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THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

MATHEMATICAL PHYSICS

Year 3

Semester 2

2014 - 2015

Special Relativity
MP352

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Time allowed: $1\frac{1}{2}$ hours

Answer **two** questions

All questions carry equal marks

1. Let $\mathcal{S}$ and $\mathcal{S}'$ be two inertial frames, with $\mathcal{S}'$ moving relative to $\mathcal{S}$ with a velocity of $0.6c$ in the positive x-direction.

   (a) An event has spacetime coordinates (x^0, x^1, x^2, x^3) in $\mathcal{S}$; what are its coordinates (x'^0, x'^1, x'^2, x'^3) in $\mathcal{S}'$?

   (b) Two 4-vectors A^μ and B_μ have the following components in $\mathcal{S}$:

   $$\begin{aligned}(A^0, A^1, A^2, A^3) &= (1, -1, 0, 0),\\ (B_0, B_1, B_2, B_3) &= (-1, 2, 0, 1).\end{aligned}$$

   Compute $||A||^2$ and $||B||^2$ and thus identify each vector as either timelike, spacelike or lightlike.

   (c) Find A'^μ and B'_μ, i.e. the components of the two 4-vectors in $\mathcal{S}'$.

   (d) Using your results from (b) and (c), confirm that

   $$||A'||^2 = ||A||^2, \quad ||B'||^2 = ||B||^2, \quad A'^\mu B'_\mu = A^\mu B_\mu.$$

2. Let G be a set and $\star$ be a binary operation on G.

   (a) List the four properties that $(G, \star)$ must satisfy in order to be a group.

   (b) Let $G = \{(x, m) | x > 0, m \in \mathbb{Z}\}$ and $\star$ be the operation

   $$(x, m) \star (y, n) = (xy, m + n).$$

   Show that $(G, \star)$ is a group.

3. Let A and B be two particles with rest masses m_A and m_B respectively, and let X be a massless particle.

   (a) Prove that if either m_A or m_B is nonzero, then the reaction $A + B \to X$ is impossible.

   (b) Show that if $m_B > m_A > 0$, the reaction $A + X \to B$ is possible and find the resulting energy of B in the rest frame of A.

   (c) Suppose both m_A and m_B are nonzero, and the reaction $A + B \to X + X$ takes place. If the total centre-of-mass energy is E_{CM}, find the energies of A, B and each X in the centre-of-mass frame.

Solutions to MP352 Exam, Spring 2014

1

P.1

This problem tests the understanding of not just Lorentz transformations, but also 4-vectors and the notation we use to describe them...

(a) We used this particular transformation so many times in lectures that it's something of a gimmee question: if the boost is of speed $v = 0.6c$ in the x-direction, then $\gamma(0.6c) = 1.25$ and so

$$\boxed{x'^0 = 1.25x^0 - 0.75x^1,\ x'^1 = 1.25x^1 - 0.75x^0,\ x'^2 = x^2,\ x'^3 = x^3}$$ [6 pts]

(b) A 4-vector with upper (contravariant) indices has a norm-squared of

$$||A||^2 = -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2$$

so for $(1,-1,0,0)$, we have $\boxed{||A||^2 = 0}$, and thus that [3pts]

vector is $\boxed{\text{lightlike}}$. A lower-index (covariant) 4-vector has [3pts]

a very similar length:

$$||B||^2 = -(B_0)^2 + (B_1)^2 + (B_2)^2 + (B_3)^2$$

so $(-1,2,0,1)$ has $\boxed{||B||^2 = 4}$ and is thus $\boxed{\text{spacelike.}}$ [3pts each]

(c) 4-vectors with upper indices transform exactly like (x^0, x^1, x^2, x^3). Thus, using the form from (a), we see that

$$\boxed{\begin{aligned} A'^0 &= 1.25(-1) - 0.75(-1) = 2 \\ A'^1 &= 1.25(-1) - 0.75(1) = -2 \\ A'^2 &= 0 \\ A'^3 &= 0 \end{aligned}}$$ [10 pts]

Lower-index 4-vectors transform as if the sign of v were reversed, i.e.

$$\boxed{\begin{aligned} B'_0 &= \gamma(v)\left(B_0 + \tfrac{v}{c}B_1\right) = 1.25(-1) + 0.75(2) = 0.25 \\ B'_1 &= \gamma(v)\left(B_1 + \tfrac{v}{c}B_0\right) = 1.25(2) + 0.75(-1) = 1.75 \\ B'_2 &= B_2 = 0 \\ B'_3 &= B_3 = 1 \end{aligned}}$$ [10 pts]

(d) Now we check that the invariants $||A||^2$, $||B||^2$ and $A\cdot B = A^\mu B_\mu$ are indeed invariant: it's easy to see that

$$\boxed{||A'||^2 = -(2)^2 + (-2)^2 = 0 = ||A||^2}$$ [3pts]

Also,

2

$$\boxed{\|B'\|^2 = -(0.25)^2 + (1.75)^2 + 1 = 4 = \|B\|^2}$$ [3 pts]

as expected. Finally,

$$A^\mu B_\mu = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 = (1)(-1) + (-1)(2) + 0 + 0 = -3$$

and

$$A'^\mu B'_\mu = (-2)(0.25) + (-2)(1.75) + 0 + 0 = -0.5 + 3.50 = -3$$

so $\boxed{A'^\mu B'_\mu = A^\mu B_\mu}$, as expected. [6 pts]

P.2

Although this question looks a bit out of place, I used some of this course to introduce the concepts of groups to the students, both in a general way and in the context of this course (e.g. SO(3,1), the Lorentz group). Since this is the only place some students might see group theory before getting their degrees, I figured it was worth including.

(a) We went through these with several examples, so hopefully most students will remember what the four basic properties are:

1. CLOSURE: if a and b are in G, then $a \star b$ is also in G	[5 pts]
2. ASSOCIATIVITY: for any $a, b, c \in G$, $(a \star b) \star c = a \star (b \star c)$	[5 pts]
3. IDENTITY: there must exist an element e in G such that, for all $a \in G$, $a \star e = e \star a = a$.	[5 pts]
4. INVERSE: for each $a \in G$, there exists $a^{-1} \in G$ such that $a \star a^{-1} = a^{-1} \star a = e$	[5 pts]

(b) In lecture, we showed that $\mathbb{R}\setminus\{0\}$ was a group under multiplication and $\mathbb{Z}$ a group under addition; this just abstracts that a bit by putting the two together.

1. CLOSED? If x, y are both positive, then so is xy. If m and n are both integers, so is $m+n$. Thus, $(xy, m+n) \in G$, so $\boxed{(G, \star) \text{ satisfies closure.}}$ [6 pts]

2. ASSOCIATIVE? Multiplication and addition are both associative, so

$$((x,k) \star (y,m)) \star (z,n) = (xy, k+m) \star (z,n)$$
$$= (xyz, k+m+n) = (x,k) \star (yz, m+n) = (x,k) \star ((y,m) \star (z,n))$$

and so $\boxed{\star \text{ is indeed associative}}$. [8 pts]

3. IDENTITY? We want the element (e_1, e_2) which satisfies $(x,m)*(e_1,e_2) = (e_1,e_2)*(x,m) = (x,m)$. We see this means $(xe_1, m+e_2) = (x,m)$, so $e_1 = 1$ and $e_2 = 0$. Thus, $\boxed{(1,0) \text{ is the identity element.}}$ [8 pts]

4. INVERSE? $(x,m)^{-1}$ is some pair (y,n) satisfying $(x,m)*(x,m)^{-1} = (x,m)^{-1}*(x,m) = e = (1,0)$ or $xy = 1$, $m+n=0$. Thus, $y = 1/x$, $n = -m$ but we also have to state that because $x>0$, $1/x$ exists and is positive, and since $m \in \mathbb{Z}$, $-m$ is also an integer. Therefore $(1/x, -m)$ is both an inverse and in G, so $\boxed{(x,m)^{-1} \in G \text{ does exist.}}$ [8 pts]

P.3

In this problem, we test concepts of "rest frame", "center-of-mass frame" as well as invariance of $\|P\|^2$ and massless particles.

(a) If A or B has a nonzero mass, then we can go into its rest frame. Suppose without loss of generality we assume $m_B > 0$. Then the total 4-momentum in the rest frame of B is

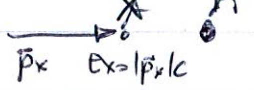
$$P^\mu_{in} = \begin{pmatrix} E_A/c + m_B c \\ \vec{p}_A \end{pmatrix}$$

where $E_A = \sqrt{|\vec{p}_A|^2 c^2 + m_A^2 c^4}$. Thus, $\|P\|^2 = -\left(\frac{E_A}{c} + m_B c\right)^2 + |\vec{p}_A|^2$

$$= -\frac{E_A^2}{c^2} - 2m_B E_A - m_B^2 c^2 + |\vec{p}_A|^2 = -(2m_B E_A + m_A^2 c^2 + m_B^2 c^2)$$

But if X is massless, then $\|P_{out}\|^2 = 0$. Thus, if this reaction can occur, $2m_B E_A + m_A^2 c^2 + m_B^2 c^2 = 0$. But $m_B > 0$, and since $m_A \geq 0$ and $E_A \geq 0$, we see there is no way this can be satisfied. $\boxed{\text{Thus, if either particle is massive, this reaction cannot occur.}}$ (10 pts)

(b) Now we assume $m_B > m_A > 0$. Since $m_A > 0$, A has a rest frame, and if we compute the 4-momentum in the rest frame of A, we have



$$P^\mu_{in} = \begin{pmatrix} |\vec{p}_X| + m_A c \\ \vec{p}_X \end{pmatrix}$$

$$\Rightarrow \|P_{in}\|^2 = -(|\vec{p}_X| + m_A c)^2 + |\vec{p}_X|^2 = -2|\vec{p}_X| m_A c - m_A^2 c^2.$$

Now, $\|P_{in}\|^2 = \|P_{out}\|^2$, and we can choose P_{out} in any frame we like: if we use B's rest frame, then $P^\mu_{out} = \begin{pmatrix} m_B c \\ \vec{0} \end{pmatrix}$, so $\|P_{out}\|^2 = -m_B^2 c^2$. Thus, $\|P_{in}\|^2 = \|P_{out}\|^2$

gives $|\vec{p}_X| = \frac{m_B^2 - m_A^2}{2m_A} c$. (Note that if $m_B \le m_A$, $|\vec{p}_X| \le 0$, which is not possible, and so $m_B > m_A$ is necessary). Now, in A's rest frame, the total 3-momentum going in is $\vec{p}_X$, so $\vec{p}_X$ is also the total 3-momentum going out, i.e. B's momentum. Thus, the energy of B in this frame is

$$E_B = \sqrt{|\vec{p}_B|^2c^2 + m_B^2c^4} = \sqrt{\left(\frac{m_B^2 - m_A^2}{2m_A}\right)^2 c^4 + m_B^2c^4}$$

$$= \sqrt{\frac{m_B^4 - 2m_B^2m_A^2 + m_A^4 + 4m_A^2m_B^2}{4m_A^2}}\, c^2$$

$$= \sqrt{\left(\frac{m_A^2 + m_B^2}{2m_A}\right)^2}\, c^2 = \boxed{\frac{m_A^2 + m_B^2}{2m_A} c^2}.$$ [10 pts]

(c) Here, we're told that the total CM energy is E_{CM}. Finding the energies of the two Xs is easy: in the CM frame, these must have equal and opposite momenta, $\vec{p}_X$ and $-\vec{p}_X$. Since their energies are both $|\vec{p}_X|c$, this means

$$P^{\mu}_{out,CM} = \begin{pmatrix} 2|\vec{p}_X| \\ 0 \end{pmatrix}.$$ But $\|P_{out,CM}\|^2 = -E_{CM}^2/c^2 = -4|\vec{p}_X|^2$,

so $|\vec{p}_X| = \frac{E_{CM}}{2c}$, or $\boxed{E_X = \frac{1}{2}E_{CM}}$ [10 pts]

A & B have equal and opposite momenta in this frame as well, so

$E_A = \sqrt{|\vec{p}_A|^2c^2 + m_A^2c^4}$ and $E_B = \sqrt{|\vec{p}_A|^2c^2 + m_B^2c^4}$. Thus,

$\sqrt{|\vec{p}_A|^2c^2 + m_A^2c^4} + \sqrt{|\vec{p}_A|^2c^2 + m_B^2c^4} = E_{CM}$. Thus,

$$\left(\sqrt{|\vec{p}_A|^2c^2 + m_A^2c^4}\right)^2 = \left(E_{CM} - \sqrt{|\vec{p}_A|^2c^2 + m_B^2c^4}\right)^2$$

or

$$|\vec{p}_A|^2c^2 + m_A^2c^4 = E_{CM}^2 - 2E_{CM}\sqrt{|\vec{p}_A|^2c^2 + m_B^2c^4} + |\vec{p}_A|^2c^2 + m_B^2c^4.$$

This says

$$m_A^2c^4 = E_{CM}^2 - 2E_{CM}E_B + m_B^2c^4$$

so

$$\boxed{E_B = \frac{E_{CM}^2 + m_B^2c^4 - m_A^2c^4}{2E_{CM}}}$$ [10 pts]

Finally,

$$E_A = E_{CM} - E_B = \frac{2E_{CM}^2 - (E_{CM}^2 + m_B^2c^4 - m_A^2c^4)}{2E_{CM}}$$

$$= \boxed{\frac{E_{CM}^2 + m_A^2c^4 - m_B^2c^4}{2E_{CM}}}$$ [10 pts]