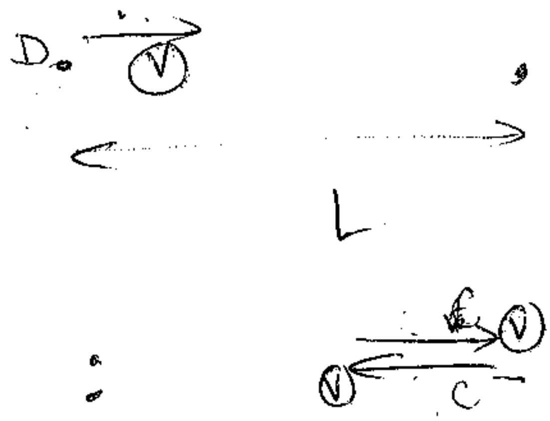
* TWIN PARADOX, again

FIRST & SIMULTANEOUS events, p. 73

DONE PREVIOUSLY

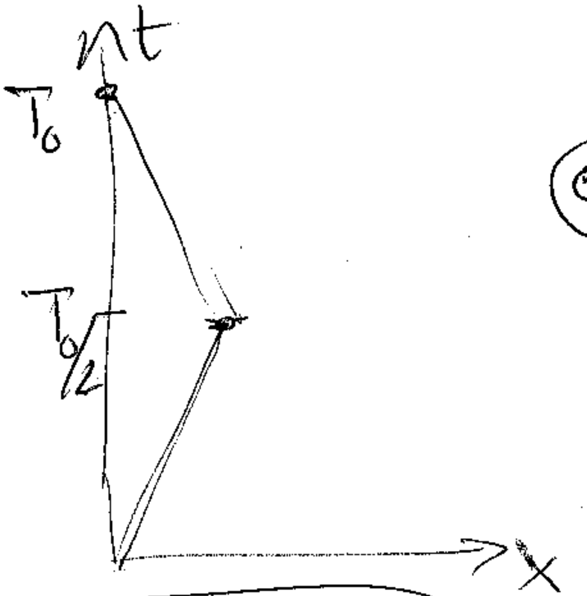


In D's frame,

C returns after

time $T_0 = \frac{2L}{v}$

How much does C age?



(c=1)

C's proper time interval

$$= \frac{T_0/2}{\gamma_v} + \frac{T_0/2}{\gamma_v}$$

$$= \frac{L/v}{\gamma_v} + \frac{L/v}{\gamma_v}$$

$$= \frac{2L}{\gamma_v v} < \frac{2L}{v}$$

= proper time for outgoing journey + proper time for return journey

C is younger than D when they meet again.

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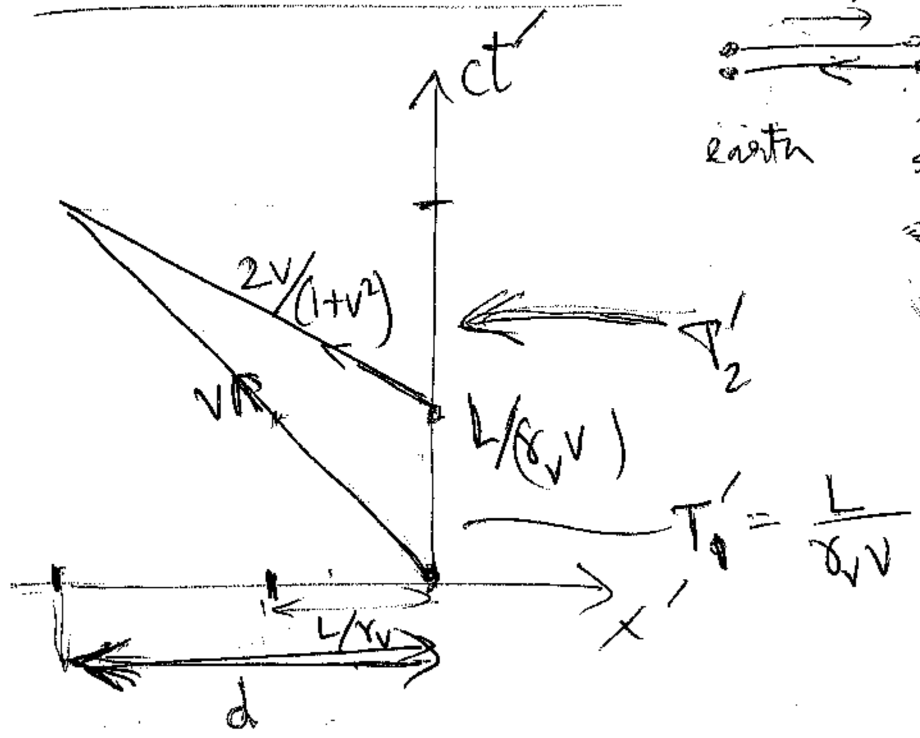
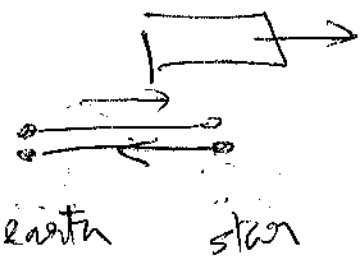
Q Would this look different from C's frame?

Ans C doesn't have a unique frame
$\Rightarrow$ question ill-defined.

C's outgoing frame:





$$\underbrace{T_2' = \frac{d}{v} - \frac{L}{\gamma_v v} = \frac{d}{2v/(1+v^2)}}$$

$$\Downarrow$$

$$d = 2L\gamma_v \quad (> \gamma_v)$$

$$\Rightarrow T_2' = \frac{d}{2v/(1+v^2)} = \frac{L}{v}\gamma_v(1+v^2)$$

C's proper time $= T_1' + \dfrac{T_2'}{\gamma\left(\frac{2v}{1+v^2}\right)}$

$$= \frac{L}{\gamma_v v} + \frac{\frac{L}{v}\,\gamma_v\,(1+v^2)}{\gamma_v^{\,2}(1+v^2)} \qquad \left\{ \begin{array}{l} \gamma\left(\frac{2v}{1+v^2}\right) \\ = \gamma_v\gamma_v(1+v^2) \end{array} \right.$$

$$= \frac{L}{\gamma_v v} + \frac{L}{\gamma_v v} = \frac{2L}{\gamma_v v}$$

$\Rightarrow$ Same as the calculation done in the earth frame, as it should be. (Propertimes are invariant.)

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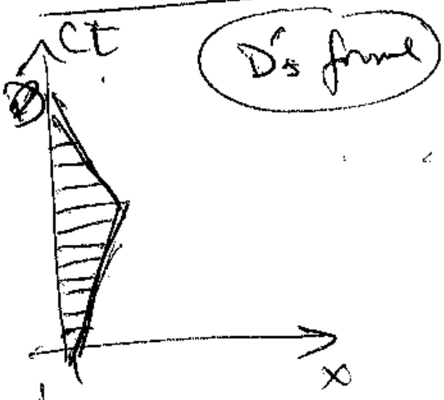
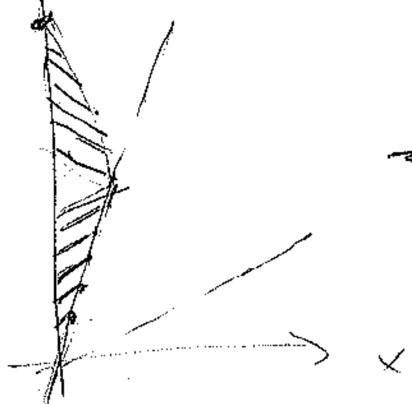
* SIMULTANEOUS EVENTS for C & D



Horizontal lines! Lines of simultaneity according to D at uniform intervals. (C aging slower by factor γ_v)



Lines of simultaneity according to C. (C's frame changes on turnaround.) According to C, D's age "jumps" when she turns around. Except for this jump, C sees D aging slower ~~according~~ by factor γ_v.

74 Electrodynamics in relativistic formulation

* Unlike mechanics, Electrodynamics is already relativistically invariant.
  We will formulate using 4-vectors, 4-tensors, study transformations of $\vec{E}$ & $\vec{B}$.

* Review:
  Maxwell's eqs: $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$, $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$
  $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$; Charge is conserved:
  $$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$
  Fields act on charges (Lorentz force law):
  $$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$
  + $\boxed{\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \quad \vec{B} = \vec{\nabla}\times\vec{A}}$

* Relativistic formulation

* q is a 4-scalar
* $A^\mu = \left(\frac{\phi}{c}, \vec{A}\right)$ is a 4-vector
* $J^\mu = \left(\mu_0 c\rho, \mu_0\vec{J}\right) = \mu_0\left(c\rho, \vec{J}\right)$
  is a 4-vector
* Often use Latin indices (i,j,k) for 3-vectors, greek indices for 4-vectors

* $\vec{E}$ & $\vec{B}$ fields combine to form an antisymmetric 4-tensor (electromagnetic field tensor):

$$F^{\mu\nu} = \begin{bmatrix} 0 & -\frac{1}{c}E_x & -\frac{1}{c}E_y & -\frac{1}{c}E_z \\ +\frac{1}{c}E_x & 0 & -B_z & B_y \\ +\frac{1}{c}E_y & B_z & 0 & -B_x \\ +\frac{1}{c}E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta} = \begin{bmatrix} 0 & +\frac{1}{c}E_x & +\frac{1}{c}E_y & +\frac{1}{c}E_z \\ -\frac{1}{c}E_x & 0 & -B_z & B_y \\ -\frac{1}{c}E_y & B_z & 0 & -B_x \\ -\frac{1}{c}E_z & -B_y & B_x & 0 \end{bmatrix}$$

Overall
* Sign, constants vary in literature. * DUAL TENSOR introduce

* Continuity equation: $\partial_\mu J^\mu = 0$

With $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ $\left[\partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right]$

* Maxwell's (~~[illegible]~~ 1st & 4th) equations:

$$\partial_\mu F^{\mu\nu} = J$$

[examples, p. 78]

* New notation for 4-force: $K^\mu = \frac{\partial p^\mu}{\partial \tau}$

(to avoid confusion with $F^{\mu\nu}$) $= \left(\frac{\gamma_v}{c}\frac{dE}{dt}, \gamma_v \vec{F}\right)$

Lorentz force law:

$$K^\mu = q F^{\mu\alpha} V_\alpha$$ [V^α is 4-vector]

worked out on p. 79

* Fields from potentials:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Ex: $F_{0\nu} = \partial_0 A_\nu - \partial_\nu A_0$

$\Rightarrow +\frac{1}{c}E_i = -\frac{\partial_t}{c}A_i - \partial_i\left(\frac{\phi}{c}\right)$

$\Rightarrow E_i = -\frac{\partial \phi}{\partial x_i} - \frac{\partial A_i}{\partial t}$

$\Rightarrow \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$

If $\nu = 1,2,3\ (i)$

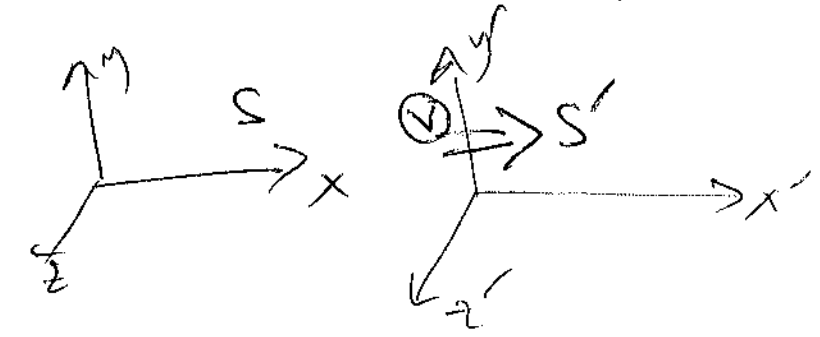
$A_\nu = -A^\nu$

$= -A_i$

$F^{\mu\nu}$ is an antisymmetric tensor, $F^{\mu\nu} = -F^{\nu\mu}$

$\longrightarrow$ 6 independent elements ($\vec{E}$ & $\vec{B}$)

* Transformation of fields



How do $\vec{E}, \vec{B}$ transform to $\vec{E}', \vec{B}'$?

We could find this using the transformation of $F^{\mu\nu}$:

$$F'^{\alpha\beta} = \Lambda^\alpha_{\ \mu} \Lambda^\beta_{\ \nu} F^{\mu\nu}$$

or using the transformations of ~~a~~ A^μ:

$$A'^0 = \gamma_v\left(A^0 - \frac{v}{c}A^1\right), \quad A'^1 = \gamma_v\left(A^1 - \frac{v}{c}A^0\right)$$

$$\Rightarrow \frac{\phi'}{c} = \gamma_v\left(\frac{\phi}{c} - \frac{v}{c}A_1\right), \quad A_1' = \gamma_v\left(A_1 - \frac{v}{c^2}\phi\right)$$

and $A'^2 = A^2$, $A'^3 = A^3$

Then $\vec{E}' = -\vec{\nabla}'\phi' - \frac{\partial \vec{A}'}{\partial t'} = \ldots$

Result is:

More general p. 80

$$E_x' = E_x \quad E_y' = \gamma\left(E_y - vB_z\right), \quad E_z' = \gamma\left(E_z + vB_y\right)$$

$$B_x' = B_x \quad B_y' = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \quad B_z' = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

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* From $\partial_\mu F^{\mu\nu} = J^\nu$,

setting $\nu = 0$:

$$\frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{10}}{\partial x^1} + \frac{\partial F^{20}}{\partial x^2} + \frac{\partial F^{30}}{\partial x^3} = J^0$$

$$\Rightarrow \frac{\partial 0}{\partial (ct)} + \frac{1}{c}\frac{\partial E_x}{\partial x} + \frac{1}{c}\frac{\partial E_y}{\partial y} + \frac{1}{c}\frac{\partial E_z}{\partial z} = \mu_0 c \rho$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \mu_0 c^2 \rho = \frac{\rho}{\epsilon_0} \quad \text{Maxwell-I}$$

Setting $\nu = 1, 2, 3$ gives ~~element~~ components of Maxwell-IV:

$\underline{\nu = 1}$

$$\frac{\partial F^{01}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{21}}{\partial x^2} + \frac{\partial F^{31}}{\partial x^3} = J^1$$

$$\Rightarrow -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial (0)}{\partial x} + \frac{\partial B_z}{\partial y} + \frac{\partial(-B_y)}{\partial z} = \mu_0 J_x$$

$$\Rightarrow \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x + \frac{1}{c^2}\frac{\partial E_x}{\partial t}$$

x-component of $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

2nd and 3rd equations ~~come from~~:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

If μ, λ, ν are the threes spatial indices $\Rightarrow$ $\vec{\nabla}\cdot\vec{B} = 0$

If one of them is 0, $\Rightarrow$ $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

---

* Lorentz force law

$$K^\mu = q F^{\mu\alpha} V_\alpha$$

$$\begin{cases} K^\mu = \left(\frac{\gamma_v}{c}\frac{dE}{dt}, \gamma_v \vec{F}\right) \\ V^\alpha = (\gamma_v c, \gamma_v \vec{v}) \end{cases}$$

With $\mu = 1$:

$$K^1 = q\left(F^{10}V_0 + F^{11}V_1 + F^{12}V_2 + F^{13}V_3\right)$$

$V_1 = g_{1\mu}V^\mu = -\gamma_v v_x$

$V_2 = -\gamma_v v_y$

$V_3 = -\gamma_v v_z$

$$= q\Big[\left(+\tfrac{1}{c}E_x\right)(\gamma_v c) + 0(\gamma_v v_x) + (-B_z)(\gamma_v v_y) + (B_y)(\gamma_v v_z)\Big]$$

$$\gamma_v F_x = \gamma_v q\left[E_x + v_y B_z - v_z B_y\right]$$

$$= \gamma_v q\left[\vec{E} + (\vec{v}\times\vec{B})\right]_x$$

$\longrightarrow$ x-component of $\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$

 from p. 27

* Transformation of fields: if relative vel. of Σ' is $\vec{V}$ (w.r.t. Σ), then

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \qquad \vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

$$\vec{E}'_{\perp} = \gamma_V \left(\vec{E}_{\perp} + \vec{V} \times \vec{B}_{\perp} \right)$$

$$\vec{B}'_{\perp} = \gamma_V \left(\vec{B}_{\perp} - \frac{1}{c^2} \vec{V} \times \vec{E}_{\perp} \right)$$

---

* Invariants of the EM field:

$$F_{\alpha\beta}F^{\alpha\beta} = 2\left(\vec{B}\cdot\vec{B} - \frac{1}{c^2}\vec{E}\cdot\vec{E} \right)$$

$$= 2\left(|\vec{B}|^2 - \frac{1}{c^2}|\vec{E}|^2 \right)$$

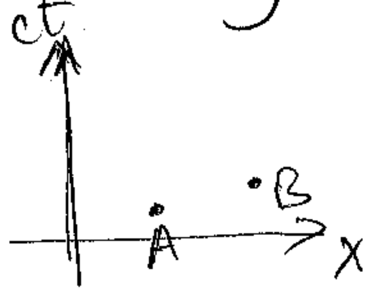
Using the dual tensor: $F_{\alpha\beta}G^{\alpha\beta}\ \cancel{\propto}\ \vec{E}\cdot\vec{B}\ \cancel{\left(-\frac{4}{c}\right)}$

invariants $\vec{E}\cdot\vec{B}$ and $|\vec{B}|^2 - \frac{1}{c^2}|\vec{E}|^2$.

Dual Tensor: $G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & E_x/c \\ -B_z & -E_y/c & -E_x/c & 0 \end{pmatrix}$

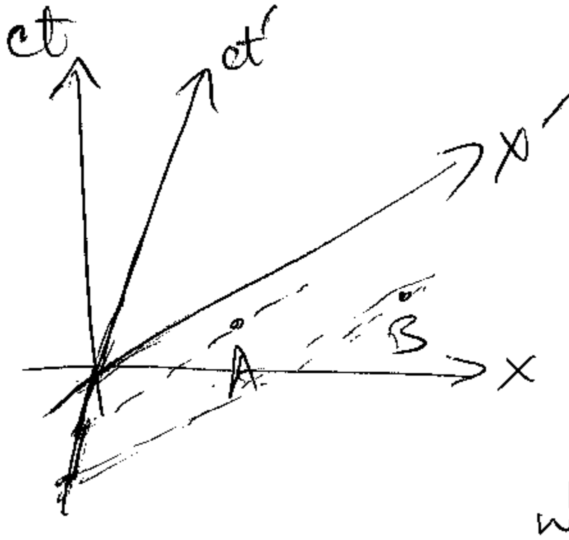
* CAUSALITY

If two events ~~are~~ have spacelike interval, one cannot affect the other. They cannot be causally related.



Looks like A happens before B; A could affect B.

$\longrightarrow (t_A < t_B)$



But: can choose a frame ~~where~~ for which A & B are SIMULTANEOUS, or even ~~the~~ one for which B happens before A

$(t'_A > t'_B)$

$\Longrightarrow$ A & B cannot be causally related.

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* COLLISIONS

* Relativistic energy & momentum are conserved!

$$\sum_i E_{i,\text{before}} = \sum_j E_{j,\text{after}}$$

$$\sum_i \vec{p}_{i,\text{before}} = \sum_j \vec{p}_{j,\text{after}}$$

One energy ~~equation~~ equations. Energies E_i are

$= mc^2$ {particles at rest}

$= hf = \frac{hc}{\lambda}$ {photons

$= \sqrt{p^2c^2+m^2c^4} = \gamma_v m c^2$ {particles with mass, moving

choose: work with momentum or speed?

Momentum: 3 equations (3-vector)

* Can combine energy-momentum conservation:

$$\sum_i \vec{\vec{p}}_{i,\text{before}} = \sum_j \vec{\vec{p}}_{j,\text{after}}$$

or $\sum_i p^\mu_{i,\text{before}} = \sum_j p^\mu_{j,\cancel{\text{before}}\,\text{after}}$

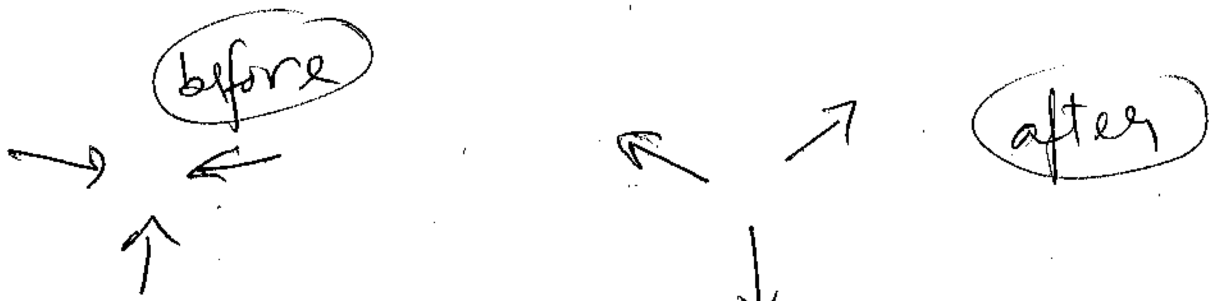
* Center of mass frame

Momenta, energy ~~co~~ conservation equations can be written in different frames.

Often useful to use

CENTER of MASS FRAME

→ frame in which total 3-momentum = 0



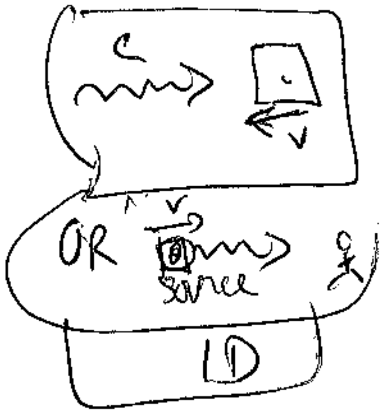
Typical CM frame picture.

Also called CENTER-of-MOMENTUM frame.

In the CoM frame the TOTAL 4-momentum is $p^{\mu}_{TOT} = \left(\frac{E_{TOT}}{c}, 0, 0, 0 \right)$.

* The WAVE 4-VECTOR & the DOPPLER EFFECT

We learned $f = \sqrt{\frac{c+v}{c-v}}\, f_0$



Consider wave

$$\phi(\vec r, t) = A \cos\left(\vec k \cdot \vec r - \omega t\right)$$

Could be $\vec E$ or $\vec B$-field, or any other wave.

$\vec k = \frac{2\pi}{\lambda} \hat k$ wavevector

$\omega = 2\pi f$ angular frequency

$= \frac{2\pi u}{\lambda}$ $\Big\{$ u = speed of wave, could be c.

$|\vec k| = \frac{\omega}{u}$

$\vec K = \left(\frac{\omega}{c}, \vec k\right)$ is a 4-vector.

$$\omega t - \vec k \cdot \vec r = \vec X * \vec K$$

Thus wave is $\phi = A \cos\left(\vec X * \vec K\right)$

Wave 4-vector in frame of observer

$$= (k^0, \vec{k}) = \left(\frac{\omega}{c}, \vec{k}\right) = \left(\frac{\omega}{c}, k_x, k_y, k_z\right)$$

In frame of source

$$k'^0 = \gamma_v \left(k^0 - \frac{v}{c} k^1\right)$$

$$\Rightarrow \quad \frac{\omega_0}{c} = \gamma_v \left(\frac{\omega}{c} - \frac{v}{c} k \cos\theta\right)$$

$$= \gamma_v \left(\frac{\omega}{c} - \frac{v}{c}\frac{\omega}{c} \cos\theta\right)$$

$$\Rightarrow \quad \omega = \frac{\omega_0}{\gamma_v\left(1 - \frac{v}{c}\cos\theta\right)} = \frac{\sqrt{c^2 - v^2}}{(c - v\cos\theta)} \omega_0$$

$$\Rightarrow \quad f = \frac{\sqrt{c^2 - v^2}}{(c - v\cos\theta)} f_0$$

For $\theta = 0$, reduces to $f = \sqrt{\frac{c+v}{c-v}} f_0$

TRANSVERSE Doppler effect: $\theta = \frac{\pi}{2}$

$$f = \sqrt{1 - \frac{v^2}{c^2}}\, f_0 = \frac{f_0}{\gamma_v}$$