-momentum.