Due Friday, November 27th

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1. Unitary matrices and operators.

   (a) **[2 pts.]** Look up and report the definition of a unitary matrix.

   (b) **[1 pt.]** Show that the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is its own inverse.

   Hint: no need to calculate an inverse explicitly; use the fact/definition that a matrix multiplied by its inverse gives the identity.

   (c) **[5 pts.]** Show that the three Pauli matrices are unitary.
   (Comment: We have seen previously that they are also hermitian. These matrices are both hermitian and unitary!)

   (d) **[3 pts.]** Show that the two-dimensional vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ has the same norm as $\sigma_y \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, i.e., operating with σ_y keeps the norm unchanged.

   (Note: This is a special case of the fact that a unitary operation keeps the norm unchanged.)

   (e) **[4 pts.]** If $\hat{U}$ is a unitary operator, show that the state $|\psi\rangle = \hat{U}|\phi\rangle$ has the same norm as the state $|\phi\rangle$.

2. The state $|\mu_1\rangle$ is an eigenstate of $\hat{M}$ with corresponding eigenvalue m_1.

   (a) **[2 pts.]** Show that $|\mu_1\rangle$ is also an eigenstate of $\hat{M}^2$, and find the corresponding eigenvalue. Show that $|\mu_1\rangle$ is also an eigenstate of $\hat{M}^n$, where n is an integer, and find the corresponding eigenvalue.

   (b) **[5 pts.]** How is the operator $e^{\hat{M}}$ defined?
   How is $e^{\alpha\hat{M}}$ defined, if α is a complex number?
   Is $|\mu_1\rangle$ is an eigenstate of $e^{\alpha\hat{M}}$? If so, find the corresponding eigenvalue.

3. Review the properties of the Dirac delta function.

   We will consider a particle in one dimension subject to a negative (attractive) delta potential at $x = 0$:

$$V(x) \;=\; -\,\lambda\delta(x)$$

   This system has a single bound state at some negative energy, $E < 0$. We will find the energy and wavefunction of this bound state.

   (a) **[5 pts.]** Consider the left half-line ($x < 0$). Write down the general solution of the Schroedinger equation in this region. Remember: we are considering $E < 0$.

   Argue why one of the terms of the general solution can be dropped.

   (b) **[2 pts.]** Similarly find the general solution on the right half-line ($x > 0$), and identify the term that should be dropped.

   (c) **[3 pts.]** Explain why the derivative of the wavefunction, $\psi'(x)$, need not be continuous at $x = 0$.

   (d) **[6 pts.]** Let ϵ be an infinitesimally small positive number. By integrating both sides of the Schroedinger equation from $-\epsilon$ to ϵ, show that the derivative of the wavefunction has the following discontinuity around the $x = 0$ point:

$$\psi'(\epsilon) - \psi'(-\epsilon) \;=\; -\,\frac{2m\lambda}{\hbar^2}\psi(0) \;+\; \text{infinitesimal term}$$

   (e) **[8 pts.]** Determine the energy of the bound state and the constants in the wavefunction.

   You can use continuity of $\psi(x)$ and the normalization of the full wavefunction. In addition, the relation above for the discontinuity of $\psi'(x)$ can be used as a boundary condition:

$$\psi'_R(0) - \psi'_L(0) \;=\; -\,\frac{2m\lambda}{\hbar^2}\psi_L(0)$$

   where $\psi_L(x)$ and $\psi_R(x)$ are the wavefunctions on the left half-line and right half-line respectively.

   (f) **[4 pts.]** Plot the wavefunction as a function of position. Make sure the (dis)continuity at $x = 0$ is clearly visible.