RELATIVISTIC DYNAMICS

* Two main results: $\begin{cases} \text{Momentum: } \vec{p} = \gamma_v m \vec{v} \\ \text{Energy: } E = \gamma_v mc^2 \end{cases}$

- $\vec{v}$ is the velocity of the particle/object (previously referred to velocity of Σ' relative to Σ)
- $\gamma_v = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ increases with speed.
- Momentum and energy depend on frame (as in Newtonian mechanics)
- In frame attached to particle, $p = 0$ $\{$ as in Newtonian mechanics

  but $E = \gamma_0 mc^2 = mc^2$ $\{$ NEW

* "Deriving" $\vec{p} = \gamma_v m \vec{v}$

Conservation of momentum should be valid in all reference frames.

If $\vec{p} = m\vec{v}$ defines momentum, conservation of momentum is invariant under GALILEAN TRANSFORMATN but not under Lorentz Transformation:

Before | After

$\overset{\circ}{m_A} \xrightarrow{} u_A$ $\overset{\circ}{m_B} \xrightarrow{} u_B$ | $\overset{\circ}{m_A} \xrightarrow{} w_A$ $\overset{\circ}{m_B} \xrightarrow{} w_B$

$V \leftarrow$ another frame

$$m_A u_A + m_B u_B = m_A w_A + m_B w_B \quad \text{---} \ (\alpha)$$

From moving frame: $u'_A = u_A + v$, $u'_B = u_B + v$
$w'_A = w_A + v$, ... (Galilean)

$$\Rightarrow m_A u'_A + m_B u'_B = m_A w'_A + m_B w'_B \quad \text{---} \ (\beta)$$

is valid. — if (α) is valid

But for LT's, $u'_A = \dfrac{u_A + v}{1 + \frac{u_A v}{c^2}}$, ...

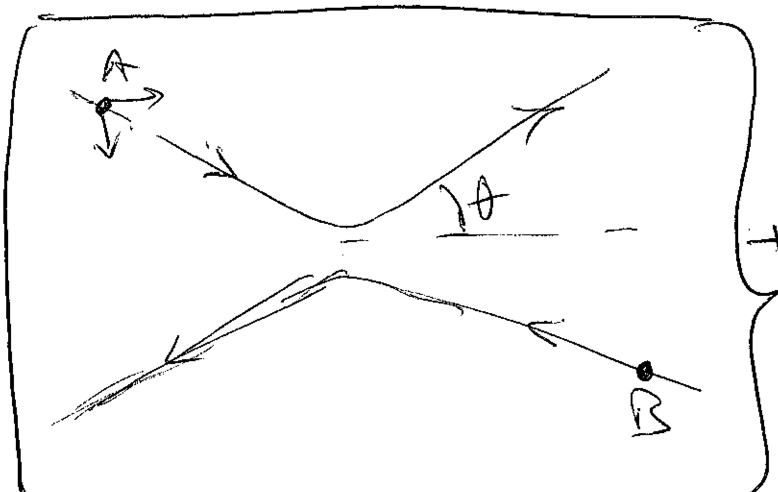
We would require

$$m_A\left(\frac{u_A + v}{1 + \frac{u_A v}{c^2}}\right) + m_B\left(\frac{u_B + v}{1 + \frac{u_B v}{c^2}}\right) = \dots$$

DOES NOT follow from (α)

Need new "momentum", $\vec{p} = \tilde{m}(v)\,\vec{v}$, whose conservation law is preserved under Lorentz Transformat^{n}s.

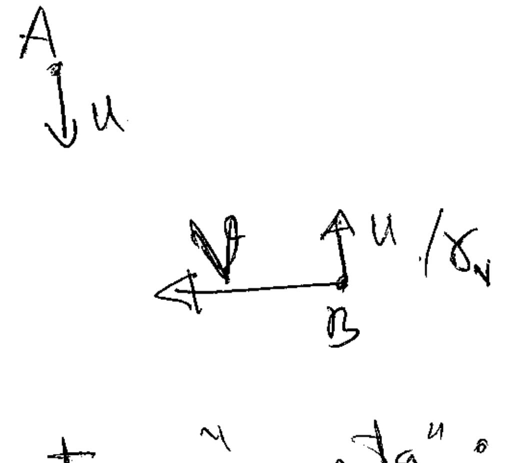
Consider collision, equal masses, with <u>very small</u> angle θ.



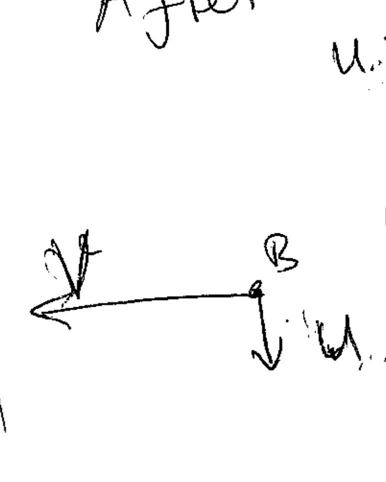
Equal, opposite, y-velocities, much smaller than x-velocities

Use frame moving in x-direction with A.

<u>Before</u>



After



$$\boxed{\gamma_V = \frac{1}{\sqrt{1 - V^2/c^2}}}$$

Equate y-"momenta":

$$-\tilde{m}(u)\,u + \tilde{m}\left(\sqrt{V^2 + \left(\tfrac{u}{\gamma_V}\right)^2}\right)\frac{u}{\gamma_V} = \tilde{m}(u) - \tilde{m}\left(\sqrt{V^2 + \left(\tfrac{u}{\gamma_V}\right)^2}\right)\frac{u}{\gamma_V}$$

Since $u \ll V$, use $\tilde{m}(u) \approx \tilde{m}(0) = m$

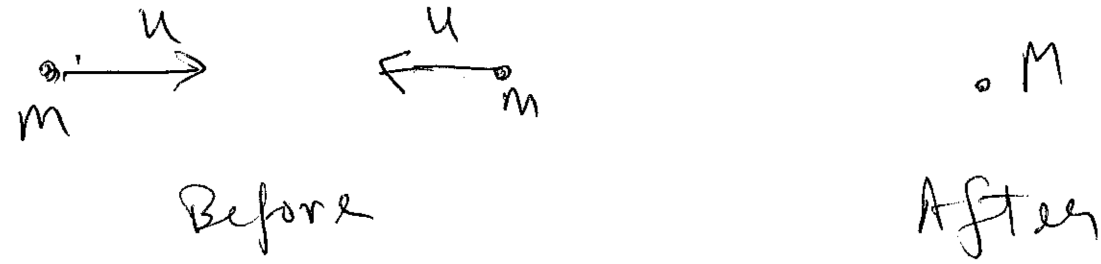
$\tilde{m}\left(\sqrt{V^2 + (u/\gamma_V)^2}\right) \approx \tilde{m}(V)$

$$\Rightarrow \tilde{m}(V) = \gamma_V \tilde{m}(0) \Rightarrow \boxed{\vec{p} = \gamma_V m \vec{v}}$$
$$= \gamma_V m$$

$$\boxed{\tilde{m}(V) = \frac{m}{\sqrt{1 - V^2/c^2}} = \gamma_V m}$$ "Relativistic mass"

In the past, $\gamma_V m$ was referred to as "mass"
→ no longer in physics texts

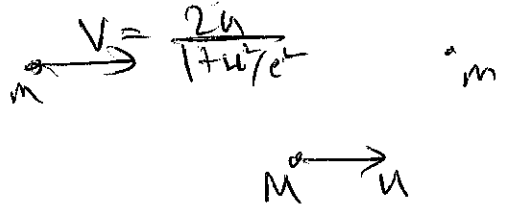
Example, momentum conservat^n



Two particles collide and stick together.

Find M. $\boxed{M \neq 2m!}$

Considering frame of an incoming particle, we find:



$$\gamma(V) = \gamma\left(\frac{2u}{1+u^2/c^2}\right) = \frac{c^2+u^2}{c^2-u^2}$$

$$\underbrace{M}_{= \frac{\gamma_V V}{\gamma_u u}} = 2\gamma(u) m = \frac{2m}{\sqrt{1-u^2/c^2}}$$

Relativistic energy : $E = \gamma m c^2$

This form justified because it's

conservation is consistent along multiple frames and with $\vec{p} = \gamma m \vec{v}$.

In previous Example, this form gives conserved energy in both frames. (Exercise!)

* KINETIC ENERGY

= the energy due to motion.

For relativistic particle, $K(v) = \gamma(v) mc^2 - \gamma(0) mc^2$

$$K = [\gamma(v) - 1] mc^2 = \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right] mc^2$$

For small v/c, $\frac{1}{\sqrt{1 - v^2/c^2}} \simeq 1 + \frac{1}{2} v^2/c^2$

$$\Rightarrow K \xrightarrow[\text{small } v/c]{} \left(\frac{1}{2}\frac{v^2}{c^2}\right) mc^2 = \frac{1}{2} mv^2$$

Nonrelativistic result recovered for $v \ll c$

Another perspective $E = mc^2 + K$

$$\approx mc^2 + \frac{1}{2}mv^2$$

mc^2 is the "rest energy", the energy that can be extracted from a body without any motion.

* Derivation of K without invoking $E = \gamma mc^2$

$$K = \int \vec{F}\cdot d\vec{r} = \int \frac{d\vec{p}}{dt}\cdot d\vec{r}$$

(integral from rest to speed v)

With nonrelativistic $\vec{p} = m\vec{v}$, this integral gives $K = \frac{1}{2}mv^2$

With relativistic $\vec{p} = \dfrac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$, this integral can be performed to give

$$K = \gamma mc^2 - mc^2 \quad \text{(done later)}$$

$\Rightarrow$ alternate justification of $E = \gamma mc^2$

Nonrelat: $K = \int_a^b \frac{d}{dt}(m\vec{v}) \cdot d\vec{r} = m\int_a^b \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt$

$$= m\int_0^V \vec{v} \cdot d\vec{v} = m\int_0^V v\, dv = m\frac{V^2}{2}$$

<u>Relativistic</u> $K = \int_a^b \frac{d\vec{p}}{dt} \cdot \vec{v}\, dt = \int_a^b \vec{v} \cdot d\vec{p}$

Indefinite integral $= \int \vec{v} \cdot d\vec{p} = \int d(\vec{v} \cdot \vec{p}) - \int \vec{p} \cdot d\vec{v}$

$$= \int d(\vec{v} \cdot \vec{p}) - \int \frac{m v\, dv}{\sqrt{1 - v^2/c^2}} \qquad \boxed{\int \frac{x\, dx}{\sqrt{1 - x^2/a^2}} = -a^2\sqrt{1 - x^2/a^2}}$$

$$= \vec{v} \cdot \vec{p} + mc^2\sqrt{1 - \frac{v^2}{c^2}}$$

$$= \frac{mv^2}{\sqrt{1 - v^2/c^2}} + mc^2\sqrt{1 - \frac{v^2}{c^2}} = mv^2\gamma(v) + \frac{mc^2}{\gamma(v)}$$

$$K = \left[mv^2\gamma(v) + \frac{mc^2}{\gamma(v)} \right]_0^V$$

$$= mV^2\gamma(V) - 0 + \frac{mc^2}{\gamma(V)} - \frac{mc^2}{1}$$

$$= m\gamma(V)\left[V^2 + \frac{c^2}{\gamma^2(V)}\right] - mc^2 \qquad \left\{ \begin{aligned} & V^2 + \frac{c^2}{\gamma^2} \\ & = V^2 + c^2\left(1 - \frac{V^2}{c^2}\right) \\ & = c^2 \end{aligned} \right.$$

$$= \gamma mc^2 - mc^2$$

(Very Important) Relationship between E, p, m

$$\left.\begin{aligned} E &= \gamma m c^2 \\ p &= \gamma m v \end{aligned}\right\} \Longrightarrow \boxed{\begin{aligned} E^2 &= p^2c^2 + m^2c^4 \\ &= (pc)^2 + (mc^2)^2 \end{aligned}}$$

(To show, eliminate v and $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ from above equations.)

E.g., $p = \frac{mv}{\sqrt{1-v^2/c^2}} \Longrightarrow$

$\gamma = \sqrt{1 + \frac{p^2}{m^2c^2}}$

$p = mc\sqrt{\gamma^2 - 1}$ SHOW!

* Can a massless particle carry energy/momentum?

In nonrelat. physics, NO

$\vec{p} = m\vec{v}$, $E = \frac{1}{2}mv^2 \longrightarrow$ So $\vec{p} = 0$, $E = 0$ when $m = 0$

But with relativity, $\vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$, $E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$

So if $v \to c$ ($\gamma \to \infty$), $\vec{p}$ & E can be FINITE for $m \to 0$.

* PHOTONS are particles with $m=0$.

Particles of light $\Rightarrow$ Travel with speed c.

Light has dual wave / particle nature (quantum mechanics)

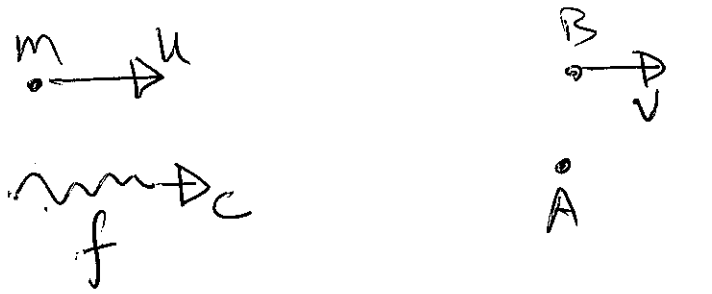
For photons (or other massless particles),

$$E = pc \quad \text{when } m=0$$

Photons : $\boxed{E = hf}$, $\boxed{p = \frac{E}{c} = \frac{hf}{c} = h/\lambda}$

$$= \frac{hc}{\lambda}$$

Observing massive/massless particles from moving frames



From A

Particle has speed u and mass m

From B

Particle has speed $\frac{u-v}{1-uv/c^2}$ and mass m

From A Photon has speed c and frequency f

From B Photon has speed c and freq. $f\sqrt{\frac{c-v}{c+v}}$

* Four Relations between E, p, m, & v !

$$E = \gamma(v)\, mc^2, \quad \vec{p} = \gamma(v)\, m\vec{v}, \quad E^2 = p^2c^2 + m^2c^4,$$

$$\frac{\vec{p}}{E} = \frac{\vec{v}}{c^2}$$

<u>$c=1$ units</u> Often we / textbooks / publication set $c=1$.

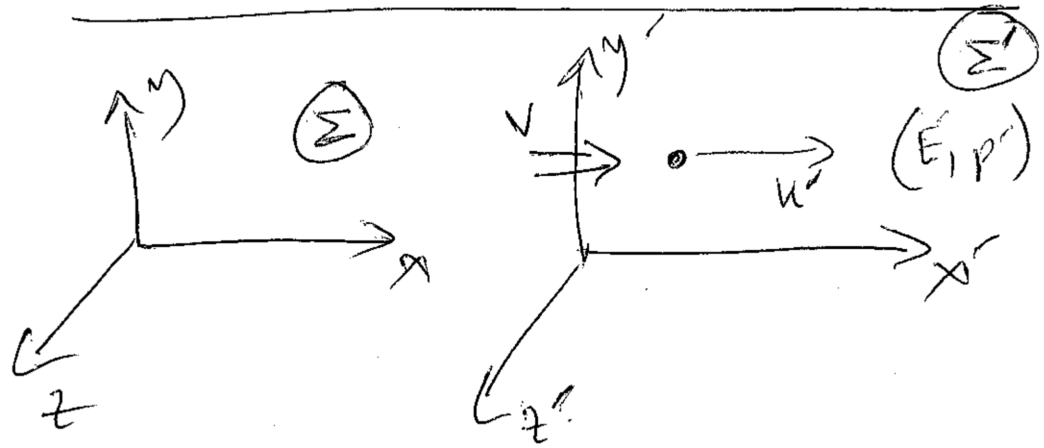
Then energy, momentum, mass are expressed in same units. Our four relations become

$$E = \gamma m, \quad p = \gamma m v, \quad E^2 = p^2 + m^2,$$

$$\frac{\vec{p}}{E} = \vec{v}$$

If you do this (e.g., in exam), state so clearly.

* TRANSFORMATIONS of E and p



u', E', p' → observed from Σ'

u, E, p

Relationship between (E, p) and (E', p') ?

We know $u = \dfrac{u'+v}{1+\frac{u'v}{c^2}}$, $\Rightarrow$ $\gamma(u) = \gamma(u')\gamma(v) \times \left(1 + u'v/c^2\right)$

Definitions: $E = \gamma(u) mc^2$, $p = \gamma(u) mu$

$E' = \gamma(u') mc^2$, $p' = \gamma(u') = mu'$

$$E = \gamma(u) mc^2 = \gamma(u')\gamma(v)\left(1 + \frac{u'v}{c^2}\right) mc^2$$

$$= \gamma(v)\left(\gamma(u') mc^2 + v\,\gamma(u') mu'\right)$$

$$= \gamma(v)\left(E' + vp'\right)$$

$$p = \gamma(u) mu = \gamma(u')\gamma(v)\left(1 + \frac{u'v}{c^2}\right) m \frac{u'+v}{1 + u'v/c^2}$$

$$= \gamma(u')\gamma(v)\, m(u'+v) = \gamma(v)\left[\gamma(u') mu' + \frac{v}{c^2}\gamma(u') mc^2\right]$$

(42)

$$\boxed{\begin{aligned} E &= \gamma(v)\left[E' + vp'\right] \\ p &= \gamma(v)\left[p' + \frac{v}{c^2}E'\right] \end{aligned}} \Rightarrow \left\{\begin{aligned} E' &= \gamma(v)\left[E - vp\right] \\ p' &= \gamma(v)\left[p - \frac{v}{c^2}E\right] \end{aligned}\right.$$

Reminder

$$\left.\begin{aligned} x &= \gamma(v)\left(x' + vt'\right) \\ t &= \gamma(v)\left(t' + \frac{v}{c^2}x'\right) \end{aligned}\right\} \text{ or } \begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \frac{\gamma v}{c} \\ \frac{\gamma v}{c} & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

Strong analogy!

$$\begin{pmatrix} E/c \\ p \end{pmatrix} = \begin{pmatrix} \gamma & \frac{\gamma v}{c} \\ \frac{\gamma v}{c} & \gamma \end{pmatrix} \begin{pmatrix} E'/c \\ p' \end{pmatrix}$$

$\Rightarrow \left(E/c, p_x, p_y, p_z\right)$ has similar transformation as $\left(ct, x, y, z\right)$. Such objects are called 4-vectors.

Common notation:

$p = (p^0, p^1, p^2, p^3)$

$p^0 = E/c$

$p^1 = p_x$, $p^2 = p_y$, $p^3 = p_z$

$x = (x^0, x^1, x^2, x^3)$

$x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

FORCE (1D)

$$F = \frac{dp}{dt} \neq ma \quad \left[a = \dot{v} = \frac{dv}{dt}\right]$$

Correct relativistic expression: $F = \gamma_v^3 ma$

Proof.

$$\frac{dp}{dt} = \frac{d}{dt}(\gamma_v m v) = m(\dot{\gamma}_v v + \gamma_v \dot{v})$$

$$= m\left(\left(\frac{\gamma_v^3 va}{c^2}\right)v + \gamma a\right)$$

$$= m\gamma_v a\left(\gamma_v^2 \frac{v^2}{c^2} + 1\right)$$

$$= m\gamma_v a \gamma_v^2 = \gamma_v^3 ma$$

$$\dot{\gamma}_v = \frac{d}{dt}\frac{1}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{(-1/2)\cdot(-2v\dot{v}/c^2)}{(1 - v^2/c^2)^{3/2}}$$

$$= \left(\frac{1}{\sqrt{1 - v^2/c^2}}\right)^3 v\dot{v}$$

$$= \gamma_v^3 va$$

Exercise Show that $F = \frac{dE}{dx}$

4-VECTORS, DEFINITION

The 4-Tuplet $A = (A^0, A^1, A^2, A^3)$ is a "4-vector" if

(1) $(A^1, A^2, A^3) = \vec{A}$ is a vector, and

(2) The A^μ transform under Lorentz transformations the same way as $(c\,dt, dx, dy, dz)$

do. If the boost is along x-direction, then A is a 4-vector if it transforms as

$$A^0 = \gamma_v\left(A^{0'} + \left(\frac{v}{c}\right)A^{1'}\right)$$
$$A^1 = \gamma_v\left(A^{1'} + \frac{v}{c}A^{0'}\right)$$
$$A^2 = A^{2'}, \quad A^3 = A^{3'}$$

* Note 1

$\vec{A} = (A^1, A^2, A^3)$ being a vector means $\vec{A}$ transforms as a vector under rotations. E.g., for rotation of x,y plane (around z-axis), the $A^{1,2,3}$ should transform as

$$\left.\begin{aligned} A^{1'} &= A^1 \cos\theta - A^2 \sin\theta \\ A^{2'} &= -A^1 \sin\theta + A^2 \cos\theta \end{aligned}\right\} \begin{aligned} A^{0'} &= A^0 \\ A^{3'} &= A^3 \end{aligned}$$

* Note 2 We've talked about transformations of (ct, x, y, z), because we were thinking of distances from the origin. It's more appropriate to talk about transformat^{n}s of $c\Delta t, \Delta x, \Delta y, \Delta z$, or $c\,dt, dx, dy, dz$