Due on Friday, October 16th

The questions indicated as [**SELF**] are for self-study and additional exercise.

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1. (a) [**SELF**] Derive, using 'integration by parts', the result

$$\int_0^\infty du\; u\; e^{-bu} = \frac{1}{b^2}\,.$$

You might need this integral later on in this assignment.

(b) [**SELF**] Teach yourself how to calculate definite and indefinite integrals using a computer algebra program (e.g., MATHEMATICA, MAPLE or MAXIMA). Or: how to use an online integrator (search on google for 'online integrator'). Check the result above.

2. Normalization. Assume the constants α_i to be real and positive.

(a) [**8 pts.**] Consider a single particle confined to the half-line $x \geq 0$. The inner product is given by $\langle\phi|\xi\rangle = \int_0^\infty \phi^*(x)\xi(x)dx$. Find the constant α_1 so that the wavefunction

$$\psi(x) \;=\; \alpha_1\sqrt{x}e^{-x}$$

is normalized.

(b) [**3 pts.**] Consider a spin-1/2 (two-state) system. Find the constant α_2 so that the following state is normalized:

$$|\psi\rangle \;=\; \alpha_2\begin{pmatrix} 2i \\ -1+i \end{pmatrix} \quad .$$

(c) [**7 pts.**] Consider a system with an infinite-dimensional Hilbert space, for which any state can be expressed as a linear combination of the set of states $\{|\phi_n\rangle\}$, with $n = 0, 1, 2, \ldots$. This set of 'basis states' satisfy $\langle\phi_m|\phi_n\rangle = \delta_{mn}$. Find the constant α_3 so that the following state is normalized:

$$|\psi\rangle \;=\; \alpha_3\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}\,|\phi_n\rangle\,.$$

(d) [**4 pts.**] For a system with a 4-dimensional Hilbert space, find the constant α_4 so that the following wavefunction is normalized:

$$|\psi\rangle \;=\; \alpha_4\begin{pmatrix} -2 \\ -1+3i \\ i \\ -i \end{pmatrix}\,.$$

(e) **[8 pts.]** For a single particle confined in the x-y plane, find the constant α_5 so that the following wavefunction is normalized:

$$\psi(x,y) \;=\; \alpha_5 e^{-(x^2+y^2)/2} \, .$$

Note: for this system the inner product is given by

$$\langle\phi|\xi\rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \quad \phi^*(x,y)\,\xi(x,y).$$

3. Discrete energy values and blackbody radiation:

(a) **[10 pts.]** In thermal equilibrium, the probability of a mode having energy E is proportional to $e^{-E/k_BT} = e^{-\beta E}$. Here we have defined $\beta = k_BT$.

Show that the average energy of the mode is k_BT or $1/\beta$ if the energy E is a continuous variable, and is $hf/(e^{\beta hf}-1)$ if the energy is discrete and quantized in units of hf.

You might have to use the summation formulae

$$\sum_{n=0}^{\infty} x^n \;=\; \frac{1}{1-x} \qquad \text{and} \qquad \sum_{n=0}^{\infty} n x^n \;=\; \frac{x}{(1-x)^2} \qquad \text{for } |x|<1.$$

(b) **[5+5 pts.]** Planck obtained the correct blackbody radiation spectrum by replacing the classical expression by the quantum expression:

$$\frac{1}{\beta} \;\longrightarrow\; \frac{hf}{e^{\beta hf}-1} \, .$$

Show that the quantum expression reduces to the classical expression at small frequencies. Show this in two ways:

– by taking the limit $f \to 0$ of the quantum expression.
  (You might have to use L'Hopital's rule.)
– by expanding the exponential in the denominator to lowest order.

(c) **[SELF]** Express the quantum expression above as a power series in f. The first term should be the classical expression. You could try working out the Taylor series by taking successive derivatives of the full expresison (quite tedious: L'Hopital's rule needed multiple times), or you could try expanding only $e^{\beta hf}$ and then using $(1+x)^{-1} \approx 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + ...$