EINSTEIN'S TWO POSTULATES

1905 paper: "On the electrodynamics of moving bodies"

① The Relativity Principle:

The laws of physics are the same in all $\underbrace{\text{INERTIAL FRAMES}}$

frames with zero acceleration, No "fictitious" forces.

② Constancy of the speed of light:

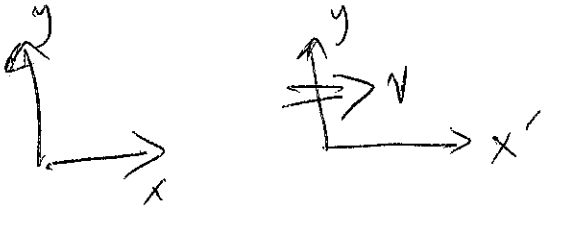
The speed c is a constant, independent of the state of the emitting body or ~~of at~~ observer.

* We consider mostly / only frames whose origins were coincident at common time $= 0$; i.e., times synchronized when origins coincident.

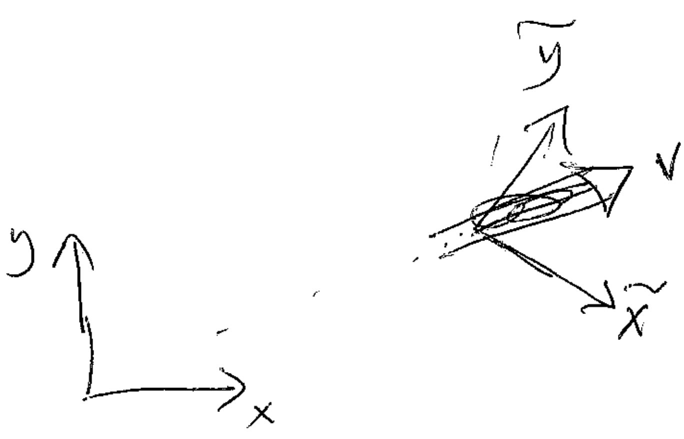
This event is $(ct, x, y, z) = (0, 0, 0, 0)$ in ALL frames. $\Rightarrow$ WE LIKE HOMOGENEOUS trafo's

* If axes are aligned as well (no rotation), we get pure BOOSTS.

* If axes are aligned and relative velocity is in common x (or x' or $\tilde{x}$) direction, then we call this standard configuration.



standard config.



valid Lorentz Trafo.

The INVARIANT INTERVAL

Consider Two events, separated by $(\Delta x, \Delta y, \Delta z, \Delta t)$
or $(c\Delta t, \Delta x, \Delta y, \Delta z)$

E.g., if two events are (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) then $\Delta x = x_2 - x_1$, etc.

Then the quantity

$$\boxed{(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \text{ is INVARIANT,}}$$

i.e., same from any frame.

If measured to $(\Delta s)^2$, $(\Delta s')^2$, $(\Delta \tilde{s})^2$ in frames $\Sigma, \Sigma', \tilde{\Sigma}$, then $(\Delta s)^2 = (\Delta s')^2 = (\Delta \tilde{s})^2$

e.g., $c^2(\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 = c^2(\Delta \tilde{t})^2 - (\Delta \tilde{x})^2 - (\Delta \tilde{y})^2 - (\Delta \tilde{z})^2$

$\Rightarrow$ "equivalent" to Lorentz Transfo

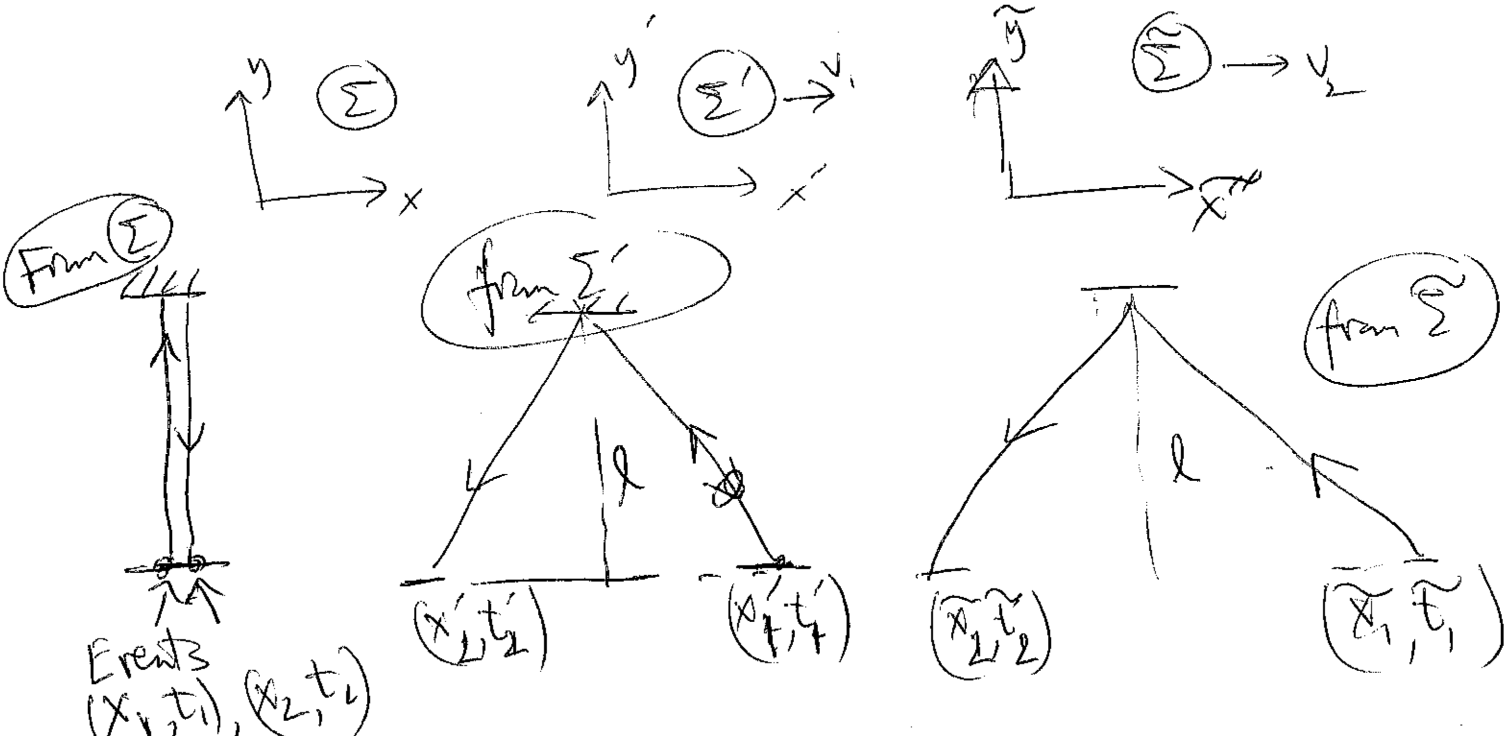
$\rightarrow$ as important as LT

$\rightarrow$ can be derived from constancy of c.

$(\Delta s)^2$ or (Δs) is called the INVARIANT INTERVAL.

* <u>Proof using invariance of c</u>

Consider $\Sigma, \Sigma', \tilde{\Sigma}$ in standard config.



In Σ: path length $= 2\ell$, $\Delta t = \frac{2\ell}{c}$

$\Delta x = 0$

In Σ': path length $= 2\sqrt{\ell^2 + \left(\frac{\Delta x'}{2}\right)^2}$

$= \sqrt{4\ell^2 + (\Delta x')^2}$

Speed of light $= \frac{\text{path length}}{\Delta t'} = \frac{\sqrt{4\ell^2 + (\Delta x')^2}}{\Delta t'}$

$\Rightarrow c = \frac{4\ell^2 + (\Delta x')^2}{(\Delta t)^2} \quad \Rightarrow c^2(\Delta t)^2 - (\Delta x')^2 = 4\ell^2$

Exactly same calculation gives in $\tilde{\Sigma}$:

$$c^2(\Delta\tilde{t})^2 - (\Delta\tilde{x})^2 = 4l^2$$

Thus $$c^2(\Delta t)^2 - (\Delta x)^2 = c^2(\Delta t')^2 - (\Delta x')^2$$
$$= c^2(\Delta\tilde{t})^2 - (\Delta\tilde{x})^2 = 4l^2$$

* Generalize to arbitrary spatial direction:

$$c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = c^2(\Delta t)^2 - |\Delta\vec{r}|^2 \text{ is invariant.}$$

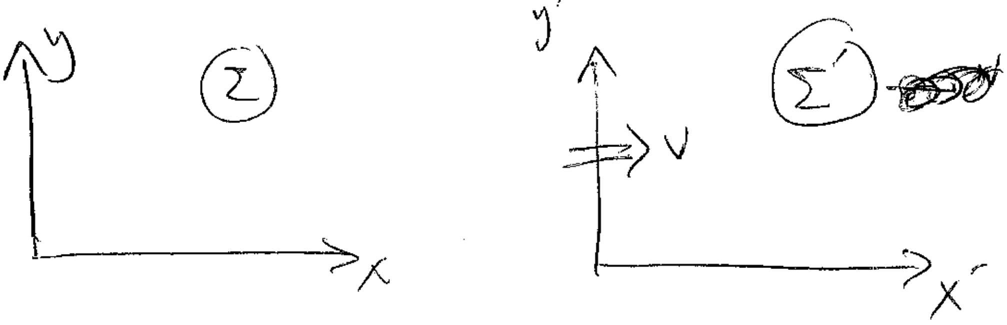
* Invariance of $(\Delta s)^2$ $\xleftrightarrow[\text{derived from each other.}]{\text{can be}}$ Lorentz ~~Transfo~~ boost

* ~~In standard configuration, if one of the two frames~~

* Since we consider $(0,0,0,0)$ to be the same event in all frames (clocks synchronized when origins coincident), if this is taken to be one of the events, then

$$c^2t^2 - x^2 - y^2 - z^2 \text{ is invariant.}$$

* Derivation of LT equations for pure boost (standard configuration)



Relating coordinates of same event in two frames:
(ct', x', y', z') to (ct, x, y, z).

- Should reduce to Galilean transfo for $\frac{v}{c} \to 0$:
$(ct' = ct,\ x' = x - vt,\ y' = y,\ z' = z)$ for $\frac{v}{c} \to 0$

- Transverse spatial directions should stay unaffected, $y' = y$, $z' = z$.
(Can be argued based on symmetry, omit.)

- Transformations should be linear. If nonlinearity, constant-velocity motion becomes accelerating motion in other $\Longrightarrow$ fictitious forces.

9b

Try
$$ct' = Act + Bx$$
$$x' = Dct + Ex$$

Σ- Coordinates of Origin of Σ' should satisfy $x = vt$, since Σ' moves with velocity v.

$\Rightarrow$ When $x' = 0$, $x = vt$

$\Rightarrow$ $0 = Dct + Evt$ $\Rightarrow$ $E = -\frac{c}{v}D$

Thus
$$ct' = Act + Bx,$$
$$x' = Dct - \frac{c}{v}Dx = -D\frac{c}{v}(x - vt)$$

Find three constants A, B, D ?

* For $\frac{v}{c} \to 0$, we have $A \to 1$, $B \to 0$ and $D \to -\frac{v}{c}$

$\boxed{\text{Use invariance of } c^2t^2 - x^2 \text{ !}}$

* Derivation 1 (Brute force)

$$c^2t'^2 - x'^2 = \left(Act + Bx\right)^2 - \left(Dct - D\frac{c}{v}x\right)^2 = c^2t^2 - x^2$$

$$\Rightarrow \quad A^2 - D^2 = 1, \quad D^2\left(\frac{c}{v}\right)^2 - B^2 = 1, \quad AB + D^2\frac{c}{v} = 0$$

Solve (take $A>0$, $D<0$): EXERCISE

$$A = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma_v, \quad B = \frac{-v/c}{\sqrt{1-\frac{v^2}{c^2}}} = -\gamma_v\frac{v}{c} = D$$

$$\Rightarrow \quad ct' = \gamma_v\left(ct - \frac{vx}{c}\right)$$

$$x' = \gamma_v\left(x - \frac{v}{c}ct\right)$$

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma_v & -\gamma_v\frac{v}{c} \\ -\gamma_v\frac{v}{c} & \gamma_v \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix}$$

* Derivation 2 (Light cone coordinates)

$$\left.\begin{aligned} x^+ &= ct + x \\ x^- &= ct - x \end{aligned}\right\}$$

$c^2t^2 - x^2 = x^+x^-$ is invariant:

$$x'^+x'^- = x^+x^-$$

light-cone coordinates

$$ct = \frac{1}{2}(x^+ + x^-), \quad x = \frac{1}{2}(x^+ - x^-)$$

9d

~~Decrease If X^+ is increased by factor to obtain X'^+, X^- must be decrease by same factor to get X'^-, i.e.,~~

$$\Rightarrow \frac{X'^+}{X^+} = \frac{X^-}{X'^-} \quad \text{Call this factor } e^{-\phi}$$

$$x'^+ = x^+ e^{-\phi}, \quad x'^- = x^- e^{+\phi}$$

Then $ct' = \frac{1}{2}(x'^+ + x'^-) = \frac{1}{2}\left(e^{-\phi}x^+ + e^{+\phi}x^-\right)$

$$= \frac{1}{2}\left[e^{-\phi}(ct+x) + e^{+\phi}(ct-x)\right]$$

$$= ct\ \tfrac{1}{2}\left(e^{\phi} + e^{-\phi}\right) - x\ \tfrac{1}{2}(ct-x)$$

$$= (ct)\cosh\phi + x(-\sinh\phi)$$

Similar: $x' = (-\sinh\phi)\,ct + (\cosh\phi)\,x$

Thus $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix}$

Compare with $x' = Dct - \left(\frac{c}{v}\right)Dx$!

$$\cosh\phi = D, \quad \sinh\phi = \left(\frac{c}{v}\right)D$$

$$\Rightarrow \boxed{\tanh\phi = \frac{v}{c}}$$

ϕ is called the RAPIDITY

$$\Rightarrow \cosh^2\phi = \frac{1}{1-\tanh^2\phi} = \frac{1}{1-v^2/c^2} = \gamma_v^2$$

$$\sinh^2\phi = \cosh^2\phi - 1 = \frac{1}{1-\frac{v^2}{c^2}} - 1 = \frac{v^2/c^2}{1-v^2/c^2}$$

$$\cosh\phi = \gamma_v, \quad \sinh\phi = \gamma_v \frac{v}{c}$$

$$\Rightarrow \begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma_v & -\gamma_v \frac{v}{c} \\ -\gamma_v \frac{v}{c} & +\gamma_v \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

Lorentz boost in x-direction, re-derived.

* Derivation 3 (Infinitesimal boosts = Generators of Lie group)

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \end{pmatrix}$$

Λ is a 2×2 matrix, which we will rederive

Invariant:

$$c^2t^2 - x^2 = (ct \;\; x)\begin{pmatrix} ct \\ -x \end{pmatrix} = (ct \;\; x)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix}$$

$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the (1+1)D metric tensor,

Full (1+3)D metric tensor is $g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

9f

Since $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \end{pmatrix}$, $\underbrace{(ct' \ \ x') = (ct \ \ x)\Lambda^T}_{\text{Transposed both sides.}}$

Invariance of $c^2t^2 - x^2$:

$$c^2t'^2 - x'^2 = c^2t^2 - x^2$$

$$\Rightarrow (ct' \ \ x')\, g \begin{pmatrix} ct' \\ x' \end{pmatrix} = (ct \ \ x)\, g \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$\Rightarrow (ct \ \ x)\, \Lambda^T g \Lambda \begin{pmatrix} ct \\ x \end{pmatrix} = (ct \ \ x)\, g \begin{pmatrix} ct \\ x \end{pmatrix}$$

for any (ct, x). $\Rightarrow$ $\boxed{\Lambda^T g \Lambda = g}$

- Matrix statement of invariant interval
- Definition of Lorentz Transfo

For zero velocity, $\Lambda = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

For infinitesimal boost :

$$\Lambda = I + \epsilon K$$

$\left\{ \begin{array}{l} \epsilon \text{ is infinitesimal number.} \\ K \text{ is } 2 \times 2 \text{ matrix} \end{array} \right.$

Substitute in $\Lambda^T g \Lambda = g$

$$\Rightarrow (I + \epsilon K^T)\, g\, (I + \epsilon K) = g$$

$$\Rightarrow \quad g + \epsilon K^T g + \epsilon g K + \epsilon^2 K^T g K = g \qquad 99$$

ignore

$$\Rightarrow \quad K^T g + g K = 0$$

Try $K = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \Rightarrow \begin{pmatrix} k_1 & k_3 \\ k_2 & k_4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix} 2k_1 & k_2 - k_3 \\ k_2 - k_3 & -2k_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad k_1 = k_4 = 0 \ ; \quad k_2 = k_3$$

Thus $K = \begin{pmatrix} 0 & k_1 \\ k_1 & 0 \end{pmatrix}$.

Choose $k_1 = 1$? $\quad \Lambda = I + \epsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}$

But we know transfo elements have opposite sign; choose $k_1 = -1$: $K = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$$\Lambda = \begin{pmatrix} 1 & -\epsilon \\ -\epsilon & 1 \end{pmatrix}$$

How to go to FINITE BOOST?

(9h)

$w = N\epsilon$

Then $\Lambda(w) = \Lambda(\epsilon)\ \Lambda(\epsilon) \cdots \Lambda(\epsilon)$

$$= [\Lambda(\epsilon)]^N = [I + \epsilon K]^N$$

$$= [\exp(\epsilon K)]^N \quad \left\{ e^{\epsilon K} = 1 + \epsilon K + O(\epsilon^2) \right.$$

$$= e^{(\epsilon N)K} = e^{wK} = \exp\begin{pmatrix} 0 & -w \\ -w & 0 \end{pmatrix}$$

Expand: $\exp\begin{pmatrix} 0 & -w \\ -w & 0 \end{pmatrix} = I + \begin{pmatrix} 0 & -w \\ -w & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -w \\ -w & 0 \end{pmatrix}^2 + \cdots$

$$= \begin{pmatrix} 1 + \frac{w^2}{2} + \frac{w^4}{4!} + \cdots & -w + \frac{w^3}{3!} - \frac{w^5}{5!} + \cdots \\ -w + \frac{w^3}{3!} - \frac{w^5}{5!} + \cdots & 1 + \frac{w^2}{2} + \frac{w^4}{4!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} \cosh w & -\sinh w \\ -\sinh w & \cosh w \end{pmatrix}$$

$\rightarrow$ w is the rapidity ϕ!

Re-Derived $\Lambda_{boost} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} = \begin{pmatrix} \gamma_v & -\gamma_v \frac{v}{c} \\ -\gamma_v \frac{v}{c} & \gamma_v \end{pmatrix}$