Mathematical Physics Department

AUTUMN REPEAT EXAMINATION
2015-2016

MP352
Special Relativity

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Time allowed: 1½ hours

Answer ***two*** questions

All questions carry equal total marks

1. Let S and S' be inertial frames. Frame S' moves at velocity v with respect to S, in the common (positive) x direction. Measurements of events in the two frames, denoted respectively by (x, y, z, t) and (x', y', z', t'), are related by the Lorentz transformation
$$x' = \gamma(v)(x - vt) \; ; \quad y' = y \; ; \quad z' = z \; ; \quad t' = \gamma(v)(t - vx/c^2)$$
where $\gamma(v) = (1 - v^2/c^2)^{-1/2}$.

   (a) A particle moves in the common x direction. Its speed is u in the S frame and u' in the S' frame. Write down, in terms of u and u', the four-velocity of the particle in the S frame and in the S' frame.
   Since the four-velocity is a four-vector, it should transform according to a Lorentz transformation. Write down the transformations between the four-vector components. Be careful to indicate the relevant speed for each γ-factor.
   Use the transformation equations to obtain $\gamma(u')$ in terms of u and v.

   [20 marks]

   (b) Write down the Lorentz transformation as a 4×4 matrix Λ that transforms the four-vector (ct, x, y, z) to the four-vector (ct', x', y', z').
   By considering the velocity of the S frame with respect to the S' frame, write down the inverse 4×4 matrix Λ^{-1}, that transforms (ct', x', y', z') to (ct, x, y, z).
   Show that explicit matrix multiplication of the two 4×4 matrices you have written yields the unit matrix, i.e., $\Lambda\Lambda^{-1} = I$.

   [15 marks]

   (c) A body of mass m is at rest in the frame S'.
   Write down its four-momentum in the frame S'.
   Write down its four-momentum in the frame S.
   Show that the norm of the four-momentum is the same in the two frames.

   [15 marks]

2. (a) A neutral pion has mass M and is travelling with speed v when it decays into two photons. The photons are seen to emerge at equal angles θ on either side of the original velocity direction. Show that $v = c\cos\theta$.

   If you introduce any new symbols when writing down conservation equations, please define them clearly.

   [20 marks]

   (b) Use the fact that rapidities are additive to derive the relativistic formula for addition of velocities along the same direction.

   In other words, if frame S' has speed $v_1 = c\tanh\phi_1$ relative to frame S, and frame S'' has speed $v_2 = c\tanh\phi_2$ relative to frame S' (in the same direction), use the fact that the rapidity of S'' relative to S is $(\phi_i + \phi_2)$, to find the speed of S'' relative to S.

   You may need to derive an identity concerning the hyperbolic tangent of the sum of two quantities. You can derive such an identity using the relations

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$$
$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

   and the definition $\tanh z = \sinh z / \cosh z$.

   [15 marks]

   (c) A space traveller, A, travels from earth to an intersteller station at speed $\frac{4}{5}c$, and then returns promptly, travelling back at the same speed. Back at earth, she finds that her twin brother B has aged 50 years during her absence. How much has A aged during her trip?

   From A's perspective, it was B who was travelling away from her and then back toward her; so she might wonder why B is not younger than her due to time dilation, since the laws of physics are the same in all inertial frames. Explain why this argument does not hold, i.e., why the situation is not symmetrical between the two twins.

   [15 marks]

3. (a) $A = (A_0, A_1, A_2, A_3)$ and $B = (B_0, B_1, B_2, B_3)$ are four-vectors. Under a boost (Lorentz transformation), the components of A transform as

$$A_0' = \gamma(v)\left(A_0 - \frac{v}{c}A_1\right)\,, \quad A_1' = \gamma(v)\left(A_1 - \frac{v}{c}A_0\right)\,, \quad A_2' = A_2\,, \quad A_3' = A_3$$

Write down the transformations for the components of B, under the same boost.

Show that the inner product, defined as

$$A \cdot B \;=\; A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3$$

is invariant under the same boost.

[20 marks]

(b) Two balls, each having mass m_0, approach each other with equal but opposite velocities of magnitude $v = 0.8c$. Their collision is perfectly inelastic, so they stick together and form a single body of mass M. What is the velocity of the final body and what is its mass M?

Please provide answers first in terms of the symbol v, and only at the end use the value $v = 0.8c$.

[15 marks]

(c) The kinetic energy of motion (T) of a particle is the relativistic total energy minus the rest energy.

Express the momentum p of the particle as a function of T and the rest mass M. Your expression should contain M and T, not speeds or velocities.

[15 marks]

(d) Inertial frame S' moves at velocity v with respect to another inertial frame S, in the common x direction. An object moves with constant velocity $\vec{u}'$ (components u'_x, u'_y, and u'_z) relative to S'. (If the particle changes x-position by $\Delta x'$ in time interval $\Delta t'$ as observed in the S' frame, then $u'_x = \Delta x'/\Delta t'$.)

Use the (inverse) Lorentz transformation to show that the velocity components (u_x, u_y, u_z) of the object relative to the S frame are

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}\,, \; u_y = \frac{u'_y}{\gamma(v)\,(1 + u'_x v/c^2)}\,, \; u_z = \frac{u'_z}{\gamma(v)\,(1 + u'_x v/c^2)}\,.$$

[20 pts.]

(e) A subatomic particle decays 1.5×10^{-8} seconds after it is created, as observed in its own reference frame. If this particle is created with speed $v = \frac{4}{5}c$ with respect to the earth, how far will it travel before it decays, as seen from the earth frame?

You can use the approximation $c \approx 3 \times 10^8$m/s.

[15 pts.]

(f) The relativistic energy of a particle of rest mass m is mc^2 when the particle is at rest and $\gamma(v)mc^2$ when the particle moves with speed v. Show that the kinetic energy due to motion is approximated by the well-known non-relativistic expression when the speed v is much smaller than the speed of light.

[15 pts.]

SOLUTIONS & HINTS

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1. Question 1

   (a) **[20 marks]** A particle moves in the common x direction. Its speed is u in the S frame and u' in the S' frame. Write down the four-velocity of the particle in the S frame and in the S' frame.

   Since the four-velocity is a four-vector, it should trannsform according to a Lorentz transformation. Write down the transformations between the four-vector components. Be careful to indicate the relevant speed for each γ.

   Use the transformation equations to obtain $\gamma(u')$ in terms of u and v.

   [Sample Answser:]

   In the S frame:

$$(\gamma_u c, \gamma_u u, 0, 0)$$

   In the S' frame:

$$(\gamma_{u'} c, \gamma_{u'} u', 0, 0)$$

   Since the four-velocity is a four-vector, it should transform according to a Lorentz transformation.

$$\begin{aligned} \gamma_{u'} c &= \gamma_v \left(\gamma_u c - \left(\frac{v}{c}\right) \gamma_u u \right) \\ \gamma_{u'} u' &= \gamma_v \left(\gamma_u u - \left(\frac{v}{c}\right) \gamma_u c \right) \end{aligned}$$

   The first equation gives the desired expression for $\gamma_{u'}$:

$$\gamma_{u'} \;=\; \gamma_u \gamma_v \left(1 - \frac{uv}{c^2}\right) \;=\; \frac{1 - uv/c^2}{\sqrt{(1 - u^2/c^2)(1 - v^2/c^2)}}$$

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(b) [**15 marks**] Write down the Lorentz transformation as a 4×4 matrix Λ that transforms the four-vector (ct, x, y, z) to the four-vector (ct', x', y', z').
By considering the velocity of the S frame with respect to the S' frame, write down the inverse 4×4 matrix Λ^{-1}, that transforms (ct', x', y', z') to (ct, x, y, z).
Show that explicit matrix multiplication of the two 4×4 matrices you have written yileds the unit matrix, i.e., $\Lambda\Lambda^{-1} = I$.

[**Sample Answser:**]
The transformation equations from (ct, x, y, z) to (ct', x', y', z') are:

$$\begin{aligned} ct' &= \gamma_v ct + (-\gamma_v v/c)x + (0)y + (0)z \\ x' &= (-\gamma_v v/c)ct + (\gamma_v)x + (0)y + (0)z \\ y' &= (0)ct + (0)x + (1)y + (0)z \\ z' &= (0)ct + (0)x + (0)y + (1)z \end{aligned}$$

By inspection of the above equation one can write down the 4×4 transformation matrix

$$\Lambda_1 = \begin{pmatrix} \gamma_v & -\gamma_v v/c & 0 & 0 \\ -\gamma_v v/c & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The inverse transformation is obtained by replacing v by $-v$:

$$\Lambda_2 = \begin{pmatrix} \gamma_v & +\gamma_v v/c & 0 & 0 \\ +\gamma_v v/c & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The product of the two matrices is

$$\begin{aligned} \Lambda_1\Lambda_2 &= \begin{pmatrix} \gamma_v & -\gamma_v v/c & 0 & 0 \\ -\gamma_v v/c & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_v & +\gamma_v v/c & 0 & 0 \\ +\gamma_v v/c & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma_v^2 - \gamma_v^2 v^2/c^2 & +\gamma_v^2 v/c - \gamma_v^2 v/c & 0 & 0 \\ +\gamma_v^2 v/c - \gamma_v^2 v/c & \gamma_v^2 - \gamma_v^2 v^2/c^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

In the last step we used $\gamma_v^2 - \gamma_v^2 v^2/c^2 = \gamma_v^2(1 - v^2/c^2) = 1$.

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(c) [**15 marks**] A body of mass m is at rest in the frame S'.

Write down its four-momentum in the frame S'.

Write down its four-momentum in the frame S.

Show that the norm of the four-momentum is the same in the two frames.

[**Sample Answser:**]

In the frame S':

The body has momentum 0 and energy $\gamma_0 mc^2 = mc^2$. Hence the four-momentum is

$$(\frac{mc^2}{c}, 0, 0, 0) \;=\; (mc, 0, 0, 0)$$

The norm is

$$(mc)^2 - 0^2 - 0^2 - 0^2 = m^2c^2$$

In the frame S:

The body has velocity v in the x-direction, hence three-momentum components

$$(\gamma_v mv, 0, 0)$$

The energy of the body is $\gamma_v mc^2$.

The four-momentum is thus

$$(\frac{\gamma_v mc^2}{c}, \gamma_v mv, 0, 0) \;=\; (\gamma_v mc, \gamma_v mv, 0, 0)$$

The norm is

$$(\gamma_v mc)^2 - (\gamma_v mv)^2 - 0^2 - 0^2 \;=\; (\gamma_v mc)^2(1 - v^2/c^2) \;=\; \gamma_v^2 m^2 c^2 \gamma_v^{-2} \;=\; m^2c^2$$

The norm is thus the same in the two frames.

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2. Question 2

(a) **[20 marks]** A neutral pion has mass M and is traveling with speed v when it decays into two photons. The photons are seen to emerge at equal angles θ on either side of the original velocity. Show that $v = c\cos\theta$.

If you introduce any new symbols when writing down conservation equations, please define them clearly.

[Sample Answser:]

With clear pictures (before and after collision), students should be able to write down energy and momentum conservation equations. In this case, it's best to write them in terms of speeds, since the desired result concerns the speed.

Energy conservation:

$$\gamma_v Mc^2 \;=\; hf_1 + hf_2$$

I've introduced notation f_1 and f_2 for the frequencies of the two photons. When you introduce new notation in your assignments of exams, please do state clearly what you have introduced.

Momentum conservation in the original direction of pion motion:

$$\gamma_v Mv \;=\; \frac{hf_1}{c}\cos\theta + \frac{hf_2}{c}\cos\theta$$

Momentum conservation in the direction perpendicular to the original velocity:

$$0 \;=\; \frac{hf_1}{c}\sin\theta - \frac{hf_2}{c}\sin\theta$$

The last equation gives $f_1 = f_2$. The first equation then yields $hf_1 = \frac{1}{2}\gamma_v Mc^2$, which, when put into the second equation, gives

$$\gamma_v Mv \;=\; 2(\tfrac{1}{2}\gamma_v Mc^2)\cos\theta \qquad \Longrightarrow \qquad v = c\cos\theta$$

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(b) [**15 marks**] Use the fact that rapidities are additive to derive the relativistic formula for addition of velocities along the same direction.

In other words, if frame S' has speed $v_1 = c\tanh\phi_1$ relative to frame S, and frame S'' has speed $v_2 = c\tanh\phi_2$ relative to frame S' (in the same direction), use the fact that the rapidity of S'' relative to S is $(\phi_i + \phi_2)$, to find the speed of S'' relative to S.

You may need to derive an identity concerning the hyperbolic tangent of the sum of two quantities. You can derive such an identity using the relations

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$$
$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

and the definition $\tanh z = \sinh z / \cosh z$.

[Sample Answser:]

Since the rapidity of S'' relative to S is $(\phi_i + \phi_2)$, the speed of S'' relative to S is

$$v \;=\; c\tanh(\phi_i + \phi_2)$$

Now using the given hyperbolic identities

$$\tanh(z_1 + z_2) \;=\; \frac{\sinh(z_1 + z_2)}{\cosh(z_1 + z_2)} \;=\; \frac{\sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2}{\cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2}$$
$$= \frac{\dfrac{\sinh z_1 \cosh z_2}{\cosh z_1 \cosh z_2} + \dfrac{\cosh z_1 \sinh z_2}{\cosh z_1 \cosh z_2}}{1 + \dfrac{\sinh z_1 \sinh z_2}{\cosh z_1 \cosh z_2}} \;=\; \frac{\tanh z_1 + \tanh z_2}{1 + \tanh z_1 \tanh z_2}$$

Thus

$$v \;=\; c \times \frac{\tanh\phi_1 + \tanh\phi_2}{1 + \tanh\phi_1 \tanh\phi_2} \;=\; c\frac{\dfrac{v_1}{c} + \dfrac{v_2}{c}}{1 + \dfrac{v_1}{c}\dfrac{v_2}{c}} \;=\; \frac{v_1 + v_2}{1 + \dfrac{v_1 v_2}{c^2}}$$

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(c) [**15 marks**] A space traveler, A, travels from earth to an intersteller station at speed $\frac{4}{5}c$, and then returns promptly, traveling back at the same speed. Back at earth, she finds that her twin brother B has aged 50 years during her absence. How much has A aged during her trip? .... Explain why the situation is not symmetrical between the two twins.

[Sample Answser:]
The traveler, A, ages less, so she ages

$$\frac{50\text{yrs}}{\gamma} = \frac{50\text{yrs}}{1/\sqrt{1-(4/5)^2}} = \frac{50\text{yrs}}{5/3} = 30\text{yrs}$$

The situation is not symmetric, because A undergoes acceleration when she changes direction at the interstellar station, while B sits in an intertial frame.

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3. Question 3

(a) **[20 marks]** $A = (A_0, A_1, A_2, A_3)$ and $B = (B_0, B_1, B_2, B_3)$ are four-vectors. Under a boost (Lorentz transformation), the components of A transform as

$$A'_0 = \gamma(v)\left(A_0 - \frac{v}{c}A_1\right) , \quad A'_1 = \gamma(v)\left(A_1 - \frac{v}{c}A_0\right) , \quad A'_2 = A_2 , \quad A'_3 = A_3$$

Write down the transformations for the components of B, under the same boost.

Show that the inner product, defined as

$$A \cdot B \;=\; A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3$$

is invariant under the same boost.

[**Sample Answser:**]

Exactly the same transformations for the components of B:

$$B'_0 = \gamma(v)\left(B_0 - \frac{v}{c}B_1\right) , \quad B'_1 = \gamma(v)\left(B_1 - \frac{v}{c}B_0\right) , \quad B'_2 = B_2 , \quad B'_3 = B_3$$

Invariance:

$$\begin{aligned}
A' \cdot B' \;&=\; A'_0B'_0 - A'_1B'_1 - A'_2B'_2 - A'_3B'_3 \\
&=\; \gamma_v\left(A_0 - \frac{v}{c}A_1\right)\gamma_v\left(B_0 - \frac{v}{c}B_1\right) \\
&\quad - \gamma_v\left(A_1 - \frac{v}{c}A_0\right)\gamma_v\left(B_1 - \frac{v}{c}B_0\right) - A_2B_2 - A_3B_3 \\
&=\; A_0B_0\gamma_v^2\left(1 - v^2/c^2\right) + A_0B_1\left(-\gamma_v\frac{v}{c} + \gamma_v\frac{v}{c}\right) \\
&\quad + A_1B_0\left(-\gamma_v\frac{v}{c} + \gamma_v\frac{v}{c}\right) + A_1B_1\gamma_v^2\left(v^2/c^2 - 1\right) - A_2B_2 - A_3B_3 \\
&=\; A_0B_0 + 0 + 0 - A_1B_1 - A_2B_2 - A_3B_3 \;=\; A \cdot B
\end{aligned}$$

proving the invariance.

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(b) [**15 marks**] Two balls, each having mass m_0, approach each other with equal but opposite velocities of magnitude $v = 0.8c$. Their collision is perfectly inelastic, so they stick together and form a single body of mass M. What is the velocity of the final body and what is its mass M?

Please provide answers first in terms of the symbol v, and only at the end use the value $v = 0.8c$.

[Sample Answser:]

The final velocity is zero from symmetry. Alternately, you could assume a final velocity V_f, and write down the momentum conservation equation:

$$\gamma(v)mv - \gamma(v)mv \;=\; \gamma(V_f)MV_f$$

which gives $V_f = 0$.

The energy conservation equation gives

$$\gamma(v)mc^2 + \gamma(v)mc^2 \;=\; \gamma(0)Mc^2 \quad \Longrightarrow \quad M \;=\; 2\gamma(v)m \;=\; \frac{2m}{\sqrt{1 - v^2/c^2}}$$

Using $v = 0.8c$ yields

$$M = \frac{10}{3}m$$

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(c) [**15 marks**] The kinetic energy of motion (T) of a particle is the relativistic total energy minus the rest energy.

Express the momentum p of the particle as a function of T and the rest mass M. Your expression should contain M and T, not speeds or velocities.

[**Sample Answser:**]

$$\begin{aligned} T &= E - Mc^2 = \sqrt{p^2c^2 + M^2c^4} - Mc^2 \\ \Longrightarrow \quad p^2c^2 &= (T + Mc^2)^2 - M^2c^4 = T^2 + 2TMc^2 \\ \Longrightarrow \quad p &= \frac{1}{c}\sqrt{T^2 + 2TMc^2} = \sqrt{\frac{T^2}{c^2} + 2TM} \end{aligned}$$

Looks simpler if you use $c = 1$ units:

$$\begin{aligned} T &= E - M = \sqrt{p^2 + M^2} - M \\ \Longrightarrow \quad p^2 &= (T + M)^2 - M^2 = T^2 + 2TM \\ \Longrightarrow \quad p &= \sqrt{T^2 + 2TM} \end{aligned}$$

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