Some partial solutions and/or hints are given below. Watch out for misprints. If you catch an error, please let me know.

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1. Unitary matrices and operators.

(a) Look up and report the definition of a unitary matrix.

Solution/Hints/Discussion →
A matrix is unitary if its hermitian conjugate (adjoint) is its inverse. U is unitary if $U^\dagger = U^{-1}$, i.e., if $U^\dagger U = UU^\dagger = I$.

Comment (1): The Hermitian conjugate is also known as the adjoint, or the conjugate transpose.
Comment (2): Sometimes, the conjugate transpose is also denoted with a star instead of with a dagger symbol, i.e., U^* instead of $U^\dagger$. (E.g., as of Nov. 2020 the star notation is used in the wikipedia page for 'Unitary Matrix'.) I strongly recommend using the dagger symbol, because in physics the star often represents the complex conjugate without transpose.

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(b) Show that the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is its own inverse.

Solution/Hints/Discussion →
A matrix A is its own inverse if $AA = I$. The matrix σ_x is its own inverse, because

$$\sigma_x\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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(c) Show that the three Pauli matrices are unitary.

Solution/Hints/Discussion →
We want to show

$$\sigma_x\sigma_x^\dagger = \sigma_x^\dagger\sigma_x = I \qquad \sigma_y\sigma_y^\dagger = \sigma_y^\dagger\sigma_y = I \qquad \sigma_z\sigma_z^\dagger = \sigma_z^\dagger\sigma_z = I$$

First for σ_x:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_x^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x \qquad \Longrightarrow \qquad \begin{cases} \sigma_x\sigma_x^\dagger = \sigma_x\sigma_x = I \\ \sigma_x^\dagger\sigma_x = \sigma_x\sigma_x = I \end{cases}$$

using the fact that $\sigma_x^2 = I$, as shown previously.
Now for σ_y:

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y \qquad \Longrightarrow \qquad \begin{cases} \sigma_y\sigma_y^\dagger = \sigma_y\sigma_y = I \\ \sigma_y^\dagger\sigma_y = \sigma_y\sigma_y = I \end{cases}$$

where we have used the fact that

$$\sigma_y\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

And finally for σ_z:

$$\sigma_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_z \qquad \Longrightarrow \qquad \begin{cases} \sigma_z\sigma_z^\dagger = \sigma_z\sigma_z = I \\ \sigma_z^\dagger\sigma_z = \sigma_z\sigma_z = I \end{cases}$$

where we have used the fact that

$$\sigma_z\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

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(d) Show that the two-dimensional vector $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has the same norm as $\sigma_y\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, i.e., operating with σ_y keeps the norm unchanged.

Solution/Hints/Discussion →

$$\sigma_y\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \;=\; \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \;=\; \begin{pmatrix}-i\beta\\ i\alpha\end{pmatrix}$$

whose norm is

$$|-i\beta|^2 + |i\alpha|^2 \;=\; |\beta|^2 + |\alpha|^2 \;=\; |\alpha|^2 + |\beta|^2$$

This is the same as the norm of $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$. Hence the norm is unchanged by operating with σ_y.

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(e) If $\hat{U}$ is a unitary operator, show that the state $|\psi\rangle = \hat{U}|\phi\rangle$ has the same norm as the state $|\phi\rangle$.

Solution/Hints/Discussion →

$$|\psi\rangle = \hat{U}|\phi\rangle \qquad \Longrightarrow \qquad \langle\psi| = \langle\phi|\hat{U}^\dagger$$

Thus the norm of $|\psi\rangle$ is

$$\langle\psi|\psi\rangle \;=\; \langle\phi|\hat{U}^\dagger\hat{U}|\phi\rangle \;=\; \langle\phi|\phi\rangle$$

i.e., is equal to the norm of $|\phi\rangle$. Note we have used $\hat{U}^\dagger\hat{U} = 1$.

Comment:

Note that this is abstract and general, and makes no assumption about whether the Hilbert space is finite-dimensional or infinite-dimensional. We don't need a specific formula for the norm such as $\int dx\psi^*(x)\psi(x)$ or $\sum_j \psi_j^*\psi_j$. The general bra-ket expression $\langle\psi|\psi\rangle$ is good enough.

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2. The state $|\mu_1\rangle$ is an eigenstate of $\hat{M}$ with corresponding eigenvalue m_1.

(a) Show that $|\mu_1\rangle$ is also an eigenstate of $\hat{M}^2$, and find the corresponding eigenvalue. Show that $|\mu_1\rangle$ is also an eigenstate of $\hat{M}^n$, where n is an integer, and find the corresponding eigenvalue.

Solution/Hints/Discussion →
Since $\hat{M}|\mu_1\rangle = m_1|\mu_1\rangle$, we have

$$\begin{aligned}\hat{M}^2|\mu_1\rangle &= \hat{M}\left(\hat{M}|\mu_1\rangle\right) = \hat{M}(m_1|\mu_1\rangle) = m_1\left(\hat{M}|\mu_1\rangle\right) \\ &= m_1(m_1|\mu_1\rangle) = m_1^2|\mu_1\rangle\end{aligned}$$

Thus $|\mu_1\rangle$ is also an eigenstate of $\hat{M}^2$, with the eigenvalue m_1^2.
Similarly, $|\mu_1\rangle$ is also an eigenstate of $\hat{M}^n$, for any positive integer n, with the eigenvalue m_1^n.

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(b) How is the operator $e^{\hat{M}}$ defined?
How is $e^{\alpha\hat{M}}$ defined, if α is a complex number?
Is $|\mu_1\rangle$ an eigenstate of $e^{\alpha\hat{M}}$? If so, find the corresponding eigenvalue.

Solution/Hints/Discussion →

$$\begin{aligned}e^{\hat{M}} &= \hat{1} + \hat{M} + \frac{1}{2!}\hat{M}^2 + \frac{1}{3!}\hat{M}^3 + \cdots \\ e^{\alpha\hat{M}} &= \hat{1} + \alpha\hat{M} + \frac{\alpha}{2!}\hat{M}^2 + \frac{\alpha^2}{3!}\hat{M}^3 + \cdots\end{aligned}$$

Is $|\mu_1\rangle$ is an eigenstate of $e^{\alpha\hat{M}}$? Let's try:

$$\begin{aligned}e^{\alpha\hat{M}}|\mu_1\rangle &= \left(\hat{1} + \alpha\hat{M} + \frac{\alpha}{2!}\hat{M}^2 + \frac{\alpha^2}{3!}\hat{M}^3 + \cdots\right)|\mu_1\rangle \\ &= |\mu_1\rangle + \alpha m_1|\mu_1\rangle + \frac{\alpha}{2!}m_1^2|\mu_1\rangle + \frac{\alpha^2}{3!}m_1^3|\mu_1\rangle + \cdots \\ &= \left(1 + \alpha m_1 + \frac{\alpha}{2!}m_1^2 + \frac{\alpha^2}{3!}m_1^3 + \cdots\right)|\mu_1\rangle = e^{\alpha m_1}|\mu_1\rangle\end{aligned}$$

Yes, an eigenstate, with the eigenvalue $e^{\alpha m_1}$.

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3. We will consider a particle in one dimension subject to a negative (attractive) delta potential at $x = 0$:

$$V(x) \;=\; -\lambda\delta(x)$$

This system has a single bound state at some negative energy, $E < 0$. We will find the energy and wavefunction of this bound state.

Comment →

The wikipedia page titled "Delta potential" is recommended reading. Also described there is the motivation for studying this artificial-looking potential.

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(a) Consider the left half-line ($x < 0$). Write down the general solution of the Schroedinger equation in this region. Remember: we are considering $E < 0$.
Argue why one of the terms of the general solution can be dropped.

Solution/Hints/Discussion →

Away from $x = 0$, the potential is zero, so the Schroedinger equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi$$

$$\Longrightarrow \qquad \psi''(x) - \alpha^2\psi(x) \;=\; 0 \qquad \text{with} \quad \alpha = \sqrt{-\frac{2mE}{\hbar^2}}$$

The general solution is

$$\psi_L(x) \;=\; A_L e^{\alpha x} + B_L e^{-\alpha x}$$

I'm using the indices L and R for the left half-line and right half-line respectively.
The second term diverges for $x \to -\infty$, and so is unphysical, and so needs to be dropped:

$$\psi_L(x) \;=\; A_L e^{\alpha x}$$

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(b) Similarly find the general solution on the right half-line ($x > 0$), and identify the term that should be dropped.

Solution/Hints/Discussion $\rightarrow$

On the right side, same general solution:

$$\psi_R(x) \;=\; A_R e^{\alpha x} + B_R e^{-\alpha x}$$

The first term diverges for $x \rightarrow +\infty$, and so is unphysical, and so needs to be dropped:

$$\psi_R(x) \;=\; B_R e^{-\alpha x}$$

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(c) Explain why the derivative of the wavefunction, $\psi'(x)$, need not be continuous at $x = 0$.

Solution/Hints/Discussion $\rightarrow$

The derivative needs to be continuous if the potential is everywhere finite. The potential here is $-\infty$ at $x = 0$, i.e., not finite everywhere. Hence the derivative $\psi'(x)$ does not need to be continuous.

We can also think about this by looking directly at the Schoroedinger equation:

$$-\frac{\hbar^2}{2m}\psi''(x) \;-\; \lambda\delta(x)\psi(x) = E\psi(x)$$

$$\Longrightarrow \qquad -\frac{\hbar^2}{2m}\psi''(x) \;=\; \lambda\delta(x)\psi(x) \;+\; E\psi(x)$$

Since the right hand side is not finite at $x = 0$, this means that $\psi''(x)$ is not finite at $x = 0$. This is consistent with $\psi'(x)$ having a jump (discontinuity) at $x = 0$. (If a function has a jump, its derivative is infinite at that point.) Hence it is expected that $\psi'(x)$ will have a jump at $x = 0$.

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(d) Let ϵ be an infinitesimally small positive number. By integrating both sides of the Schroedinger equation from $-\epsilon$ to ϵ, show that the derivative of the wavefunction has the following discontinuity around the $x = 0$ point:

$$\psi'(\epsilon) - \psi'(-\epsilon) \ = \ -\frac{2m\lambda}{\hbar^2}\psi(0) \ + \ \text{infinitesimal term}$$

Solution/Hints/Discussion →

The Schroedinger equation is

$$-\frac{\hbar^2}{2m}\psi''(x) \ - \ \lambda\delta(x)\psi(x) = E\psi(x)$$
$$\implies \quad \psi''(x) \ = \ -\frac{2m\lambda}{\hbar^2}\delta(x)\psi(x) - -\frac{2mE}{\hbar^2}\psi(x)$$

Integrating both sides from $x = -\epsilon$ to $x = +\epsilon$, one obtains

$$\int_{-\epsilon}^{\epsilon}\psi''(x)dx \ = \ -\frac{2m\lambda}{\hbar^2}\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)dx - \frac{2mE}{\hbar^2}\int_{-\epsilon}^{\epsilon}\psi(x)dx$$

The first integral is equal to $\psi'(\epsilon) - \psi'(-\epsilon)$ because $\frac{d}{dx}\psi'(x) = \psi''(x)$. (Fundamental Theorem of Calculus.)

The second integral is $\psi(0)$ due to the property of the Dirac delta function.

Since $\psi(x)$ is continuous, the last integral can be approximated for very small ϵ as $\psi(0)2\epsilon$, which is an infinitesimal quantity.

Thus

$$\psi'(\epsilon) - \psi'(-\epsilon) \ = \ -\frac{2m\lambda}{\hbar^2}\psi(0) \ + \ \text{infinitesimal term}$$

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(e) Determine the energy of the bound state and the constants in the wavefunction.

You can use continuity of $\psi(x)$ and the normalization of the full wavefunction. In addition, the relation above for the discontinuity of $\psi'(x)$ can be used as a boundary condition:

$$\psi'_R(0) - \psi'_L(0) \ = \ -\frac{2m\lambda}{\hbar^2}\psi_L(0)$$

where $\psi_L(x)$ and $\psi_R(x)$ are the wavefunctions on the left half-line and right half-line respectively.

Solution/Hints/Discussion →
The wavefunction is $\psi_L(x) = A_L e^{\alpha x}$ on the left and $\psi_R(x) = B_R e^{-\alpha x}$ on the right.
Continuity of the wavefunction:

$$\psi_L(0) = \psi_R(0) \qquad \Longrightarrow \qquad A_L = B_R$$

Normalization:

$$\int_{-\infty}^{0} |\psi_L(x)|^2\, dx \ + \ \int_{0}^{\infty} |\psi_R(x)|^2\, dx \ = \ 1$$

$$\Longrightarrow \qquad |A_L|^2 \int_{-\infty}^{0} e^{2\alpha x} dx \ + \ |A_L|^2 \int_{0}^{\infty} e^{-2\alpha x} dx \ = \ 1$$

$$\Longrightarrow \qquad |A_L|^2 \frac{1}{2\alpha} \ + \ |A_L|^2 \frac{1}{2\alpha} \ = \ 1$$

$$\Longrightarrow \qquad |A_L|^2 \ = \ \alpha \ = \ \sqrt{-\frac{2mE}{\hbar^2}}$$

Discontinuity of ψ':

$$-\alpha A_L - \alpha A_L \ = \ -\frac{2m\lambda}{\hbar^2} A_L \qquad \begin{cases} \text{using } \psi'_L(0) = A_L(\alpha)e^0 = \alpha A_L \\ \text{and } \psi'_R(0) = B_R(-\alpha)e^0 = - - \alpha A_L \end{cases}$$

$$\Longrightarrow \qquad \alpha \ = \ \frac{m\lambda}{\hbar^2} \qquad \Longrightarrow \qquad E = -\frac{\hbar^2\alpha^2}{2m} = -\frac{m\lambda^2}{2\hbar^2}$$

Thus we have found the energy and the overall constant to be

$$E = -\frac{m\lambda^2}{2\hbar^2} \qquad A_L \ = \ \sqrt{\alpha} \ = \ \sqrt{\frac{m\lambda}{\hbar^2}}$$

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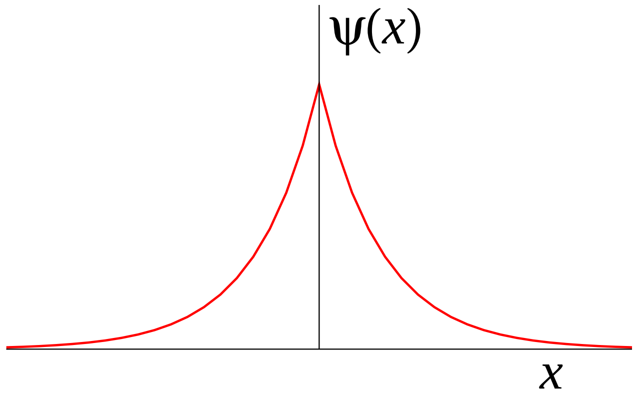
(f) Plot the wavefunction as a function of position. Make sure the (dis)continuity at $x = 0$ is clearly visible.

Solution/Hints/Discussion $\rightarrow$



Note that the wavefunction from either side DOES NOT diverge at $x = 0$, because both functions $e^{\alpha x}$ and $e^{-\alpha x}$ go to unity at $x = 0$. i.e., the wavefunction is very finite.

The wavefunction $\psi(x)$ itself is continuous – there is no jump.

The derivative of the wavefunction is discontinuous – this appears as a 'kink' in the curve at $x = 0$.

Exercise: Plot the derivative, $\psi'(x)$, as a function of position.