MATHEMATICAL PHYSICS

SEMESTER 2, REPEAT

2017–2018

MP352
Special Relativity

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Time allowed: 2 hours

Answer **ALL** questions

1. Consider the set of 4×4 matrices Λ with real elements which satisfy the following relation
$$\Lambda^T g \Lambda = g \,, \tag{1}$$
where g is the metric tensor. These matrices represent Lorentz transformations of spacetime points (ct, x, y, z).

   (a) Write down the metric tensor g in the the two common conventions, and mention which you will use in the following.

   [3 marks]

   (b) Under what conditions is a matrix of this set orthochronous?
   Explain what a non-orthochronous matrix represents physically.

   [6 marks]

   (c) Write down the 4×4 matrix transforming spacetime coordinates (ct, x, y, z) of an event under a Lorentz boost in the z direction. Show that this matrix satisfies condition (1).

   [8 marks]

   (d) Show that, if a transformation of spacetime coordinates (ct, x, y, z) preserves the Minkowski norm, then it must satisfiy condition (1).

   [13 marks]

2. Consider intertial frames Σ and Σ'. Frame Σ' moves at velocity v with respect to Σ, in the common (positive) x direction. Measurements of an event in the two frames, (ct, x, y, z) and (ct', x', y', z'), are related by the Lorentz transformation

$$ct' = \gamma_v(ct - vx/c) \ ; \quad x' = \gamma_v(x - vt) \ ; \quad y' = y \ ; \quad z' = z$$

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

(a) A photon has velocity $\vec{u} = (\frac{4}{5}c, \frac{3}{5}c, 0)$ relative to Σ.
Find the velocity $\vec{u'}$ of the photon relative to Σ'.
Explain how your result is consistent with the constancy of the speed of light.

[18 marks]

(b) A particle moves with velocity $(w, 0, 0)$ relative to Σ. Write down the four-velocity of the particle as observed from the Σ frame and as observed from the Σ' frame.

[7 marks]

(c) In the Σ frame, two events occur at times $t_1 = \dfrac{L}{c}$ and $t_2 = \dfrac{L}{2c}$, and have spatial coordinates respectively

$$(x_1 = 4L, y_1 = 0, z_1 = 0) \quad \text{and} \quad (x_2 = 2L, y_2 = 0, z_2 = -6L).$$

For a certain value of v, these two events are simultaneous in Σ'. Find this value of v.

[10 marks]

3. (a) A high-energy particle of mass M is travelling with speed v when it decays into two photons. The photons are seen to emerge at equal angles θ on either side of the original velocity. Show that $v = c\cos\theta$.

   Also find the energy of either photon. Express this energy in terms M and θ.

   [17 marks]

   (b) Show that the four-momentum of a photon is light-like.

   Show that the four-momentum of a particle with nonzero mass is time-like.

   [8 marks]

   (c) On a spacetime diagram (ct versus x) diagram, show examples of physical worldlines and un-physical (tachyonic) worldlines.

   Sketch the worldline of a particle subject to a constant force in positive x direction. The particle starts from rest at the origin at time $t = 0$. State in words the initial and asymptotic (late-time) speeds of the particle. Both should be clear from your sketch.

   [10 marks]

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SAMPLE PARTIAL ANSWERS

MAY BE INCOMPLETE

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1. Question 1.

Consider the set of 4×4 matrices Λ with real elements which satisfy the following relation

$$\Lambda^T g \Lambda = g\,,$$

where g is the metric tensor. These matrices represent Lorentz transformations of spacetime points (ct, x, y, z).

(a) Question 1(a)

Write down the metric tensor g in the the two common conventions, and mention which you will use in the following.

[3 marks]

[**Sample Answer:**]

In the two common conventions,

$$g = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

respectively. Sometimes, the first is referred to as the $(+ - --)$ convention and the second as the $(- + ++)$ convention.

In class we used the first (negative-trace) convention. These solutions will continue to use this.

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(b) Question 1(b)

Under what conditions is a matrix of this set orthochronous?

Explain what a non-orthochronous matrix represents physically.

[6 marks]

[**Sample Answer:**]

A Lorentz transformation matrix is orthochronous if the top-left element is positive, i.e.,

$$\Lambda^0_0 \ > \ 0$$

Physically, non-orthochronous transformations involve changing the sign (direction) of time. These transformations are unphysical.

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(c) Question 1(c)

Write down the 4×4 matrix transforming spacetime coordinates (ct, x, y, z) of an event under a Lorentz boost in the z direction. Show that this matrix satisfies condition (1).

[8 marks]

[**Sample Answer:**]

If the speed of the boost is denoted as v, then in $c = 1$ units, the transformation matrix is

$$B \ = \ \begin{pmatrix} \gamma_v & 0 & 0 & -\gamma_v v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_v v & 0 & 0 & \gamma_v \end{pmatrix}$$

Now showing that the matrix B satisfies the condition (1):

$$
\begin{aligned}
BgB &= B\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma_v & 0 & 0 & -\gamma_v v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_v v & 0 & 0 & \gamma_v \end{pmatrix} \\
&= B\begin{pmatrix} \gamma_v & 0 & 0 & -\gamma_v v \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ +\gamma_v v & 0 & 0 & -\gamma_v \end{pmatrix} \\
&= \begin{pmatrix} \gamma_v & 0 & 0 & -\gamma_v v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_v v & 0 & 0 & \gamma_v \end{pmatrix}\begin{pmatrix} \gamma_v & 0 & 0 & -\gamma_v v \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ +\gamma_v v & 0 & 0 & -\gamma_v \end{pmatrix} \\
&= \begin{pmatrix} \gamma_v^2 - \gamma_v^2 v^2 & 0 & 0 & -\gamma_v^2 v + \gamma_v^2 v \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\gamma_v^2 v + \gamma_v^2 v & 0 & 0 & \gamma_v^2 v^2 - \gamma_v^2 \end{pmatrix}
\end{aligned}
$$

The off-diagonal terms vanish, and the two nontrivial-looking diagonal terms are

$$\gamma_v^2(1-v^2) = \frac{1}{(1-v^2)}(1-v^2) = 1$$

and

$$\gamma_v^2(v^2-1) = \frac{1}{(1-v^2)}(v^2-1) = -1$$

so that

$$BgB = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g$$

i.e., B satisfies the equation defining Lorentz transformations.

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(d) Question 1(d)

Show that, if a transformation of spacetime coordinates (ct, x, y, z) preserves the Minkowski norm, then it must satisfiy condition (1).

[13 marks]

[**Sample Answer:**]

First note that

$$c^2t^2 - x^2 - y^2 - z^2 = \begin{pmatrix} ct & x & y & z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

or, calling the 4-vector X,

$$c^2t^2 - x^2 - y^2 - z^2 = X^T g X$$

If this quantity stays invariant under the transformation $X' = \Lambda X$, then for any X we must have

$$\begin{aligned} X'^T g X' &= X^T g X \\ \Longrightarrow \quad (\Lambda X)^T g (\Lambda X) &= X^T g X \\ \Longrightarrow \quad (X^T \Lambda^T) g (\Lambda X) &= X^T g X \\ \Longrightarrow \quad X^T (\Lambda^T g \Lambda) X &= X^T g X \end{aligned}$$

Since this is true for every four-vector X, we must have

$$\boxed{\Lambda^T g \Lambda = g}$$

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2. Question 2

Consider intertial frames Σ and Σ'. Frame Σ' moves at velocity v with respect to Σ, in the common (positive) x direction. Measurements of an event in the two frames, (ct, x, y, z) and (ct', x', y', z'), are related by the Lorentz transformation

$$ct' = \gamma_v(ct - vx/c) \ ; \quad x' = \gamma_v(x - vt) \ ; \quad y' = y \ ; \quad z' = z$$

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

(a) Question 2(a)

A photon has velocity $\vec{u} = (\frac{4}{5}c, \frac{3}{5}c, 0)$ relative to Σ.

Find the velocity $\vec{u'}$ of the photon relative to Σ'.

Explain how your result is consistent with the constancy of the speed of light.

[18 marks]

[Sample Answer:]

Best to first (re-)derive the velocity addition formulae, rather than try to reproduce them from memory:

$$\begin{aligned} u'_x &= \frac{dx'}{dt} = \frac{\gamma_v(dx - vdt)}{\gamma_v(dt - vdx/c^2)} = \frac{u_x - v}{1 - u_x v/c^2} \\ u'_y &= \frac{dy'}{dt} = \frac{dy}{\gamma_v(dt - vdx/c^2)} = \frac{u_y}{\gamma_v(1 - u_x v/c^2)} \end{aligned}$$

Now use these, with $u_x = \frac{4}{5}c$ and $u_y = \frac{3}{5}c$:

$$\begin{aligned} u'_x &= \frac{u_x - v}{1 - \frac{1}{c^2}u_x v} = \frac{\frac{4}{5}c - v}{1 - \frac{1}{c^2}(\frac{4}{5}c)v} = \frac{4c - 5v}{5c - 4v}c \\ u'_y &= \frac{u_y}{\gamma_v(1 - u_x v/c^2)} = \frac{\frac{3}{5}c}{\gamma_v\left(1 - \frac{1}{c^2}(\frac{4}{5}c)v\right)} = \frac{3\sqrt{c^2 - v^2}}{(5c - 4v)}c \\ u_z &= 0 \end{aligned}$$

Consistency: The photon should have the same speed c in any frame, i.e., we should have $|\vec{u}| = |\vec{u'}| = c$. Check that this is true for the $\vec{u'}$

obtained above:

$$|\vec{u'}|^2/c^2 \;=\; \frac{1}{c^2}\left((u'_x)^2 + (u'_y)^2 + (u'_z)^2\right) \;=\; \left(\frac{4c-5v}{5c-4v}\right)^2 + 0^2 + \left(\frac{3\sqrt{c^2-v^2}}{5c-4v}\right)^2$$
$$= \frac{(4c-5v)^2 + 9\,(c^2-v^2)}{(5c-4v)^2} \;=\; \frac{16c^2+25v^2-40cv+9\,(c^2-v^2)}{(5c-4v)^2}$$
$$= \frac{25c^2+16v^2-40cv}{(5c+3v)^2} \;=\; \frac{(5c-4v)^2}{(5c+3v)^2} \;=\; 1$$

Thus the calculated velocity is consistent with the constancy of the speed of light in different frames.

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(b) Question 2(b)

A particle moves with velocity $(w, 0, 0)$ relative to Σ. Write down the four-velocity of the particle as observed from the Σ frame and as observed from the Σ' frame.

[7 marks]

[**Sample Answer:**]

In the Σ frame, the four-velocity is

$$\gamma_w(c, w, 0, 0)$$

In the Σ' frame, the 3-velocity is still in the x direction, but its magnitude is

$$w' = \frac{w - v}{\gamma_v(1 - \frac{wv}{c^2})}$$

Thus the four-velocity is

$$\gamma_{w'}(c, w', 0, 0) = \gamma\left(\frac{w - v}{\gamma_v(1 - \frac{wv}{c^2})}\right) \times \left(c, \frac{w - v}{\gamma_v(1 - \frac{wv}{c^2})}, 0, 0\right)$$

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(c) Question 2(c)

In the Σ frame, two events occur at times $t_1 = \dfrac{L}{c}$ and $t_2 = \dfrac{L}{2c}$, and have spatial coordinates respectively

$$(x_1 = 4L, y_1 = 0, z_1 = 0) \quad \text{and} \quad (x_2 = 2L, y_2 = 0, z_2 = -6L).$$

For a certain value of v, these two events are simultaneous in Σ'. Find this value of v.

[10 marks]

[Sample Answer:]
The LT also holds for the difference of events, thus

$$\Delta t' = \gamma_v(\Delta t - v\Delta x/c^2) \ .$$

(Alternatively, can write the LT for the two events and then subtract.)
If the events are simultaneous in frame Σ', we have $\Delta t' = 0$, so that

$$\gamma_v(\Delta t - v\Delta x/c^2) = 0$$
$$\implies \frac{v}{c^2} = \frac{\Delta t}{\Delta x} = \frac{t_1 - t_2}{x_1 - x_2} = \frac{L/2c}{2L} = \frac{1}{4c} \qquad \implies \qquad v = c/4$$

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3. Question 3

(a) Question 3(a)
A high-energy particle of mass M is travelling with speed v when it decays into two photons. The photons are seen to emerge at equal angles θ on either side of the original velocity. Show that $v = c\cos\theta$. Also find the energy of either photon. Express this energy in terms M and θ.

[17 marks]

[**Sample Answer:**]
BEFORE AND AFTER PICTURES
If the pictures are drawn reasonably, the examinees should be able to write down energy and momentum conservation equations. It's sensible to write them in terms of speeds, since the desired result concerns the speed.
Energy conservation:

$$\gamma_v Mc^2 \; = \; hf_1 + hf_2$$

I've introduced notation f_1 and f_2 for the frequencies of the two photons.
Momentum conservation in the original direction of pion motion:

$$\gamma_v Mv \; = \; \frac{hf_1}{c}\cos\theta + \frac{hf_2}{c}\cos\theta$$

Momentum conservation in the direction perpendicular to the original velocity:

$$0 \; = \; \frac{hf_1}{c}\sin\theta - \frac{hf_2}{c}\sin\theta$$

The last equation gives $f_1 = f_2$. Of course, this is also implied in the wording of the problem. ("The energy of either photon" implies that the energies and hence the frequencies of the two photons are equal.) The first equation then yields $hf_1 = \frac{1}{2}\gamma_v Mc^2$, which, when put into the second equation, gives

$$\gamma_v Mv \; = \; 2(\tfrac{1}{2}\gamma_v Mc)\cos\theta \qquad \Longrightarrow \qquad v = c\cos\theta$$

The energy of either photon is

$$hf \; = \; \frac{1}{2}\gamma_v Mc^2 \; = \; \frac{Mc^2}{2\sqrt{1-(v/c)^2}} \; = \; \frac{Mc^2}{2\sqrt{1-\sin^2\theta}} \; = \; \frac{Mc^2}{2\cos\theta}$$

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(b) Question 3(b)

Show that the four-momentum of a photon is light-like.

Show that the four-momentum of a particle with nonzero mass is time-like.

[8 marks]

[Sample Answer:]

In the metric used in class (trace -2 metric), time-like, space-like, null mean that the norm-squared is positive, negative, zero respectively.

The 4-momentum is of a particle is

$$(E/c, p_x, p_y, p_z)$$

whose norm-square is

$$p_\mu p^\mu \;=\; (E/c)^2 - p_x^2 - p_y^2 - p_z^2 \;=\; (E/c)^2 - p^2$$

For a photon, energy and momentum are related by $E = pc$ (because $E = hf$ and $p = hf/c$); thus

$$p_\mu p^\mu \;=\; p^2 - p^2 \;=\; 0$$

Norm-squared being zero indicates that the 4-vector is light-like for a photon.

For a particle with nonzero mass, $E^2 = p^2c^2 + m^2c^4$; hence the norm-square is

$$p_\mu p^\mu \;=\; (E/c)^2 - p^2 \;=\; (p^2c^2 + m^2c^4)/c^2 - p^2 \;=\; m^2/c^2 > 0$$

Norm-squared being positive indicates that the 4-vector is time-like.

Alternatively, the 4-momentum of photon could be written explicitly in the frame in which the photon motion is in the (say) x direction: $hf/c, hf/c, 0, 0$. The norm-squared is then

$$(hf/c)^2 - (hf/c)^2 - 0^2 - 0^2 = 0$$

Since the norm is an invariant, it is zero in every inertial frame. Hence the 4-momentum of a photon is light-like or null.

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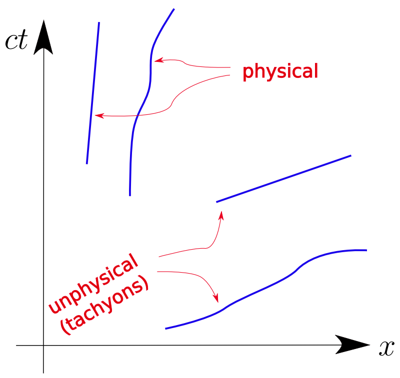
(c) Question 3(c)

On a spacetime diagram (ct versus x) diagram, show examples of physical worldlines and an un-physical (tachyonic) worldlines.

Sketch the worldline of a particle subject to a constant force in positive x direction. The particle starts from rest at the origin at time $t = 0$. State in words the initial and asymptotic (late-time) speeds of the particle. Both should be clear from your sketch.

[10 marks]

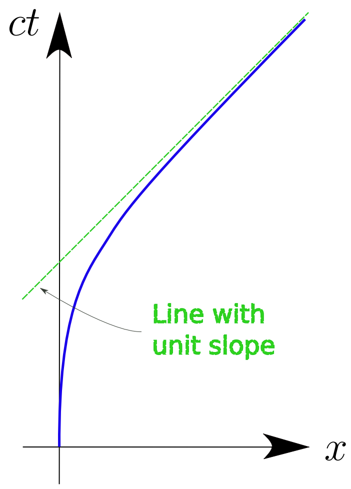
[Sample Answer:]



Physical worldlines have slope > 1, reflecting speeds less than c. Tachyonic worldlines have slope < 1, reflecting speeds larger than c.

Since the initial speed is zero, the initial slope should be infinity — the worldline starts out vertical.

At very long times, since the particle continues to be accelerated, it approaches (but never reaches) the speed of light, hence its slope approaches but never reaches unit slope, i.e., the slope of a diagaonal (45°) line.



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