Due on Friday November 13th.

[**SELF**] means: will not be marked. Working them out may be helpful for doing the other problems.

– – – – – – – – – – – – – – – ⋆ – – – – – – – – – – – – – –

1. Dirac notation (bra-ket notation).

   Work through an introduction to Dirac notation (Nash notes Sec. 2.3, wikipedia page, a q.m. textbook,...) Some links are on the module webpage.

   The bra vector $\langle\phi|$ is sometimes called the dual vector or co-vector of the ket vector $|\phi\rangle$.

   (a) [**1 pt.**] If $|\phi\rangle = c\,|\chi\rangle$, where c is a complex number, express $\langle\phi|$ in terms of $\langle\chi|$.

   (b) [**1 pt.**] If $|\phi\rangle = \hat{A}\,|\chi\rangle$, where $\hat{A}$ is an operator (not necessarily hermitian), express $\langle\phi|$ in terms of $\hat{A}$ and $\langle\chi|$.

   (c) [**SELF**] Is $\langle\chi|\phi\rangle$ a number, an operator, or a ket vector, or a bra vector? What about $|\chi\rangle\langle\phi|$?

   (d) [**2 pts.**] If $|\phi\rangle$ is not normalized, what would you multiply it by, in order to obtain a normalized state vector? Are you multiplying by a number, an operator, or another ket?

   (e) [**2 pts.**] If $|\chi\rangle$ is normalized and $|\phi\rangle = \gamma\,|\chi\rangle$, where γ is a complex number, find the value of $\langle\chi|\phi\rangle$.

   (f) [**1 pt.**] If $\langle\psi|\lambda\rangle = b$, then what is $\langle\lambda|\psi\rangle$?

2. Basis vectors.

   A physical system is described by a three-dimensional Hilbert space. Any state vector can be expressed as a linear combination of the three state vectors $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$. We say that the set $\{|\phi_j\rangle\}$ is a basis set (a set of basis vectors) for this space.

   We choose our three basis vectors to be **orthonormal**.

   (a) [**2 pts.**] Find out (look up) what orthonormal means. Our basis set $\{|\phi_j\rangle\}$ is orthonormal: Express what this means in terms of inner products. In particular, what is $\langle\phi_1|\phi_1\rangle$, and what is $\langle\phi_1|\phi_2\rangle$?

(b) **[4 pts.]** Consider the normalized state

$$|W\rangle = \gamma\,|\phi_1\rangle + \gamma\,|\phi_2\rangle + \gamma\,|\phi_3\rangle\,.$$

The real part of γ is $\sqrt{3}$ times larger than its imaginary part. Determine γ.

(c) **[2 pts.]** Consider the states

$$|\psi\rangle = \frac{1}{\sqrt{3}}\,|\phi_1\rangle - \frac{1}{\sqrt{3}}\,|\phi_2\rangle + \frac{i}{\sqrt{3}}\,|\phi_3\rangle$$

$$|\theta\rangle = \frac{i}{2}\,|\phi_1\rangle + \frac{\sqrt{3}}{2}\,|\phi_3\rangle$$

Show that these two states are normalized.

(d) **[1 pt.]** Express $\langle\psi|$ in terms of the bra vectors corresponding to the basis vectors, i.e., in terms of the set $\{\langle\phi_j|\}$.

(e) **[2 pts.]** Calculate $\langle\psi|\theta\rangle$ and $\langle\theta|\psi\rangle$.

(f) **[4 pts.]** Let's define the operator $\hat{A} = |\theta\rangle\langle\psi|$. Write out the operator in terms of the basis vectors and their duals, i.e., in terms of operators $|\phi_i\rangle\langle\phi_j|$.

(g) **[2 pts.]** Find the state obtained by operating with $\hat{A}$ on the state $|\phi_1\rangle$.

(h) **[3 pts.]** Find the state obtained by operating with $\hat{A}$ on the state $|W\rangle$, i.e., express $\hat{A}\,|W\rangle$ in terms of the basis vectors.

(i) **[2 pts.]** Let's introduce the shorthand notation $A_{ij} = \langle\phi_i|\,\hat{A}\,|\phi_j\rangle$. Find the values of the nine quantities A_{ij}. These are generally known as 'matrix elements' of the operator $\hat{A}$.

3. Vector/matrix notation. (Building on previous problem)

We are using the set $\{|\phi_j\rangle\}$ as a basis set. This can be expressed in vector notation as

$$|\phi_1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad |\phi_2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad |\phi_3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

The size of the vectors is the number of orthonormal basis vectors required to represent all states, which is also the dimension of the Hilbert space. (In this case: three.)

(a) **[3 pts.]** Write down the states $|W\rangle$, $|\psi\rangle$ and $|\theta\rangle$ (from previous problem) as vectors.

(b) **[2 pts.]** Write down the row vectors which represent the dual (bra) vectors $\langle\psi|$ and $\langle\theta|$.

(c) **[2 pts.]** Using these representations, calculate $\langle\psi|\theta\rangle$ through matrix multiplication between a row vector and a column vector. (Check with the value that you calculated previously using bra-ket notation.)

(d) **[3 pts.]** Construct the matrix representing the operator $\hat{A} = |\theta\rangle\langle\psi|$, by multiplying the column vector representing $|\theta\rangle$ and the row vector representing $\langle\psi|$. (The order of vectors is important!)

(e) **[SELF]** Is it clear why $A_{ij} = \langle\phi_i|\hat{A}|\phi_j\rangle$ are called the "matrix elements"?

(f) **[3 pts.]** Using matrix-vector multiplication, find the state obtained by operating with $\hat{A}$ on the state $|W\rangle$.

4. **[8 pts.]** Let's use the convention that we use symbols with hats to denote operators, symbols without hats to denote numbers, and the usual ket and bra to denote states and their duals. Which of the following statements could be meaningful, and which are nonsensical? Explain why in each case.

(a) $\hat{B} = \alpha_1|\phi_1\rangle + \alpha_2|\phi_2\rangle$

(b) $|\theta\rangle\langle\chi| = \hat{A}$

(c) $\langle\phi|\chi\rangle = \hat{A}$

(d) $|\theta\rangle\langle\chi| = c_1$

(e) $\langle\phi|\chi\rangle = c_2$

(f) $|\theta\rangle\langle\phi|\chi\rangle = \hat{Y}$

(g) $|\theta\rangle\langle\phi|\chi\rangle = c_3$

(h) $\hat{B} = \langle\chi|\hat{Y}|\theta\rangle$