Due on Friday November 20th.

— — — — — — — — — — — — — — — ⋆ — — — — — — — — — — — — — —

1. A particle in a 1D harmonic oscillator: eigenstates.

For a single particle in a harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2$, the time-independent Schroedinger equation has an infinite number of eigenvalue/eigenfunction pairs. The lowest three energy eigenvalues are $E_0 = \dfrac{1}{2}\hbar\omega$, $E_1 = \dfrac{3}{2}\hbar\omega$, $E_2 = \dfrac{5}{2}\hbar\omega$. The corresponding eigenfunctions are respectively

$$\phi_0(x) = A_0\, e^{-x^2/2\sigma^2}, \qquad \phi_1(x) = A_1\, x\, e^{-x^2/2\sigma^2},$$

$$\phi_2(x) = A_2 \left(\frac{2x^2}{\sigma^2} - 1\right) e^{-x^2/2\sigma^2}. \qquad \text{Here } \sigma^2 = \frac{\hbar}{m\omega}.$$

(a) **[2+1 pts.]** Sketch (by hand) plots of the three functions $\phi_n(x)$ for $n = 0$, $n = 1$, $n = 2$. You may use a computer program if you want to figure out how these functions look like, but please submit hand-drawn sketches.

By looking at the three plots and observing the pattern, guess and sketch the plots of the next two eigenstates, $\phi_3(x)$ and $\phi_4(x)$.

(b) **[3 pts.]** Look up (or calculate, or ask a computer algebra system or online integrator) the integrals

$$\int_{-\infty}^{+\infty} x^{2n} e^{-x^2}\, dx$$

for $n = 0$, $n = 1$, $n = 2$. Report the three results.

The first one ($n = 0$, integral over a Gaussian) is important enough that you might consider learning it for life. (You should certainly know how to derive it!)

(c) **[SELF]** Calculate the integrals

$$\int_{-\infty}^{+\infty} x^{2n} \exp\left[-\frac{x^2}{2\sigma^2}\right] dx$$

for $n = 0$, $n = 1$, $n = 2$, based on the results of the previous question.

(d) **[2 pts.]** Use symmetry arguments to evaluate the integral

$$\int_{-\infty}^{+\infty} x^{2n+1} \exp\left[-\frac{x^2}{2\sigma^2}\right] dx$$

for any integer n. The integrand is an **odd** function.

(e) **[2+2+3 pts.]** Show that ϕ_0 and ϕ_1 are orthogonal to each other, as are ϕ_1 and ϕ_2, and finally that ϕ_0 and ϕ_2 are orthogonal to each other.

(f) **[7 pts.]** Show in general that, if $|w_1\rangle$ and $|w_2\rangle$ are eigenstates of a hermitian operator corresponding to distinct eigenvalues, then they are orthogonal to each other. (This is an important result.)

(g) **[2 pts.]** Calculate A_0 so that ϕ_0 is normalized.

(h) **[SELF]** Calculate A_1 and A_2 so that ϕ_1 and ϕ_2 are normalized.

(i) **[2 pts.]** Construct a solution of the TDSE, based on the three eigenfunctions of the time-independent SE.

(j) **[3 pts.]** At time $t = 0$, the particle in the harmonic oscillator finds itself in the state

$$\psi(x,0) \;=\; \frac{1}{\sqrt{2}}\phi_0(x) \;-\; \frac{1}{2}\phi_1(x) \;+\; \frac{1}{2}\phi_2(x)\;.$$

What is the wavefunction (state) $\psi(x,t)$ at a later time t?

(k) **[6 pts.]** If the $\phi_n(x)$ functions are normalized, then show that $\psi(x,0)$ is normalized, and that $\psi(x,t)$ remains normalized at later times t.

(l) This is a good time to have a first browse through Nash Chapter 4 (The harmonic oscillator), or the wikipedia page on "Quantum Harmonic Oscillator".

2. Let $|\phi_n\rangle$ be the orthonormalized energy eigenstate of a particle of mass m in an infinite square well of width a, with corresponding energy eigenvalue

$$E_n \;=\; \frac{n^2\pi^2\hbar^2}{2ma^2}\,, \qquad n = 1,2,3,...$$

The particle is prepared to be in the state

$$|\psi\rangle \;=\; \sum_{n=1}^{\infty}\frac{\alpha}{n^2}\,|\phi_n\rangle$$

where α is a positive real number.

(a) **[SELF]** You might need the series

$$\sum_{n=1}^{\infty}\frac{1}{n^4} \;=\; \frac{\pi^4}{90}\,, \qquad \sum_{n=1}^{\infty}\frac{1}{n^2} \;=\; \frac{\pi^2}{6}$$

Feel free to have fun deriving these two series sums.

(b) **[SELF]** Show that $|\psi\rangle$ is normalized if $\alpha = \sqrt{90/\pi^4}$.

(c) **[8 pts.]** Find the expectation value of the Hamiltonian for this state.

(d) **[7 pts.]** What is the probability that a measurement of the energy gives $2\pi^2\hbar^2/ma^2$, and if this energy is found, what state is the particle in immediately afterwards?

Hint: This question is about measurement in quantum mechanics. For guidance, you could try reading the first few sections of the wikipedia page on "Measurement in Quantum Mechanics", and Sections 3.2 and 3.3 of Nash notes. There is also a paragraph summarizing measurement in the writeup "Essentials of QM", linked on the webpage.