Due on Friday, October 23rd.

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1. Matrices (or operators) might not commute with each other. This means, if $\hat{A}$ and $\hat{B}$ are matrices (or operators), $\hat{A}\hat{B}$ might be unequal to $\hat{B}\hat{A}$. We define the **commutator** of two matrices (or two operators) $\hat{A}$ and $\hat{B}$ as

$$[\hat{A}, \hat{B}] \;=\; \hat{A}\hat{B} - \hat{B}\hat{A}$$

Sometimes operators acting on functions are written with a hat, as I've done here. Operators acting on finite vectors, i.e., matrices, are usually written without a hat. Let's use the hat notation for this problem.

(a) **[2 pts.]** Show that $[\hat{B}, \hat{A}] = -[\hat{A}, \hat{B}]$.

(b) **[SELF]** Show that $[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$.

(c) **[4 pts.]** Show that $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$.

2. (a) **[0 pts.]** Look up the three Pauli matrices, denoted as σ_x, σ_y, σ_z, and write them down, (Each one is a 2×2 matrix.)

(b) **[4 pts.]** Calculate $[\sigma_x, \sigma_y]$ and express it in terms of σ_z.

(c) **[2 pts.]** Look up (or calculate or guess correctly) $[\sigma_y, \sigma_z]$ in terms of σ_x, and $[\sigma_z, \sigma_x]$ in terms of σ_y. No need to show calculations; just report the results.

(d) **[7 pts.]** Find the eigenvalues and (normalized) eigenvectors of σ_x.

(e) **[SELF]** Find the eigenvalues and (normalized) eigenvectors of σ_z.

(f) **[SELF]** Is it easier to find eigenvalues & eigenvectors if the matrix is diagonal? Why?

3. (a) In Charles Nash's notes (linked on module webpage): Read Section 5 of Chapter III, on 'Expectation Values'. Begins on page 42.

(b) **[0 pts.]** Define the expectation value of an observable represented by operator $\hat{A}$, when the system state is $|\psi\rangle$.

(c) **[6 pts.]** The state $|\phi\rangle$ is an eigenstate of $\hat{A}$, with the corresponding eigenvalue being λ. What is the expectation value of $\hat{A}$ in this state? (Prove/derive your answer.)

4. For a spin-1/2 system, the operators for the components of spin are

$$S_x = \frac{\hbar}{2}\sigma_x \qquad S_y = \frac{\hbar}{2}\sigma_y \qquad S_z = \frac{\hbar}{2}\sigma_z$$

   (a) [**6 pts.**] Find the eigenvalues and eigenvectors of S_z. How are they related to the eigenvalues and eigenvectors of σ_z?

   (b) [**6 pts.**] If our two-level system is in the state $|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then find the expectation values of S_z and S_x.

   (c) [**SELF**] Which one of the expectation values could you have guessed?

   (d) [**SELF**] Find the commutation relations between the operators S_x, S_y and S_z. (You may use the results for commutators of Pauli matrices.)

5. Photoelectric effect: When electromagnetic radiation of frequency 2×10^{15} Hz shines on a sample of gold, the emitted electrons have a kinetic energy of 3.0 eV. When this frequency is increased to 10^{16} Hz, the electron kinetic energy is 36.1 eV.

   (a) [**0 pts.**] How much is 1 eV of energy, in SI units?

   (b) [**6 pts.**] Pretending that we do not know the value of Planck's constant (h), use the above data to estimate h in SI units (J-s), Use Einstein's energy conservation equation for the photoelectric effect.

   (c) [**SELF**] Use the above data to find the work function of gold, in eV and also in Joules. (You don't necessarily need the value of h for this, but if you want you can use $h = 6.626 \times 10^{-34}$ J-s.) What does the work function physically represent?

6. [**7 pts.**] Look up the properties of the Dirac delta function. If a is a positive real number, write down the values of the following integrals.

$$\text{(a)} \int_{-a}^{a} dx\ \delta(x) \qquad \text{(b)} \int_{a}^{2a} dx\ \delta(x) \qquad \text{(c)} \int_{-a}^{a} dx\ \delta(x) f(x)$$

$$\text{(d)} \int_{a}^{2a} dx\ \delta(x) f(x) \qquad \text{(e)} \int_{-a}^{a} dx\ \delta(x-2a)$$

$$\text{(f)} \int_{-a}^{a} dx\ \delta(x-2a) f(x) \qquad \text{(g)} \int_{0}^{3a} dx\ \delta(x-2a) f(x)$$

   Hint: plotting the integrand in the region within the limits of integration is often helpful.