Maynooth University
National University of Ireland Maynooth

MATHEMATICAL PHYSICS

SEMESTER 2

2017–2018

MP352
Special Relativity

Dr. M. Haque, Prof. D. A. Johnston and Dr. J.-I. Skullerud

Time allowed: 2 hours

Answer **ALL** questions

1. Consider the set of 4×4 matrices Λ with real elements which satisfy the relation

$$\Lambda^T g \Lambda = g \,, \quad \text{where} \quad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

is the metric tensor. These matrices represent Lorentz transformations of spacetime points (ct, x, y, z).

(a) Find the possible values of the determinant of a matrix belonging to this set.

[5 marks]

(b) What additional property must such a matrix satisfy, in order to represent a *proper* Lorentz tranformation?

What does a non-proper Lorentz transformation mean physically?

[7 marks]

(c) Show that the set of matrices which satisfiy condition (1) forms a group under matrix multiplication. Verify all four group properties (closure, associativity, identity, inverse).

[18 marks]

2. Let Σ and Σ' be inertial frames. Frame Σ' moves at velocity v with respect to Σ, in the common (positive) x direction. Measurements of an event in the two frames, (ct, x, y, z) and (ct', x', y', z'), are related by the Lorentz transformation

$$ct' = \gamma_v(ct - vx/c) \ ; \quad x' = \gamma_v(x - vt) \ ; \quad y' = y \ ; \quad z' = z$$

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

(a) A body of mass m is at rest in the frame Σ'.

Write down its four-momentum in the frame Σ'.

Write down its four-momentum in the frame Σ.

Show that the norm of the four-momentum is the same in the two frames.

[10 marks]

(b) A photon has velocity $\vec{u'} = (0, 0, c)$ relative to Σ'.

Find the velocity of the photon relative to Σ.

Explain how your result is consistent with the constancy of the speed of light.

[12 marks]

(c) Represent the (ct, x) axes and the (ct', x') axes on a single spacetime diagram, such that the ct and x axes are perpendicular to each other.

Show two events on this joint diagram which are simultaneous when measured from Σ'. Show which of these events happens earlier according to Σ.

Use the LT to find out how x' units are related to x units on this diagram. (Hint: You could consider the event $(ct', x') = (0, 1)$, find its coordinates in the Σ frame, and hence obtain the distance of this point from the origin in x units.)

[13 marks]

3. (a) Explain using equations or inequalities what it means for a four-vector to be time-like, space-like, and light-like.

Find the four-velocity of a particle with nonzero mass m and velocity $\vec{u} = (c/2, c/2, c/2)$. Find out whether this four-vector is time-like, space-like, or light-like.

[14 marks]

(b) In the lab frame, two identical balls, each having mass m_0, collide with equal but opposite velocities of magnitude v. Their collision is perfectly inelastic, so they stick together and form a single body.

Find the mass of the final body in terms of m_0 and v.

Inertial frame Σ moves with one of the balls before the collision. Find the energy of the final body relative to Σ.

[11 marks]

(c) The inertial frame $\tilde{\Sigma}$ is obtained from inertial frame Σ by a boost by speed v in the positive y direction, followed by a rotation by angle θ around the z direction.

Find the matrix that transforms the coordinates of an event in Σ to its coordinates in $\tilde{\Sigma}$.

[10 marks]

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SAMPLE PARTIAL ANSWERS

MAY BE INCOMPLETE

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1. Question 1.

Consider the set of 4×4 matrices Λ with real elements which satisfy the relation

$$\Lambda^T g \Lambda = g\,, \quad \text{where} \quad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is the metric tensor. These matrices represent Lorentz transformations of spacetime points (ct, x, y, z).

(a) Question 1(a)

Find the possible values of the determinant of a matrix belonging to this set.

[5 marks]

[Sample Answer:]

Taking the determinant of both sides of the given defining equation,

$$\det\left(\Lambda^T g \Lambda\right) \;=\; \det(g) \qquad \Longrightarrow \qquad \det\left(\Lambda^T\right)\det(g)\det(\Lambda) \;=\; \det(g)$$

Direct calculation gives $\det(g) = 1$. In addition, the determinant of the transpose of a matrix is the same as the determinant of the matrix. Thus

$$\det(\Lambda)^2 \;=\; 1 \qquad \Longrightarrow \qquad \det(\Lambda) = \pm 1$$

Since the matrix has real elements, the determinant can only take real values. Hence the possible values are $+1$ and -1.

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(b) Question 1(b)

What additional property must such a matrix satisfy, in order to represent a *proper* Lorentz tranformation?

What does a non-proper Lorentz transformation mean physically?

[7 marks]

[Sample Answer:]

To be a proper Lorentz transformation, the determinant must take the positive value $+1$.

The transformation is not proper when $\det(\Lambda) = -1$.

If the determinant is negative, the transformation is either non-orthochronous (involves reflection of time), or involves a reflection of the exes.

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(c) Question 1(c)

Show that the set of matrices which satisfiy condition (1) forms a group under matrix multiplication. Verify all four group properties (closure, associativity, identity, inverse).

[18 marks]

[Sample Answer:]

For this set to be a group, the properties of Closure, Associativity, Existence of Identity, and Existence of Inverse must be satisfied.

Closure: If Λ_1 and Λ_2 are members of the set, then $\Lambda_1^T g \Lambda_1 = g$ and $\Lambda_2^T g \Lambda_2 = g$. Then

$$\Big(\Lambda_1\Lambda_2\Big)^T g\Big(\Lambda_1\Lambda_2\Big) \;=\; \Big(\Lambda_2^T\Lambda_1^T\Big)g\Big(\Lambda_1\Lambda_2\Big) \;=\; \Lambda_2^T\Big(\Lambda_1^T g\Lambda_1\Big)\Lambda_2 \;=\; \Lambda_2^T g\Lambda_2 \;=\; g$$

which means that $\Lambda_1\Lambda_2$ is also a member of the set.

Associativity: Matrix multiplication is known to be associative.

Existence of Identity: The 4×4 identity matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a member of the set, because

$$I^T g I = I g = g$$

Hence the set contains an identity element.

Note that just pointing to the 4×4 identity matrix I is not enough: one has to explicitly show that I belongs to this particular set.

Existence of Inverse: If Λ is an element of the set, $\Lambda^T g \Lambda = g$ by definition. To show that the matrix inverse Λ^{-1} also belongs to the set, multiply both sides by $(\Lambda^{-1})^T$ on the left and by Λ^{-1} on the right:

$$\big(\Lambda^{-1}\big)^T\Big(\Lambda^T g\Lambda\Big)\Lambda^{-1} = \big(\Lambda^{-1}\big)^T g\Lambda^{-1}$$

The left side is

$$\Big(\big(\Lambda^{-1}\big)^T\Lambda^T\Big)g\Big(\Lambda\Lambda^{-1}\Big) \;=\; \Big(\Lambda\Lambda^{-1}\Big)^T gI \;=\; I^T g \;=\; g$$

so that we have obtained

$$g \;=\; \left(\Lambda^{-1}\right)^{T} g \Lambda^{-1}$$

i.e., the inverse of Λ satisfies the defining equation and hence belongs to the set.

These four properties show that the set defined by $\Lambda^T g \Lambda = g$ is a group under matrix multiplication.

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2. Question 2

Let Σ and Σ' be inertial frames. Frame Σ' moves at velocity v with respect to Σ, in the common (positive) x direction. Measurements of an event in the two frames, (ct, x, y, z) and (ct', x', y', z'), are related by the Lorentz transformation

$$ct' = \gamma_v(ct - vx/c) \ ; \quad x' = \gamma_v(x - vt) \ ; \quad y' = y \ ; \quad z' = z$$

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

(a) Question 2(a)

A body of mass m is at rest in the frame Σ'.

Write down its four-momentum in the frame Σ'.

Write down its four-momentum in the frame Σ.

Show that the norm of the four-momentum is the same in the two frames.

[10 marks]

[Sample Answer:]

In the frame Σ':

The body has momentum 0 and energy $\gamma_0 mc^2 = mc^2$. Hence the four-momentum is

$$\left(\frac{mc^2}{c}, 0, 0, 0\right) \quad = \quad (mc, 0, 0, 0)$$

The norm is

$$(mc)^2 - 0^2 - 0^2 - 0^2 = m^2c^2$$

In the frame Σ:

The body has velocity v in the x-direction, hence three-momentum components

$$(\gamma_v mv, 0, 0)$$

The energy of the body is $\gamma_v mc^2$.

The four-momentum is thus

$$(\frac{\gamma_v mc^2}{c}, \gamma_v mv, 0, 0) \quad = \quad (\gamma_v mc, \gamma_v mv, 0, 0)$$

The norm is

$$(\gamma_v mc)^2 - (\gamma_v mv)^2 - 0^2 - 0^2 \ = \ (\gamma_v mc)^2(1 - v^2/c^2) \ = \ \gamma_v^2 m^2 c^2 \gamma_v^{-2} \ = \ m^2c^2$$

The norm is thus the same in the two frames.

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(b) Question 2(b)
A photon has velocity $\vec{u'} = (0, 0, c)$ relative to Σ'.
Find the velocity of the photon relative to Σ.
Explain how your result is consistent with the constancy of the speed of light.

[12 marks]

[Sample Answer:]
Best to first (re-)derive the velocity addition formulae, rather than try to reproduce them from memory:

$$u_x = \frac{dx}{dt} = \frac{\gamma_v(dx' + vdt')}{\gamma_v(dt' + vdx'/c^2)} = \frac{u'_x + v}{1 + u'_x v/c^2}$$

$$u_z = \frac{dz}{dt} = \frac{dz'}{\gamma_v(dt' + vdx'/c^2)} = \frac{u'_z}{\gamma_v(1 + u'_x v/c^2)}$$

Now use these, with $u'_x = 0$ and $u'_z = c$:

$$u_x = \frac{u'_x + v}{1 + \frac{1}{c^2}u'_x v} = \frac{0 + v}{1 + \frac{1}{c^2}(0)v} = v$$

$$u_y = 0$$

$$u_z = \frac{u'_z}{\gamma_v(1 + u'_x v/c^2)} = \frac{c}{\gamma_v\left(1 + \frac{1}{c^2}(0)v\right)} = \frac{c}{\gamma_v} = \sqrt{c^2 - v^2}$$

Consistency: The photon should have the same speed c in any frame, i.e., we should have $|\vec{u}| = |\vec{u'}| = c$. Clearly, with the given $\vec{u'}$, we have $|\vec{u'}| = c$. Let's now check that this is true for the $\vec{u}$ obtained above:

$$|\vec{u}|^2 = v^2 + \left(\sqrt{c^2 - v^2}\right)^2$$

Hence $|\vec{u}| = c$ as well. Thus the calculated velocity is consistent with the constancy of the speed of light in different frames.

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(c) Question 2(c)

Represent the (ct, x) axes and the (ct', x') axes on a single spacetime diagram, such that the ct and x axes are perpendicular to each other.

Show two events on this joint diagram which are simultaneous when measured from Σ'. Show which of these events happens earlier according to Σ.

Use the LT to find out how x' units are related to x units on this diagram. (Hint: You could consider the event $(ct', x') = (0, 1)$, find its coordinates in the Σ frame, and hence obtain the distance of this point from the origin in x units.)

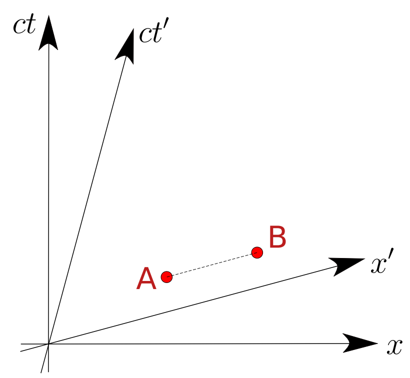
[13 marks]

[Sample Answer:]

Two sets of spacetime axes, corresponding to Σ and Σ' frames. The angle θ between the x' axis and the x axis is the same as the angle between the ct' axis and the ct axis.

The two events A and B are simultaneous in Σ', because on the spacetime diagram the line joining them is parallel to the x' axis.

From Σ, the events are not simultaneous because the line joining them is not parallel to the x axis. From Σ, event A occurs before event B.



To relate the units, following the given hint, the Σ coordinates corresponding to $(ct', x') = (0, 1)$ are noted to be

$$\begin{aligned} ct &= \gamma_v\left(ct' + \frac{v}{c}x'\right) = \gamma_v\left(0 + \frac{v}{c}\right) = \gamma_v\frac{v}{c} \\ x &= \gamma_v\left(x' + \frac{v}{c}ct'\right) = \gamma_v\left(1 + \frac{v}{c}0\right) = \gamma_v \end{aligned}$$

The distance of this point from the common origin is, in x units

$$\sqrt{\left(\gamma_v\frac{v}{c}\right)^2 + (\gamma_v)^2} = \sqrt{\gamma_v^2\left(1 + \frac{v}{c}\right)^2} = \sqrt{\frac{c^2 + v^2}{c^2 - v^2}}$$

Thus unit distance in x' units equates distance $\sqrt{(c^2 + v^2)/(c^2 - v^2)}$ in x units.

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3. Question 3

(a) Question 3(a)

Explain using equations or inequalities what it means for a four-vector to be time-like, space-like, and light-like.

Find the four-velocity of a particle with nonzero mass m and velocity $\vec{u} = (c/2, c/2, c/2)$. Find out whether this four-vector is time-like, space-like, or light-like.

[14 marks]

[**Sample Answer:**]

In the metric used in class (trace -2 metric):

Timelike means that the norm-squared is positive. This can be expressed as

$$\text{Timelike:} \qquad A_\mu A^\mu > 0 \quad \text{or} \quad g_{\mu\nu} A^\mu A^\nu > 0$$

If the elements of the 4-vector are termed A^0, A^1, A^2, A^3, then this is

$$\left(A^0\right)^2 - \left(A^1\right)^2 - \left(A^2\right)^2 - \left(A^3\right)^2 > 0$$

Similarly, spacelike means that the norm-squared is negative, and lightlike means that the norm-squared is zero:

$$\text{Spacelike:} \qquad A_\mu A^\mu < 0 \quad \text{or} \quad g_{\mu\nu} A^\mu A^\nu < 0$$
$$\text{Lightlike:} \qquad A_\mu A^\mu = 0 \quad \text{or} \quad g_{\mu\nu} A^\mu A^\nu = 0$$

The speed of the particle is

$$\sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{c}{2}\right)^2 + \left(\frac{c}{2}\right)^2} \;=\; \frac{\sqrt{3}}{2}\,c$$

The γ factor corresponding to this speed is

$$\gamma\left(\frac{\sqrt{3}}{2}\,c\right) \;=\; \frac{1}{\sqrt{1-(3/4)}} \;=\; 2$$

Hence the 4-velocity is

$$\gamma_u(c, u_x, u_y, y_z) \;=\; 2(c, c/2, c/2, c/2) \;=\; (2c, c, c, c)$$

The norm-squared is

$$(2c)^2 - c^2 - c^2 - c^2 \;=\; c^2 > 0$$

Hence the four-velocity is a timelike four-vector.

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(b) Question 3(b)

In the lab frame, two identical balls, each having mass m_0, collide with equal but opposite velocities of magnitude v. Their collision is perfectly inelastic, so they stick together and form a single body.

Find the mass of the final body in terms of m_0 and v.

Inertial frame Σ moves with one of the balls before the collision. Find the energy of the final body relative to Σ.

[11 marks]

[Sample Answer:]

Let us denote the mass of the final body (to be determined) as M. In the lab frame, the final body has zero velocity from symmetry.

(Without invoking symmetry, one could denote the final velocity by V_f, and write down the momentum conservation equation:

$$\gamma(v)m_0v - \gamma(v)m_0v \;=\; \gamma(V_f)MV_f$$

which gives $V_f = 0$. Note in the equation above, the γ on the right side is different from the γ's on the left side. When writing a γ, it is essential to be clear which speed it corresponds to.)

The energy conservation equation gives

$$\gamma(v)m_0c^2+\gamma(v)m_0c^2 \;=\; \gamma(0)Mc^2 \quad\Longrightarrow\quad M \;=\; 2\gamma(v)m_0 \;=\; \frac{2m_0}{\sqrt{1-v^2/c^2}}$$

According to frame Σ, the lab has speed v and hence so does the final body. Therefore the energy relative to frame Σ is

$$\gamma(v)Mc^2 \;=\; \gamma(v)2\gamma(v)m_0c^2 \;=\; \frac{2m_0c^2}{1-v^2/c^2}$$

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(c) Question 3(c)

The inertial frame $\tilde{\Sigma}$ is obtained from inertial frame Σ by a boost by speed v in the positive y direction, followed by a rotation by angle θ around the z direction.

Find the matrix that transforms the coordinates of an event in Σ to its coordinates in $\tilde{\Sigma}$.

[10 marks]

[**Sample Answer:**]

A boost in the y direction is

$$B = \begin{pmatrix} \gamma_v & 0 & -\gamma_v v & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_v v & 0 & \gamma_v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Here I used $c = 1$ units, i.e., v is to thought of as v/c.

A rotation around the z direction is

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The direction of rotation is not specified, so it would be equally correct if $\sin\theta$ and $-\sin\theta$ were interchanged.

The coordinates in $\tilde{\Sigma}$ are obtained by applying first the boost, and then the rotation, i.e.,

$$\tilde{X} = RBX$$

where X and $\tilde{X}$ are the event coordinates in the two frames. Thus the required transformation matrix is RB, not BR.

$$RB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_v & 0 & -\gamma_v v & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_v v & 0 & \gamma_v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_v & 0 & -\gamma_v v & 0 \\ -\gamma_v v \sin\theta & \cos\theta & \gamma_v \sin\theta & 0 \\ -\gamma_v v \cos\theta & -\sin\theta & \gamma_v \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If rotation in the other direction were to be used, the θ's would have their signs flipped, so that

$$RB = \begin{pmatrix} \gamma_v & 0 & -\gamma_v v & 0 \\ +\gamma_v v\sin\theta & \cos\theta & -\gamma_v\sin\theta & 0 \\ -\gamma_v v\cos\theta & +\sin\theta & \gamma_v\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Either of these are correct answers to the question, as the direction of rotation is not specified.

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