Mathematical Physics Department

SEMESTER 2
2015-2016

MP352
Special Relativity

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Time allowed: 1½ hours

Answer ***two*** questions

All questions carry equal total marks

1. Let S and S' be inertial frames. Frame S' moves at velocity v with respect to S, in the common (positive) x direction. Measurements of events in the two frames, denoted respectively by (x, y, z, t) and (x', y', z', t'), are related by the Lorentz transformation

$$x' = \gamma(v)(x - vt) \; ; \quad y' = y \; ; \quad z' = z \; ; \quad t' = \gamma(v)(t - vx/c^2)$$

where $\gamma(v) = \dfrac{1}{\sqrt{1 - v^2/c^2}}$.

(a) Show that the quantity $x^2 + y^2 + z^2 - c^2t^2$ is invariant under this Lorentz transformation.

[20 pts.]

(b) By algebraically solving for x and t, find the inverse Lorentz transformation, i.e,, express (x, y, z, t) in terms of (x', y', z', t').
Explain why you could have expected this result, based on the velocity of frame S relative to frame S'.

[15 pts.]

(c) The rapidity ϕ is defined such that $\tanh \phi = v/c$. Show that the Lorentz transformation (omitting the transverse directions) can be expressed as

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh \phi & -\sinh \phi \\ -\sinh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \; .$$

(Possibly useful relationships involving the hyperbolic functions: $\tanh \phi = \sinh \phi / \cosh \phi$ and $\cosh^2 \phi - \sinh^2 \phi = 1$.)

[15 pts.]

2. (a) Inertial frame S' moves at velocity v with respect to another inertial frame S, in the common x direction. An object moves with constant velocity $\vec{u}'$ (components u'_x, u'_y, and u'_z) relative to S'. (If the particle changes x-position by $\Delta x'$ in time interval $\Delta t'$ as observed in the S' frame, then $u'_x = \Delta x'/\Delta t'$.)

Use the (inverse) Lorentz transformation to show that the velocity components (u_x, u_y, u_z) of the object relative to the S frame are

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}\,, \quad u_y = \frac{u'_y}{\gamma(v)\,(1 + u'_x v/c^2)}\,, \quad u_z = \frac{u'_z}{\gamma(v)\,(1 + u'_x v/c^2)}\,.$$

[20 pts.]

(b) A subatomic particle decays 1.5×10^{-8} seconds after it is created, as observed in its own reference frame. If this particle is created with speed $v = \frac{4}{5}c$ with respect to the earth, how far will it travel before it decays, as seen from the earth frame?

You can use the approximation $c \approx 3 \times 10^8$m/s.

[15 pts.]

(c) The relativistic energy of a particle of rest mass m is mc^2 when the particle is at rest and $\gamma(v)mc^2$ when the particle moves with speed v. Show that the kinetic energy due to motion is approximated by the well-known non-relativistic expression when the speed v is much smaller than the speed of light.

[15 pts.]

3. (a) A particle of rest mass M, while at rest, decays into a particle of mass m and speed v, and a photon of frequency f moving in opposite direction. Relativistic momentum and energy are conserved in this process.

   Write down and mark clearly the equations for momentum conservation and energy conservation.

   Use these equations to show that $m = M\sqrt{(c-v)/(c+v)}$.

   [20 pts.]

   (b) Two spacecraft, P and Q, approach each other on a collision course along the same straight line. Relative to a nearby planet, they move at speeds $\frac{2}{5}c$ and $\frac{3}{5}c$ respectively. Find the speed of spacecraft Q as observed by occupants of P.

   Relative to the same planet, a third spacecraft R is seen to be chasing P (moving in the same direction as P) at speed $\frac{4}{5}c$. What is the speed of R as observed by occupants of P?

   [15 pts.]

   (c) Using the fact that a Lorentz transformation can be expressed as $\Lambda(\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$ in terms of the rapidity ϕ, show that rapidities for successive Lorentz transformations are additive.

   Mention explicitly the identities for hyperbolic functions that you have used.

   [15 pts.]

PARTIAL SOLUTIONS & HINTS

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1. (a) **[20 pts.]** Show that the quantity $x^2 + y^2 + z^2 - c^2t^2$ is invariant under this Lorentz transformation.

[Sample Answser:]

$$
\begin{aligned}
x'^2 + y'^2 + z'^2 - c^2t'^2 \;&=\; \gamma^2(x - vt)^2 + y^2 + z^2 - \gamma^2c^2(t - vx/c^2)^2 \\
&= \gamma^2\left[\left(x^2 - 2xvt + v^2t^2\right) - c^2\left(t^2 - 2xvt/c^2 + v^2x^2/c^2\right)\right] + y^2 + z^2 \\
&= \gamma^2\left[(x^2 - c^2t^2) - \frac{v^2}{c^2}(x^2 - c^2t^2)\right] + y^2 + z^2 \\
&= \gamma^2(x^2 - c^2t^2)(1 - v^2/c^2) + y^2 + z^2 \\
&= (x^2 - c^2t^2) + y^2 + z^2 \qquad \Big\{\text{since } \gamma^2 = \tfrac{1}{1 - v^2/c^2} \\
&= x^2 + y^2 + z^2 - c^2t^2
\end{aligned}
$$

which demonstrates the invariance.

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(b) **[15 pts.]** By algebraically solving for x and t, find the inverse Lorentz transformation, i.e,, express (x, y, z, t) in terms of (x', y', z', t').
Explain why you could have expected this result, based on the velocity of frame S relative to frame S'.

[Sample Answser:]
The second and third equations can be inverted directly:

$$y' = y\ ; \quad z' = z \qquad \Longrightarrow \qquad y = y'\ ; \quad z = z' \tag{1}$$

From the fourth equation one obtains

$$t' = \gamma(t - vx/c^2) \qquad \Longrightarrow \qquad t = \frac{t'}{\gamma} + \frac{vx}{c^2} \tag{2}$$

Substituting into the first equation, we obtain

$$
\begin{aligned}
x' = \gamma(x - vt) = \gamma\left(x - \frac{vt'}{\gamma} - \frac{v^2x}{c^2}\right) &= \gamma x\left(1 - \frac{v^2}{c^2}\right) - vt' \\
&= \frac{x}{\gamma} - vt'
\end{aligned}
$$

which yields

$$x = \gamma(x' + vt') \tag{3}$$

Substituting into Eq. (**??**) gives

$$t = \frac{t'}{\gamma} + \frac{v}{c^2}\gamma(x' + vt') = \gamma t'\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) + \gamma\frac{vx'}{c^2}$$

The term in brackets is $(1 - v^2/c^2) + v^2/c^2 = 1$. Thus

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \tag{4}$$

Equations (**??**), (**??**), and (**??**) together give the inverse Lorentz transformation:

$$x = \gamma(x' + vt')\ ; \quad y = y'\ ; \quad z = z'\ ; \quad t = \gamma(t' + vx'/c^2)$$

We could have expected this, because frame S moves at velocity $-v$ with respect to frame S'. So, the S coordinates are obtained from the S' coordinates the same way as the S' coordinates are obtained from the S coordinates, except that one must use $-v$ instead of v,

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(c) **[15 pts.]** The rapidity ϕ is defined such that $\tanh\phi = v/c$. Show that the Lorentz transformation (omitting the transverse directions) can be expressed as

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} .$$

(Possibly useful relationships involving the hyperbolic functions: $\tanh\phi = \sinh\phi/\cosh\phi$ and $\cosh^2\phi - \sinh^2\phi = 1$.)

[Sample Answser:]
Since $v/c = \tanh\phi$,

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\tanh^2\phi}} = \frac{1}{\sqrt{1-\dfrac{\sinh^2\phi}{\cosh^2\phi}}}$$
$$= \frac{\cosh\phi}{\sqrt{\cosh^2\phi - \sinh^2\phi}} = \cosh\phi$$

and also

$$\gamma v = \cosh\phi(c\tanh\phi) = c\sinh\phi$$

The Lorentz transformation for x and t are then

$$x' = \gamma(x - vt) = (\cosh\phi)x - (c\sinh\phi)t$$
$$t' = \gamma(t - vx/c^2) = (\cosh\phi)t - (c\sinh\phi)x/c^2$$

which can be rewritten as

$$\begin{aligned} ct' &= (\cosh\phi)ct + (-\sinh\phi)x \\ x' &= (-\sinh\phi)ct + (\cosh\phi)x \end{aligned}$$

or in matrix form:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} .$$

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2. (a) **[20 pts.]** Inertial frame S' moves at velocity v with respect to another inertial frame S, in the common x direction. An object moves with constant velocity $\vec{u}'$ (components u'_x, u'_y, and u'_z) relative to S'. (If the particle changes x-position by $\Delta x'$ in time interval $\Delta t'$ as observed in the S' frame, then $u'_x = \Delta x'/\Delta t'$.) Use the (inverse) Lorentz transformation to show that the velocity components (u_x, u_y, u_z) of the object relative to the S frame are

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}\,,\ u_y = \frac{u'_y}{\gamma(v)\,(1 + u'_x v/c^2)}\,,\ u_z = \frac{u'_z}{\gamma(v)\,(1 + u'_x v/c^2)}\,.$$

[Sample Answser:]
x-component:

$$\begin{aligned} u_x &= \frac{\Delta x}{\Delta t} = \frac{\gamma(v)(\Delta x' + v\Delta t')}{\gamma(v)\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)} && \begin{cases}\text{using the inverse}\\ \text{Lorentz transformation}\end{cases} \\ &= \frac{\frac{\Delta x'}{\Delta t'} + v}{1 + \frac{v}{c^2}\frac{\Delta x'}{\Delta t'}} && \begin{cases}\text{Dividing numerator and}\\ \text{denominator by}\gamma(v)\Delta t'\end{cases} \\ &= \frac{u'_x + v}{1 + u'_x v/c^2} \end{aligned}$$

y-component:

$$\begin{aligned} u_y &= \frac{\Delta y}{\Delta t} = \frac{\Delta y'}{\gamma(v)\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)} && \begin{cases}\text{using the inverse}\\ \text{Lorentz transformation}\end{cases} \\ &= \frac{\frac{\Delta y'}{\Delta t'}}{\gamma(v)\left(1 + \frac{v}{c^2}\frac{\Delta x'}{\Delta t'}\right)} && \begin{cases}\text{Dividing numerator and}\\ \text{denominator by}\Delta t'\end{cases} \\ &= \frac{u'_y}{\gamma(v)\,(1 + u'_x v/c^2)} \end{aligned}$$

Same for z-component.

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(b) **[15 pts.]** A subatomic particle decays 1.5×10^{-8} seconds after it is created, as observed in its own reference frame. If this particle is created at a speed $v = \frac{4}{5}c$ with respect to earth, how far will it travel before it decays, as seen from the earth frame?

You can use the approximation $c \approx 3 \times 10^8$m/s.

[Sample Answser:]

From the earth frame, the lifetime of the particle is

$$\tau = \gamma\tau_0 = \frac{1}{\sqrt{1-(4/5)^2}} 1.5 \times 10^{-8}\text{s} = \frac{5}{3} 1.5 \times 10^{-8}\text{s} = 2.5 \times 10^{-8}\text{s}$$

Therefore the distance covered is

$$\text{time} \times \text{speed} = 2.5 \times 10^{-8}\text{s} \times \left(\frac{4}{5} \times 3 \times 10^8\text{m/s}\right) = 6\,\text{meters}.$$

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Wrong →

A wrong answer:

$$\frac{1.5 \times 10^{-8}\text{s}}{\gamma} \times \left(\frac{4}{5} \times 3 \times 10^8\text{m/s}\right) = 0.9 \times 10^{-8}\text{s} \times \left(\frac{4}{5} \times 3 \times 10^8\text{m/s}\right) = 2.16\,\text{meters}.$$

This is incorrect; please explain why!!

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(c) **[15 pts.]** The relativistic energy of a particle of rest mass m is mc^2 when the particle is at rest and $\gamma(v)mc^2$ when the particle moves with speed v. Show that the kinetic energy due to motion is approximated by the well-known non-relativistic expression when the speed v is much smaller than the speed of light.

[**Sample Answser:**]

$$\begin{aligned}
&\text{The kinetic energy due to motion } = \\
&= \begin{pmatrix}\text{energy of}\\ \text{moving particle}\end{pmatrix} - \begin{pmatrix}\text{energy of}\\ \text{particle at rest}\end{pmatrix} = \gamma(v)mc^2 - mc^2 \\
&\qquad = \left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)mc^2
\end{aligned}$$

When $v \ll c$, we can expand as a power series in $(v/c)^2$:

$$\begin{aligned}
&\gamma(v)mc^2 - mc^2 = mc^2\left(1-\frac{v^2}{c^2}\right)^{-1/2} - mc^2 \\
&= mc^2\left(1 + \left(-\tfrac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2}\left(-\frac{v^2}{c^2}\right)^2 + \cdots\right) - mc^2 \\
&\qquad = \left(mc^2 + \tfrac{1}{2}mv^2 + \cdots\right) - mc^2 = \tfrac{1}{2}mv^2 + \cdots
\end{aligned}$$

The leading term of this expression for small v/c is the nonrelativistic expression for kinetic energy, $\frac{1}{2}mv^2$.

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3. (a) **[20 pts.]** A particle of rest mass M, while at rest, decays into a particle of mass m and speed v, and a photon of frequency f moving in opposite direction. Relativistic momentum and energy are conserved in this process. Write down and mark clearly the equations for momentum conservation and energy conservation. Use these to show that $m = M\sqrt{(c-v)/(c+v)}$.

[Sample Answser:]

$$\text{Momentum conservation:} \qquad 0 = \frac{hf}{c} - \gamma(v)mv$$

$$\text{Energy conservation:} \qquad Mc^2 = hf + \gamma(v)mc^2$$

Eliminating hf yields

$$Mc^2 = \gamma(v)mvc + \gamma(v)mc^2$$

$$\Longrightarrow \qquad m = \frac{Mc}{\gamma(v)\times(c+v)} = M\frac{c\sqrt{1-(v/c)^2}}{c+v} = M\sqrt{\frac{c-v}{c+v}}$$

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(b) **[15 pts.]** Two spacecrafts P and Q approach each other on a collision course, moving (relative to a nearby planet) at speeds $\frac{2}{5}c$ and $\frac{3}{5}c$ respectively. Find the speed of spacecraft Q as observed by occupants of P.

Relative to the same planet, a third spacecraft R is seen to be chasing P (moving in the same direction as P) at speed $\frac{4}{5}c$. What is the speed of R as observed by occupants of P?

[Sample Answser:]

The relativistic velocity addition formula can be used to calculate the relative velocities.

Speed of Q relative to P:

$$v_{QP} = \frac{\frac{3}{5}c + \frac{2}{5}c}{1 + \frac{\frac{3}{5}c \cdot \frac{2}{5}c}{c^2}} = \frac{c}{1 + \frac{6}{25}} = \frac{25}{31}c \approx 0.806c$$

Speed of R relative to P:

$$v_{RP} = \frac{\frac{4}{5}c - \frac{2}{5}c}{1 + \frac{\frac{4}{5}c \cdot \left(-\frac{2}{5}c\right)}{c^2}} = \frac{\frac{2}{5}c}{1 - \frac{8}{25}} = \frac{10}{17}c \approx 0.588c$$

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(c) **[15 pts.]** Using the fact that a Lorentz transformation can be expressed as $\Lambda(\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$ in terms of the rapidity ϕ, show that rapidities for successive Lorentz transformations are additive.

Mention explicitly the identities for hyperbolic functions that you have used.

[Sample Answser:]

$$\Lambda(\phi_1)\Lambda(\phi_2) = \begin{pmatrix} \cosh\phi_1 & -\sinh\phi_1 \\ -\sinh\phi_1 & \cosh\phi_1 \end{pmatrix} \begin{pmatrix} \cosh\phi_2 & -\sinh\phi_2 \\ -\sinh\phi_2 & \cosh\phi_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\phi_1\cosh\phi_2 + (-\sinh\phi_1)(-\sinh\phi_2) & \cosh\phi_1(-\sinh\phi_1) + (-\sinh\phi_1)\cosh\phi_2 \\ (-\sinh\phi_1)\cosh\phi_2 + \cosh\phi_1(-\sinh\phi_2) & (-\sinh\phi_1)(-\sinh\phi_2) + \cosh\phi_1\cosh\phi_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\phi_1+\phi_2) & -\sinh(\phi_1+\phi_2) \\ -\sinh(\phi_1+\phi_2) & \cosh(\phi_1+\phi_2) \end{pmatrix} = \Lambda(\phi_1+\phi_2)$$

This shows that successive Lorentz transformations are additive.

The identities used above are:

$$\sinh(\phi_1+\phi_2) = \sinh\phi_1\cosh\phi_2 + \cosh\phi_1\sinh\phi_2$$
$$\cosh(\phi_1+\phi_2) = \cosh\phi_1\cosh\phi_2 + \sinh\phi_1\sinh\phi_2$$

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