1. Consider a spin-1/2 system. The components of the spin are described by the operators

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Consider the state $|\phi\rangle = \begin{pmatrix} \alpha \\ -\sqrt{7}\alpha \end{pmatrix}$.

The state is normalized. The real and imaginary parts of α are equal. Find α.

[6 marks]

(b) Calculate the commutator $\left[\hat{S}_x, \hat{S}_y\right]$ and express your answer in terms of $\hat{S}_z$.

[6 marks]

(c) Calculate the uncertainty of S_y in the state $|W\rangle = \begin{pmatrix} 1/\sqrt{2} \\ (1-i)/2 \end{pmatrix}$.

[10 marks]

(d) A measurement of the x-component of the spin (S_x) yields a negative value. What is the state of the system immediately after the measurement?

[13 marks]

2. Consider a particle of mass m on an infinite one-dimensional line. It is subject to the potential

$$V(x) = \begin{cases} V_0 & x < 0 \quad \text{(region 1)} \\ 3V_0 & 0 < x < L \quad \text{(region 2)} \\ 0 & x > L \quad \text{(region 3)} \end{cases}$$

where V_0 is a positive constant. For $V_0 < E_3 V_0$, the time-independent Schroedinger equation has a solution with energy E, of the form

$$\begin{aligned} \psi(x) &= \psi_1(x) = e^{ik_1 x} + Ae^{-ik_1 x} && \text{(region 1)} \\ \psi(x) &= \psi_2(x) = Be^{\alpha x} + Ce^{-\alpha x} && \text{(region 2)} \\ \psi(x) &= \psi_3(x) = De^{ik_3 x} && \text{(region 3)} \end{aligned}$$

where k_1, k_3 and α are real and positive.

(a) Express the constants k_1, k_3 and α in terms of E and V_0.

[8 marks]

(b) The solution written above can be interpreted as a scattering situation. Identify the terms representing incident, reflected and tranmitted waves.

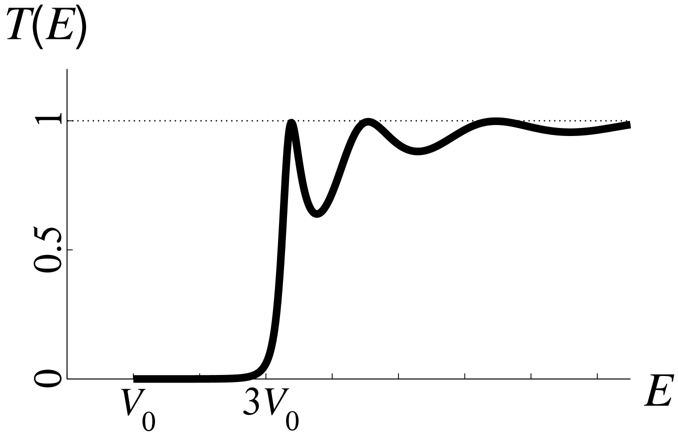
[3 marks]

(c) Define the reflection and transmission probabilities in terms of the constants appearing in the wavefunction.

[5 marks]

(d) What are the boundary conditions that the solution must satisfy at $x = 0$ and at $x = L$? Use these boundary conditions to write four equations for the constants A, B, C, D. **[8 marks]**

(e) A sketch of the transmission probability T is shown as a function of incident energy E.



Sketch a plot of the reflection probability R.

The curve starts at V_0 and not at zero. Explain why $E < V_0$ is not meaningful.

The curve includes the region $E > 3V_0$. How does the form of the solution change when $E > 3V_0$?

[11 marks]

3. Consider a particle of mass m in a one-dimensional harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$. We denote the orthonormalized energy eigenstates of the harmonic oscillator by $|n\rangle$, on which the lowering and raising operators act as follows:

$$\hat{a}_- |n\rangle = \sqrt{n}\,|n-1\rangle \qquad \hat{a}_+ |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

The ladder operators are defined as

$$\hat{a}_- = \frac{1}{\sqrt{2}}\left(\frac{1}{\sigma}\hat{x} + \frac{i\sigma}{\hbar}\hat{p}\right), \qquad \hat{a}_+ = \frac{1}{\sqrt{2}}\left(\frac{1}{\sigma}\hat{x} - \frac{i\sigma}{\hbar}\hat{p}\right),$$

where $\sigma = \sqrt{\hbar/(m\omega)}$.

(a) Calculate the commutator $[\hat{a}_-, \hat{a}_+]$.
You may need to use the relation $[\hat{x}, \hat{p}] = i\hbar$.

[7 marks]

(b) Find the uncertainty of position in the state $|n\rangle$.

[13 marks]

(c) If the system wavefunction at time $t = 0$ is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\,|0\rangle - \frac{i}{2}\,|2\rangle + \frac{1}{2}\,|3\rangle$$

then what is the wavefunction at a later time $t = T$? Your answer should contain the oscillator frequency ω.
If the energy is measured at time $t = T$, what are the possible results of the measurement? What are the probabilities for the possible results to occur?

[10 marks]

_____*_____

SAMPLE ANSWERS

_____*_____

1. Question 1.

   Consider a spin-1/2 system. The components of the spin are described by the operators

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   (a) Question 1(a)

   Consider the state $|\phi\rangle = \begin{pmatrix} \alpha \\ -\sqrt{7}\alpha \end{pmatrix}$.

   The state is normalized. The real and imaginary parts of α are equal. Find α.

   [6 marks]

   [Sample Answer:]

$$|\alpha|^2 + \left|\sqrt{7}\alpha\right|^2 = 1 \qquad \Longrightarrow |\alpha|^2 = \frac{1}{8}$$

   Since the real and imaginary parts of α are equal, we have

$$\text{either} \quad \alpha = \frac{1}{4} + \frac{i}{4} \quad \text{or} \quad \alpha = -\frac{1}{4} - \frac{i}{4}$$

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   (b) Question 1(b)

   Calculate the commutator $\left[\hat{S}_x, \hat{S}_y\right]$ and express your answer in terms of $\hat{S}_z$.

   [6 marks]

[Sample Answer:]

$$\begin{aligned}\left[\hat{S}_x, \hat{S}_y\right] &= \hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x \\ &= \frac{\hbar^2}{4}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \\ = \frac{\hbar^2}{4}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix} &= i\hbar\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = i\hbar\hat{S}_z\end{aligned}$$

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(c) Question 1(c)

Calculate the uncertainty of S_y in the state $|W\rangle = \begin{pmatrix}1/\sqrt{2}\\ (1-i)/2\end{pmatrix}$.

[10 marks]

[Sample Answer:]
We need to calculate

$$\Delta S_y = \sqrt{\langle S_y^2\rangle - \langle S_y\rangle^2}$$

In the state $|W\rangle$. The operators are

$$S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \text{and} \quad S_y^2 = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Therefore the expectation values are

$$\begin{aligned}\langle S_y\rangle = \langle W|\,\hat{S}_y\,|W\rangle &= \frac{\hbar}{2}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1+i}{2}\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1-i}{2}\end{pmatrix} \\ &= \frac{\hbar}{2}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1+i}{2}\end{pmatrix}\begin{pmatrix}\frac{-1-i}{2}\\ \frac{i}{\sqrt{2}}\end{pmatrix} \\ &= \frac{\hbar}{2}\left\{\frac{-1-i}{2\sqrt{2}} + \frac{-1+i}{2\sqrt{2}}\right\} = -\frac{\hbar}{2\sqrt{2}}\end{aligned}$$

and

$$\langle S_y^2 \rangle \;=\; \langle W| \, \hat{S}_y^2 \, |W\rangle \;=\; \langle W| \, \frac{\hbar^2}{4} \mathbb{I} \, |W\rangle \;=\; \frac{\hbar^2}{4}$$

since the state is normalized.
The uncertainty is therefore

$$\Delta S_y \;=\; \sqrt{\langle S_y^2 \rangle - \langle S_y \rangle^2} \;=\; \sqrt{\frac{\hbar^2}{4} - \left(-\frac{\hbar}{2\sqrt{2}}\right)^2} \;=\; \frac{\hbar}{2\sqrt{2}}$$

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(d) Question 1(d)

A measurement of the x-component of the spin (S_x) yields a negative value. What is the state of the system immediately after the measurement?

[13 marks]

[Sample Answer:]

The possible values that can be obtained in a measurement are the eigenvalues of the operator, and the state after the measurement is the eigenstate corresponding to the eigenvalue that is found. Thus, we need to first find the eigenvalues of the operator

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and then find the eigenstate corresponding to the negative eigenvalue.

Finding the eigenvalue: this can be done in couple of different ways. If done correctly, the eigenvalues will be $\pm\hbar/2$. These are the possible values that can be obtained in a measurement of S_x.

Here is a sample eigenvalue calculation:

An eigenvalue λ of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ must satisfy

$$\left| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \lambda I \right| = 0$$

$$\implies \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\implies \lambda^2 - 1 = 0 \qquad \implies \lambda = \pm 1$$

$$\implies \text{The eigenvalues of } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ are } \pm 1$$

$$\implies \text{The eigenvalues of } \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ are } \pm\hbar/2.$$

The result of the measurement is negative, i.e., equal to $-\hbar/2$, After the measurement, the system wavefunction 'collapses' to the eigenstate corresponding to the eigenvalue $-\hbar/2$. So, we need to calculate the eigenstate corresponding to the eigenvalue $-\hbar/2$.

Calling this eigenstate $\begin{pmatrix} \beta \\ \gamma \end{pmatrix}$, we obtain

$$\hat{S}_x \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$$

$$\implies \quad \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$$

$$\implies \quad \begin{pmatrix} \gamma \\ \beta \end{pmatrix} = -\begin{pmatrix} \beta \\ \gamma \end{pmatrix}$$

which gives $\gamma = -\beta$, so that the eigenstate is

$$\begin{pmatrix} \beta \\ -\beta \end{pmatrix}$$

The value of β can be calculated by normalization:

$$\beta = \frac{1}{\sqrt{2}} e^{i\chi} \, ; \qquad \chi \text{ is an arbitrary real number}$$

Thus the state after the measurement is

$$\frac{1}{\sqrt{2}} e^{i\chi} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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2. Question 2.

Consider a particle of mass m on an infinite one-dimensional line. It is subject to the potential

$$V(x) = \begin{cases} V_0 & x < 0 & \text{(region 1)} \\ 3V_0 & 0 < x < L & \text{(region 2)} \\ 0 & x > L & \text{(region 3)} \end{cases}$$

where V_0 is a positive constant. For $V_0 < E 3V_0$, the time-independent Schroedinger equation has a solution with energy E, of the form

$$\begin{aligned} \psi(x) &= \psi_1(x) = e^{ik_1x} + Ae^{-ik_1x} && \text{(region 1)} \\ \psi(x) &= \psi_2(x) = Be^{\alpha x} + Ce^{-\alpha x} && \text{(region 2)} \\ \psi(x) &= \psi_3(x) = De^{ik_3x} && \text{(region 3)} \end{aligned}$$

where k_1, k_3 and α are real and positive.

(a) Question 2(a)

Express the constants k_1, k_3 and α in terms of E and V_0.

[8 marks]

[Sample Answer:]

$$k_1 = \sqrt{\frac{2m(E-V_0)}{\hbar_2}}; \quad k_3 = \sqrt{\frac{2mE}{\hbar_2}}; \quad \alpha = \sqrt{\frac{2m(3V_0-E)}{\hbar_2}}$$

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(b) Question 2(b)
The solution written above can be interpreted as a scattering situation. Identify the terms representing incident, reflected and tranmitted waves.

[3 marks]

[Sample Answer:]
e^{ik_1x} $\longrightarrow$ incident wave
Ae^{-ik_1x} $\longrightarrow$ reflected wave
De^{ik_3x} $\longrightarrow$ transmitted wave

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(c) Question 2(c)
Define the reflection and transmission probabilities in terms of the constants appearing in the wavefunction.

[5 marks]

[Sample Answer:]
The reflection probability is the reflected current divided by the incident current.
The transmission probability is the transmitted current divided by the incident current.

$$\begin{aligned} \text{incident current} &= 1 \times \frac{\hbar k_1}{m} \\ \text{reflected current} &= |A|^2 \times \frac{\hbar k_1}{m} \\ \text{transmitted current} &= |D|^2 \times \frac{\hbar k_3}{m} \end{aligned}$$

Thus

$$R = |A|^2 \qquad T = |D|^2 \frac{k_3}{k_1}$$

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(d) Question 2(d)

What are the boundary conditions that the solution must satisfy at $x = 0$ and at $x = L$? Use these boundary conditions to write four equations for the constants A, B, C, D.

[8 marks]

[Sample Answer:]

Boundary conditions: the wavefunction and its derivative must be continuous at $x = 0$ and at $x = L$. Thus four boundary conditions:

$$\psi_1(0) = \psi_2(0)\,; \quad \psi_1'(0) = \psi_2'(0)\,; \quad \psi_2(L) = \psi_3(L)\,; \quad \psi_2'(L) = \psi_3'(L)$$

Each gives a relation between A, B, C, D. The four equations are

$$\begin{aligned} 1 + A &= B + C \\ ik_1 - ik_1 A &= \alpha B - \alpha C \\ Be^{\alpha L} + Ce^{-\alpha L} &= De^{ik^3 L} \\ \alpha Be^{\alpha L} - \alpha Ce^{\alpha L} &= ik_3 De^{ik^3 L} \end{aligned}$$

This is a system of linear equations for the four constants. In matrix form

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -ik_1 & -1 & 1 & 0 \\ 0 & e^{\alpha L} & e^{-\alpha L} & -e^{ik^3 L} \\ 0 & \alpha e^{\alpha L} & -\alpha e^{-\alpha L} & -ik_3 e^{ik^3 L} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} -1 \\ -ik_1 \\ 0 \\ 0 \end{pmatrix}$$

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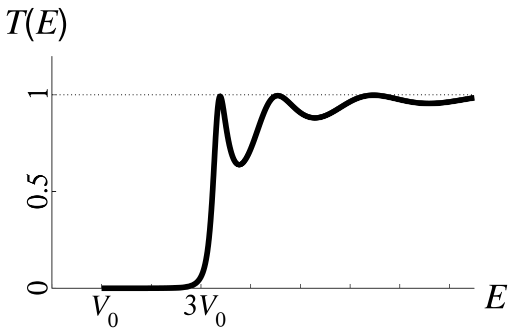
(e) Question 2(e)

A sketch of the transmission probability T is shown as a function of incident energy E.

Sketch a plot of the reflection probability R.

The curve starts at V_0 and not at zero. Explain why $E < V_0$ is not meaningful.

The curve includes the region $E > 3V_0$. How does the form of the solution change when $E > 3V_0$?



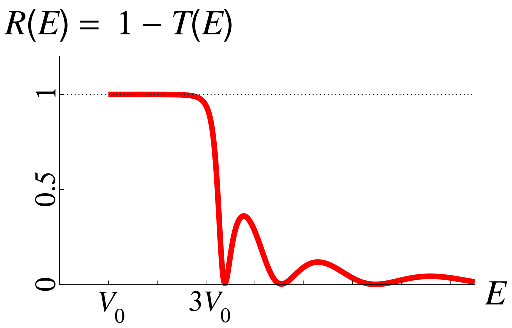
[11 marks]

[Sample Answer:]

The reflection probability is

$$R = 1 - T$$

because the sum of reflection and transmission probabilities must be one.



The curve is only defined for $E > V_0$, because in order to have a scattering situation, we need plane wave solutions in region 1, i.e., the kinetic energy needs to be positive.

For $E > 3V_0$, the solutions in region 2 are not exponentials; but also plane waves. Thus ψ_2 has to be corrected:

$$\psi_2(x) \ = \ Be^{ik^2x} + Ce^{-ik^2x}$$

Another way of saying this is that α becomes imaginary instead of real.

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3. Question 3.

Consider a particle of mass m in a one-dimensional harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$. We denote the orthonormalized energy eigenstates of the harmonic oscillator by $|n\rangle$, on which the lowering and raising operators act as follows:

$$\hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

The ladder operators are defined as

$$\hat{a}_- = \frac{1}{\sqrt{2}}\left(\frac{1}{\sigma}\hat{x} + \frac{i\sigma}{\hbar}\hat{p}\right), \qquad \hat{a}_+ = \frac{1}{\sqrt{2}}\left(\frac{1}{\sigma}\hat{x} - \frac{i\sigma}{\hbar}\hat{p}\right),$$

where $\sigma = \sqrt{\hbar/(m\omega)}$.

(a) Question 3(a)
Calculate the commutator $[\hat{a}_-, \hat{a}_+]$.
You may need to use the relation $[\hat{x}, \hat{p}] = i\hbar$.

[7 marks]

[Sample Answer:]

$$\begin{aligned}
[\hat{a}_-, \hat{a}_+] &= \frac{1}{2}\left[\left(\frac{1}{\sigma}\hat{x} + \frac{i\sigma}{\hbar}\hat{p}\right), \left(\frac{1}{\sigma}\hat{x} - \frac{i\sigma}{\hbar}\hat{p}\right)\right] \\
&= \frac{1}{2}\left(\frac{1}{\sigma^2}[\hat{x},\hat{x}] + \frac{-i}{\hbar}[\hat{x},\hat{p}] + \frac{i}{\hbar}[\hat{p},\hat{x}] + \frac{\sigma^2}{\hbar^2}[\hat{p},\hat{p}]\right) \\
&= \frac{1}{2}\left(0 + \frac{-i}{\hbar}(i\hbar) + \frac{i}{\hbar}(-i\hbar) + 0\right) = \frac{1}{2}\left(\frac{\hbar}{\hbar} + \frac{\hbar}{\hbar}\right) = 1
\end{aligned}$$

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(b) Question 3(b)

(c) Find the uncertainty of position in the state $|n\rangle$.

[13 marks]

[Sample Answer:]
Inverting the given definitions,

$$\hat{x} = \frac{\sigma}{\sqrt{2}}\left(\hat{a}_- + \hat{a}_+\right) \qquad \hat{p} = \frac{-i\hbar}{\sqrt{2}\sigma}\left(\hat{a}_- - \hat{a}_+\right)$$

Thus

$$\hat{x}^2 \;=\; \frac{\sigma^2}{2}\left(\hat{a}^\dagger + \hat{a}\right)\left(\hat{a}^\dagger + \hat{a}\right) \;=\; \frac{\sigma^2}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\right) \tag{1}$$

NOTE!! Since $\hat{a}$ and $\hat{a}^\dagger$ do not commute, $\hat{a}^\dagger\hat{a} \neq \hat{a}\hat{a}^\dagger$. The ordering of operators matter; thus $\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger \neq 2\hat{a}^\dagger\hat{a}$ This might be a common mistake.
To evaluate the uncertainty $\Delta x = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2}$ in the state $|n\rangle$, we need to calculate $\langle\hat{x}\rangle = \langle n|\,\hat{x}\,|n\rangle$ and $\langle\hat{x}^2\rangle = \langle n|\,\hat{x}^2\,|n\rangle$. We first calculate the expectation values of the required ladder operator combinations:

$$\langle n|\,\hat{a}\,|n\rangle = \sqrt{n}\langle n|n-1\rangle = 0 \qquad \langle n|\,\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\langle n|n+1\rangle = 0$$

Note we are using the orthonormality of the eigenstates, $\langle m|n+1\rangle = \delta_{mn}$. Extending this calculation, one sees that $\hat{a}^\dagger\hat{a}^\dagger$ and $\hat{a}\hat{a}$ have zero expectation value, but $\hat{a}^\dagger\hat{a}$ and $\hat{a}\hat{a}^\dagger$ give nonzero contributions.

$$\begin{gathered}
\langle n|\,\hat{a}\hat{a}\,|n\rangle = \sqrt{n}\langle n|\,\hat{a}\,|n-1\rangle = \sqrt{n}\sqrt{n-1}\langle n|n-2\rangle = 0 \\
\text{Similarly,} \qquad \langle n|\,\hat{a}^\dagger\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\sqrt{n+2}\langle n|n+2\rangle = 0 \\
\langle n|\,\hat{a}^\dagger\hat{a}\,|n\rangle = \sqrt{n}\langle n|\,\hat{a}^\dagger\,|n-1\rangle = \sqrt{n}\sqrt{n}\langle n|n\rangle = n \times 1 = n \\
\langle n|\,\hat{a}\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\langle n|\,\hat{a}\,|n+1\rangle = \sqrt{n+1}\sqrt{n+1}\langle n|n\rangle = n+1
\end{gathered}$$

Employing these, we obtain

$$\begin{aligned}
\langle\hat{x}\rangle \;&=\; \frac{\sigma}{\sqrt{2}}\left(\langle\hat{a}^\dagger\rangle + \langle\hat{a}\rangle\right) \;=\; 0 \\
\langle\hat{x}^2\rangle \;&=\; \frac{\sigma^2}{2}\left(\langle\hat{a}^\dagger\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}\rangle^\dagger + \langle\hat{a}\hat{a}\rangle\right) \;=\; \frac{\sigma^2}{2}\left[0 + n + (n+1) + 0\right] \\
\implies \qquad \langle\hat{x}^2\rangle \;&=\; \sigma^2\left(n + \frac{1}{2}\right)
\end{aligned}$$

Thus the unceratinty is

$$\Delta x = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2} = \sigma\sqrt{n + \frac{1}{2}}$$

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(d) Question 3(c)

If the system wavefunction at time $t = 0$ is

$$|\psi(0)\rangle \;=\; \frac{1}{\sqrt{2}}\,|0\rangle - \frac{i}{2}\,|2\rangle + \frac{1}{2}\,|3\rangle$$

then what is the wavefunction at a later time $t = T$? Your answer should contain the oscillator frequency ω.

If the energy is measured at time $t = T$, what are the possible results of the measurement? What are the probabilities for the possible results to occur?

[10 marks]

[Sample Answer:]

The three eigenstates appearing in the wavefunction correspond to eigenenergies

$$\epsilon_0 = \frac{1}{2}\hbar\omega\,; \quad \epsilon_2 = \frac{5}{2}\hbar\omega\,; \quad \epsilon_3 = \frac{7}{2}\hbar\omega\,.$$

Hence the wavefunction at time $t = T$ is

$$\begin{aligned} |\psi(0)\rangle \;&=\; \frac{1}{\sqrt{2}}\,|0\rangle e^{-i\epsilon_0 T/\hbar} - \frac{i}{2}\,|2\rangle e^{-i\epsilon_2 T/\hbar} + \frac{1}{2}\,|3\rangle e^{-i\epsilon_3 T/\hbar} \\ &=\; \frac{1}{\sqrt{2}}\,|0\rangle e^{-i\omega T/2} - \frac{i}{2}\,|2\rangle e^{-i3\omega T/2} + \frac{1}{2}\,|3\rangle e^{-i7\omega T/2} \end{aligned}$$

The possible results of an energy measurement are the eigenenergies corresponding to the three eigenstates appearing in the wavefunction:

$$\epsilon_0 = \frac{1}{2}\hbar\omega\,; \quad \epsilon_2 = \frac{5}{2}\hbar\omega\,; \quad \epsilon_3 = \frac{7}{2}\hbar\omega\,.$$

These appear with the probabilities

$$\left|\frac{1}{\sqrt{2}}e^{-i\omega T/2}\right|^2 \;=\; \frac{1}{2}\,, \quad \left|-\frac{i}{2}e^{-i3\omega T/2}\right|^2 \;=\; \frac{1}{4}\,, \quad \left|\frac{1}{2}e^{-i7\omega T/2}\right|^2 \;=\; \frac{1}{4}\,,$$

respectively. Note this is independent of T, as the phase factor (containing the time dependence) has modulus one and hence contributes nothing to the probability. It doesn't actually matter at which time the measurement is performed,

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