Some partial solutions and/or hints are given below.

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1. Dirac notation (bra-ket notation).

   (a) If $|\phi\rangle = c\,|\chi\rangle$, where c is a complex number, express $\langle\phi|$ in terms of $\langle\chi|$.

   Solution/Hints/Discussion →

$$|\phi\rangle = c\,|\chi\rangle \quad \Longrightarrow \quad \langle\phi| = c^*\langle\chi|$$

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   (b) If $|\phi\rangle = \hat{A}\,|\chi\rangle$, where $\hat{A}$ is an operator (not necessarily hermitian), express $\langle\phi|$ in terms of $\hat{A}$ and $\langle\chi|$.

   Solution/Hints/Discussion →

$$|\phi\rangle = \hat{A}\,|\chi\rangle \quad \Longrightarrow \quad \langle\phi| = \langle\chi|\,\hat{A}^\dagger$$

   Note: this is NOT written as $\hat{A}^\dagger\langle\chi|$. An operator acting on a dual/bra vector is written to the right of the bra, not to the left.
   Alternately, the notation

$$\langle\phi|\,\hat{A}^\dagger = \langle\left(\hat{A}\phi\right)\Big|$$

   could also be used, i.e., $\langle\phi|\,\hat{A}^\dagger$ is the dual vector of $\hat{A}\,|\phi\rangle$.

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(c) Is $\langle\chi|\phi\rangle$ a number, an operator, or a ket vector, or a bra vector? What about $|\chi\rangle\langle\phi|$?

Solution/Hints/Discussion $\rightarrow$

$\langle\chi|\phi\rangle$ is a number. It is the inner product of the vectors $|\chi\rangle$ and $|\phi\rangle$. You can think of it as a row vector multiplying a column vector, which would result in a number.

$|\chi\rangle\langle\phi|$ is an operator. In matrix/vector representation, you can think of it as a column vector multiplying a row vector, which would result in a matrix. Matrices represent operators.

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(d) If $|\phi\rangle$ is not normalized, what would you multiply it by, in order to obtain a normalized state vector? Are you multiplying by a number, an operator, or another ket?

Solution/Hints/Discussion $\rightarrow$

We would multiply by the factor

$$\lambda = \frac{1}{\sqrt{\langle\phi|\phi\rangle}}$$

This is a NUMBER. The resultling state vector $|\psi\rangle = \lambda\,|\phi\rangle = \dfrac{1}{\sqrt{\langle\phi|\phi\rangle}}\,|\phi\rangle$ is normalized:

$$\langle\psi|\psi\rangle = \lambda^2\langle\phi|\phi\rangle = \left(\frac{1}{\sqrt{\langle\phi|\phi\rangle}}\right)^2\langle\phi|\phi\rangle = \frac{1}{\langle\phi|\phi\rangle}\langle\phi|\phi\rangle = 1$$

$\sqrt{\langle\phi|\phi\rangle}$ is called the NORM of ϕ.

To be more general, we could include a multiplicative phase factor with λ, which would keep the norm unchanged:

$$\lambda = \frac{e^{ir}}{\sqrt{\langle\phi|\phi\rangle}} \qquad \begin{cases} r \text{ is a real number, so that} \\ |e^{ir}|^2 = 1 \text{ and } |\lambda|^2 = \lambda^2 \end{cases}$$

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(e) If $|\chi\rangle$ is normalized and $|\phi\rangle = \gamma\,|\chi\rangle$, where γ is a complex number, find the value of $\langle\chi|\phi\rangle$.

Solution/Hints/Discussion →

$$\langle\chi|\phi\rangle \;=\; \langle\chi|\,\gamma\,|\chi\rangle \;=\; \gamma\langle\chi|\chi\rangle \;=\; \gamma$$

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(f) If $\langle\psi|\lambda\rangle = b$, then what is $\langle\lambda|\psi\rangle$?

Solution/Hints/Discussion →

$$\langle\lambda|\psi\rangle \;=\; \langle\psi|\lambda\rangle^* \;=\; b^*$$

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2. Basis vectors.

A physical system is described by a three-dimensional Hilbert space. Any state vector can be expressed as a linear combination of the three state vectors $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$. We say that the set $\{|\phi_j\rangle\}$ is a basis set (a set of basis vectors) for this space.

We choose our three basis vectors to be **orthonormal**.

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(a) Find out (look up) what orthonormal means. Our basis set $\{|\phi_j\rangle\}$ is orthonormal: Express what this means in terms of inner products. In particular, what is $\langle\phi_1|\phi_1\rangle$, and what is $\langle\phi_1|\phi_2\rangle$?

Solution/Hints/Discussion →

Othonormal means each of the basis vectors is normalized and is orthogonal to all the other basis vectors. This can be written as

$$\langle\phi_i|\phi_j\rangle = \delta_{ij} \qquad \begin{cases}\text{where } \delta_{ij} \text{ is the Korniecker delta:} \\ = 0 \text{ when } i \neq j \text{ and } = 1 \text{ when } i = j\end{cases}$$

In particular, $\langle\phi_1|\phi_1\rangle = 1$ and $\langle\phi_1|\phi_2\rangle = 0$.

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(b) Consider the normalized state

$$|W\rangle = \gamma\,|\phi_1\rangle + \gamma\,|\phi_2\rangle + \gamma\,|\phi_3\rangle\,.$$

Determine γ. Assume γ to be purely imaginary.

Solution/Hints/Discussion →

If the vector $|S\rangle$ is expressed in terms of the basis vectors as

$$|S\rangle = \sum_j s_j\,|\phi_j\rangle$$

then using orthonormality of the set $\{|\phi_j\rangle\}$ it's easy to show

$$\langle S|S\rangle \;=\; \sum_j s_j^* s_j \;=\; \sum_j |s_j|^2$$

Please make sure that you can work out how this follows from orthonormality of the basis vectors.
Thus the norm of the vector $|W\rangle$ is

$$\langle W|W\rangle \;=\; |\gamma|^2 \;+\; |\gamma|^2 \;+\; |\gamma|^2 \;=\; 3|\gamma|^2$$

Since the state $|W\rangle$ is normalized, this has to be unity, i.e., $3|\gamma|^2 = 1$. Since γ is purely imaginary, this means

$$\gamma = \frac{i}{\sqrt{3}} \qquad \text{or} \qquad \gamma = -\frac{i}{\sqrt{3}}$$

We will use the first form below when $|W\rangle$ is referred to.

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(c) Consider the states

$$|\psi\rangle = \frac{1}{\sqrt{3}}|\phi_1\rangle - \frac{1}{\sqrt{3}}|\phi_2\rangle + \frac{i}{\sqrt{3}}|\phi_3\rangle\,; \qquad |\theta\rangle = \frac{i}{2}|\phi_1\rangle + \frac{\sqrt{3}}{2}|\phi_3\rangle$$

Show that these two states are normalized.

Solution/Hints/Discussion →
Norm of $|\psi\rangle$ is

$$\langle\psi|\psi\rangle \;=\; \left|\frac{1}{\sqrt{3}}\right|^2 \;+\; \left|-\frac{1}{\sqrt{3}}\right|^2 \;+\; \left|\frac{i}{\sqrt{3}}\right|^2 \;=\; \frac{1}{3}+\frac{1}{3}+\frac{1}{3} \;=\; 1$$

and similarly for $|\theta\rangle$.

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(d) Express $\langle\psi|$ in terms of the bra vectors corresponding to the basis vectors, i.e., in terms of the set $\{\langle\phi_j|\}$.

Solution/Hints/Discussion $\rightarrow$

$$\langle\psi| \;=\; \frac{1}{\sqrt{3}}\langle\phi_1| \;-\; \frac{1}{\sqrt{3}}\langle\phi_2| \;-\; \frac{i}{\sqrt{3}}\langle\phi_3|$$

Note that the coefficients are complex conjugated.

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(e) Calculate $\langle\psi|\theta\rangle$ and $\langle\theta|\psi\rangle$.

Solution/Hints/Discussion →
If the vectors $|S\rangle$ and $|T\rangle$ are expressed in terms of the basis vectors as

$$|S\rangle = \sum_j s_j |\phi_j\rangle \qquad \text{and} \qquad |T\rangle = \sum_j t_j |\phi_j\rangle$$

then using orthonormality of the set $\{|\phi_j\rangle\}$ it's easy to show

$$\langle S|T\rangle = \sum_j s_j^* t_j \qquad \text{and} \qquad \langle T|S\rangle = \sum_j t_j^* s_j$$

Please make sure you are comfortable using this and can work out how this follows from orthonormality.
It follows:

$$\langle\psi|\theta\rangle = \left(\frac{1}{\sqrt{3}}\right)\left(\frac{i}{2}\right) + \left(-\frac{1}{\sqrt{3}}\right)(0) + \left(-\frac{i}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{i}{2\sqrt{3}} - \frac{i}{2} = \frac{i}{2}\left(\frac{1}{\sqrt{3}} - 1\right)$$

$$\langle\theta|\psi\rangle = -\frac{i}{2}\left(\frac{1}{\sqrt{3}} - 1\right)$$

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(f) Let's define the operator $\hat{A} = |\theta\rangle\langle\psi|$. Write out the operator in terms of the basis vectors and their duals, i.e., in terms of operators $|\phi_i\rangle\langle\phi_j|$.

Solution/Hints/Discussion →

$$\hat{A} = |\theta\rangle\langle\psi| = \left(\frac{i}{2}|\phi_1\rangle + \frac{\sqrt{3}}{2}|\phi_3\rangle\right)\left(\frac{1}{\sqrt{3}}\langle\phi_1| - \frac{1}{\sqrt{3}}\langle\phi_2| - \frac{i}{\sqrt{3}}\langle\phi_3|\right)$$
$$= \frac{i}{2\sqrt{3}}|\phi_1\rangle\langle\phi_1| - \frac{i}{2\sqrt{3}}|\phi_1\rangle\langle\phi_2| + \frac{1}{2\sqrt{3}}|\phi_1\rangle\langle\phi_3|$$
$$+ \frac{1}{2}|\phi_3\rangle\langle\phi_1| - \frac{1}{2}|\phi_3\rangle\langle\phi_2| - \frac{i}{2}|\phi_3\rangle\langle\phi_3|$$

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(g) Find the state obtained by operating with $\hat{A}$ on the state $|\phi_1\rangle$.

Solution/Hints/Discussion →
Since $\langle\phi_m|\phi_n\rangle = \delta_{mn}$, the only terms in $\hat{A}$ which contribute when applied on $|\phi_1\rangle$ are those of the form $|\phi_j\rangle\langle\phi_1|$. In other words,

$$\begin{aligned}\Big(|\phi_j\rangle\langle\phi_1|\Big)|\phi_1\rangle &= |\phi_j\rangle\langle\phi_1|\phi_1\rangle = |\phi_j\rangle \\ \Big(|\phi_j\rangle\langle\phi_2|\Big)|\phi_1\rangle &= |\phi_j\rangle\langle\phi_2|\phi_1\rangle = 0 \\ \Big(|\phi_j\rangle\langle\phi_3|\Big)|\phi_1\rangle &= |\phi_j\rangle\langle\phi_3|\phi_1\rangle = 0\end{aligned}$$

Hence, only the first and fourth terms in the above expression for $\hat{A}$ will contribute to $\hat{A}|\phi_1\rangle$. Thus

$$\hat{A}|\phi_1\rangle = \frac{i}{2\sqrt{3}}|\phi_1\rangle + \frac{1}{2}|\phi_3\rangle$$

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(h) Find the state obtained by operating with $\hat{A}$ on the state $|W\rangle$, i.e., express $\hat{A}|W\rangle$ in terms of the basis vectors.

Solution/Hints/Discussion →

$$\hat{A}|W\rangle = \left(\frac{i}{2\sqrt{3}}|\phi_1\rangle\langle\phi_1| - \frac{i}{2\sqrt{3}}|\phi_1\rangle\langle\phi_2| + \frac{1}{2\sqrt{3}}|\phi_1\rangle\langle\phi_3|\right.$$
$$\left.+\frac{1}{2}|\phi_3\rangle\langle\phi_1| - \frac{1}{2}|\phi_3\rangle\langle\phi_2| - \frac{i}{2}|\phi_3\rangle\langle\phi_3|\right)\left(\frac{i}{\sqrt{3}}|\phi_1\rangle + \frac{i}{\sqrt{3}}|\phi_2\rangle + \frac{i}{\sqrt{3}}|\phi_3\rangle\right)$$

Each term of $\hat{A}$, when applied to $|W\rangle$, gives exactly one nonzero term due to orthonormality of the basis vectors. Hence:

$$\begin{aligned}\hat{A}\,|W\rangle &= -\frac{1}{6}\,|\phi_1\rangle + \frac{1}{6}\,|\phi_1\rangle + \frac{i}{6}\,|\phi_1\rangle + \frac{i}{2\sqrt{3}}\,|\phi_3\rangle - \frac{i}{2\sqrt{3}}\,|\phi_3\rangle + \frac{1}{2\sqrt{3}}\,|\phi_3\rangle \\ &= \frac{i}{6}\,|\phi_1\rangle + \frac{1}{2\sqrt{3}}\,|\phi_3\rangle\end{aligned}$$

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(i) Let's introduce the shorthand notation $A_{ij} = \langle \phi_i |\, \hat{A}\, | \phi_j \rangle$. Find the values of the nine quantities A_{ij}. These are generally known as 'matrix elements' of the operator $\hat{A}$.

Solution/Hints/Discussion →

I will write the nine quantities as elements of a matrix.

$$\begin{pmatrix} \frac{i}{2\sqrt{3}} & -\frac{i}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{i}{2} \end{pmatrix}$$

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3. Vector/matrix notation. (Building on previous problem)

We are using the set $\{|\phi_j\rangle\}$ as a basis set. This can be expressed in vector notation as

$$|\phi_1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad |\phi_2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad |\phi_3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

The size of the vectors is the number of orthonormal basis vectors required to represent all states, which is also the dimension of the Hilbert space. (In this case: three.)

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(a) Write down the states $|W\rangle$, $|\psi\rangle$ and $|\theta\rangle$ (from previous problem) as vectors.

Solution/Hints/Discussion →

$$|W\rangle = \gamma|\phi_1\rangle + \gamma|\phi_2\rangle + \gamma|\phi_3\rangle = \gamma\begin{pmatrix}1\\0\\0\end{pmatrix} + \gamma\begin{pmatrix}0\\1\\0\end{pmatrix} + \gamma\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\gamma\\\gamma\\\gamma\end{pmatrix}$$

Similarly

$$|\psi\rangle = \begin{pmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{i}{\sqrt{3}}\end{pmatrix} \qquad |\theta\rangle = \begin{pmatrix}\frac{i}{2}\\0\\\frac{\sqrt{3}}{2}\end{pmatrix}$$

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(b) Write down the row vectors which represent the dual (bra) vectors $\langle\psi|$ and $\langle\theta|$.

Solution/Hints/Discussion →

In vector notation, the dual vectors are simply the hermitian conjugates (adjoints) of the vectors. Hence bra vectors are represented by hermitian conjugates column vectors, and hence are row vectors.

$$\langle\psi| = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \end{pmatrix} \qquad \langle\theta| = \begin{pmatrix} -\frac{i}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Note the complex conjugates!!

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(c) Using these representations, calculate $\langle\psi|\theta\rangle$ through matrix multiplication between a row vector and a column vector. (Check with the value that you calculated previously using bra-ket notation.)

Solution/Hints/Discussion →

$$\begin{aligned}\langle\psi|\theta\rangle &= \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{i}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix} \\ &= \left(\frac{1}{\sqrt{3}}\right)\left(\frac{i}{2}\right) + \left(-\frac{1}{\sqrt{3}}\right)(0) + \left(\frac{i}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{i}{2\sqrt{3}} - \frac{i}{2} = \frac{i}{2}\left(\frac{1}{\sqrt{3}} - 1\right)\end{aligned}$$

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(d) Construct the matrix representing the operator $\hat{A} = |\theta\rangle\langle\psi|$, by multiplying the column vector representing $|\theta\rangle$ and the row vector representing $\langle\psi|$. (The order of vectors is important!)

Solution/Hints/Discussion →

$$\hat{A} = |\theta\rangle\langle\psi| = \begin{pmatrix} \frac{i}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{i}{2\sqrt{3}} & -\frac{i}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{i}{2} \end{pmatrix}$$

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(e) Is it clear why $A_{ij} = \langle \phi_i | \hat{A} | \phi_j \rangle$ are called the "matrix elements"?

Solution/Hints/Discussion →
Because these are exactly the elements of a matrix representation of operators.

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(f) Using matrix-vector multiplication, find the state obtained by operating with $\hat{A}$ on the state $|W\rangle$.

Solution/Hints/Discussion →

$$\hat{A}|W\rangle = \begin{pmatrix} i/2\sqrt{3} & -i/2\sqrt{3} & 1/2\sqrt{3} \\ 0 & 0 & 0 \\ 1/2 & -1/2 & -i/2 \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma \\ \gamma \end{pmatrix} = \begin{pmatrix} \gamma/2\sqrt{3} \\ 0 \\ -i\gamma/2 \end{pmatrix} = \begin{pmatrix} i/6 \\ 0 \\ 1/2\sqrt{3} \end{pmatrix}$$

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4. Let's use the convention that we use symbols with hats to denote operators, symbols without hats to denote numbers, and the usual ket and bra to denote states and their duals. Which of the following statements could be meaningful, and which are nonsensical? Explain why in each case.

   (a) $\hat{B} \;=\; \alpha_1 |\phi_1\rangle + \alpha_2 |\phi_2\rangle$

   Solution/Hints/Discussion →
   operator = state
   Nonsense

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   (b) $|\theta\rangle\langle\chi| \;=\; \hat{A}$

   Solution/Hints/Discussion →
   Operator = operator
   Meaningful

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   (c) $\langle\phi|\chi\rangle \;=\; \hat{A}$

   Solution/Hints/Discussion →
   number = operator
   Nonsense

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   (d) $|\theta\rangle\langle\chi| \;=\; c_1$

   Solution/Hints/Discussion →
   operator = number
   Nonsense

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(e) $\langle\phi|\chi\rangle \ = \ c_2$

Solution/Hints/Discussion →
number = number
Meaningful

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(f) $|\theta\rangle\langle\phi|\chi\rangle \ = \ \hat{Y}$

Solution/Hints/Discussion →
Left side is a state times a number, which is a state.
Right side is an operator.
Nonsense

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(g) $|\theta\rangle\langle\phi|\chi\rangle \ = \ c_3$

Solution/Hints/Discussion →
Left side is a state times a number, which is a state.
Right side is an number.
Nonsense

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(h) $\hat{B} \ = \ \langle\chi|\,\hat{Y}\,|\theta\rangle$

Solution/Hints/Discussion →
operator = number
Nonsense

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