Due on Friday after Study Break, November 6th.

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- Integrals you might need:

$$\int_0^{\pi} du\,\sin^2 u = \frac{\pi}{2}\,, \qquad \int_0^{\pi} du\,u\sin^2 u = \frac{\pi^2}{4}\,, \qquad \int_0^{\pi} du\,u^2\sin^2 u = \frac{\pi^3}{6} - \frac{\pi}{4}\,.$$

- Suggestions for this assignment:
  (1) (Re-)read the section on Expectation Values (3.5) in Nash's notes.
  (2) Read about spin-1/2 systems. A couple of links appear on the module webpage.

- [**SELF**] means: your own responsibility; will not be marked. However, in this problem sheet the [**SELF**] questions are rather important. Ignore at your own peril.

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1. A particle is confined in the one-dimensional region between $x = 0$ to $x = L$. It is in the stationary state $\psi = \sqrt{\frac{2}{L}}\sin(\pi x/L)$.

   (a) [**3 pts.**] Show that the wavefunction is normalized.

   (b) [**6 pts.**] Find $\langle\hat{x}\rangle$, the expectation value of the position of the particle. (You should be able to guess the answer before doing the calculation.)

   (c) [**7 pts.**] Find the uncertainty of the position, i.e., $\Delta x = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2}$.

   (d) [**6 pts.**] Find the expectation value of the momentum of the particle. Warning: $\psi^*(x)\hat{p}\psi(x) \neq \hat{p}|\psi(x)|^2$. Why not?
   (You should be able to guess the answer before doing the calculation.)

   (e) [**7 pts.**] Find the uncertainty of the momentum.

   (f) [**SELF**] How does the position uncertainty depend on L? Could you have expected this?

   (g) [**SELF**] Does the momentum uncertainty increase or decrease, if L is increased? Exp;ain whether/how this is consistent with the Heisenberg uncertainty principle.

(h) **[SELF]** If the energy (Hamiltonian) operator is $\hat{H} = \frac{\hat{p}^2}{2m}$, find the expectation value and the uncertaintly of the energy.

2. Recall for a spin-1/2 system:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Each of these matrices have two eigenvalues, in each case $+\hbar/2$ and $-\hbar/2$.

(a) **[SELF]** Why does it make sense physically that the three operators have the same set of eigenvalues? (What's the connection between measurement results and eigenvalues?)

(b) **[3 pts.]** For S_x, verify the statement about eigenvalues above i.e., calculate eigenvalues of the S_x matrix.

(c) **[2 pts.]** The eigenvector of S_x corresponding to the eigenvector $+\hbar/2$ is $|x,+\rangle = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$. Calculate α so that the state is normalized. You can choose α to be real and positive.

(d) **[4 pts.]** Calculate the expectation value of S_x in the state $|x,+\rangle$, i.e., calculate $\langle S_x \rangle = \langle x,+| S_x |x,+\rangle$. (You should be able to predict the answer before doing the calculation.)

(e) **[4 pts.]** Calculate the uncertainty of S_x in the state $|x,+\rangle$. (You should be able to guess the answer before doing the calculation.)

(f) **[4 pts.]** Calculate the expectation value of S_z in the state $|x,+\rangle$. (You might be able to guess the answer before doing the calculation.)

(g) **[4 pts.]** Calculate the uncertainty of S_z in the state $|x,+\rangle$.

3. A single particle on a one-dimensional line is in the state described by the wavefunction

$$\psi(x) = B \exp\left[-\frac{(x-a)^2}{\sigma^2}\right]$$

(a) **[SLEF]** Calculate the constant B so that $\psi(x)$ is normalized.

(b) **[SELF]** Find the expectation value of the position of the particle. (You should be able to guess the answer before doing the calculation. Also, you should be able to do the definite integral by symmetry.)

(c) **[SELF]** Find the uncertainty of the position. (Before doing the calculation, first guess how Δx will depend on the parameters a and σ.)

(d) **[SELF]** Find the expectation value of the momentum of the particle.

(e) **[SELF]** Find the uncertainty of the momentum. How is the uncertaintly of the momentum related to the uncertainty of the position?

4. The wavefunction of a particle on a one-dimensional line is given by

$$\psi(x,t) = C(t)\exp\left[-\frac{(x-vt)^2}{(s_0+\xi t)^2}\right]$$

Expectation values (and uncertainties) are functions of time, since the wavefunction is time-dependent.

You should be able to answer the following based on the previous problem, without new lengthy calculations.

(a) **[SELF]** Calculate the time-dependent (but space-independent) quantity $C(t)$ so that the wavefunction is normalized at all times.

(b) **[SELF]** Express and plot $\langle x(t)\rangle$ as a function of time.

(c) **[SELF]** Plot the uncertainty of the position, as a function of time.

(d) **[SELF]** Plot the uncertainty of the momentum, as a function of time.