Due on Friday, December 4th.

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1. Potential scattering.

   Consider a particle of mass m on a one-dimensional line. The potential is zero to the left of $x = -a$ and has a constant positive value to the right of $x = a$. The potential has unknown features within the region $x \in [-a, a]$.

$$V(x) = \begin{cases} 0 & \text{for } x < -a \\ \text{unknown} & \text{for } -a < x < a \\ V_0 & \text{for } x > a \end{cases}$$

   The Schroedinger equation for this system is found/known to have a solution with energy E with the following forms in the regions outside $[-a, a]$:

$$\psi(x) = \alpha e^{ik_1 x} + \beta e^{-ik_1 x} \qquad \text{for } x < -a$$
$$\psi(x) = \gamma e^{ik_2 x} \qquad \text{for } x > a$$

   where k_1 and k_2 are positive real constants, and

$$\alpha = \frac{k_1 + ik_2}{k_1 - ik_2} \quad ; \quad \beta = \frac{ik_1}{k_1 - ik_2} \quad ; \quad \gamma = \frac{\sqrt{k_1 k_2}}{k_1 - ik_2}$$

   (a) **[4 pts.]** Express k_1 in terms of E.

   (b) **[2 pts.]** Is E larger or smaller than V_0? Explain why.

   (c) **[5 pts.]** Express k_2 in terms of E and V_0.

   (d) **[3 pts.]** If we want to interpret this as a scattering problem, which parts of the wavefunction would you interpret as incident wave, reflected wave, and transmitted wave?

   (e) **[4 pts.]** Calculate the reflection coefficient.

   (f) **[5 pts.]** Calculate the transmission coefficient.
   (Careful: remember momentum factors.)

   (g) **[2 pts.]** Check that the sum of the two coefficients makes sense.

2. Suppose $\psi_0(x)$ is a normalized wavefunction with

$$\langle \hat{x} \rangle_{\psi_0} \;=\; \langle \psi_0 | \hat{x} | \psi_0 \rangle \;=\; x_0 \qquad \langle \hat{p} \rangle_{\psi_0} \;=\; \langle \psi_0 | \hat{p} | \psi_0 \rangle \;=\; p_0$$

where x_0 and p_0 are constants. The subscript indicates the state in which the expectation values are being calculated. Define a new wavefunction

$$\psi_{new}(x) \;=\; e^{iqx/\hbar}\, \psi_0(x)$$

where q is a real quantity having dimensions of momentum.

(a) **[6 pts.]** What is the expectation value $\langle \hat{x} \rangle_{\psi_{new}}$ in the state $\psi_{new}(x)$?

(b) **[7 pts.]** What is the expectation value $\langle \hat{p} \rangle_{\psi_{new}}$ in the state $\psi_{new}(x)$?

(c) **[SELF]** Based on your results, interpret in one sentence the physical significance of multiplying a wavefunction by the factor $e^{iqx/\hbar}$.

3. Consider a particle of mass m on a one-dimensional line. The state of the particle is described by wavefunction $\psi(x)$.

(a) **[0 pts.]** Look up and write down correctly the definition of the probability current density.

(b) **[4 pts.]** We write the wavefunction in amplitude-phase form:

$$\psi(x) \;=\; A(x) e^{i\theta(x)}$$

where both $A(x)$ and $\theta(x)$ are real functions.
Show that the probability current density is given by

$$j(x) \;=\; [A(x)]^2 \, \frac{\hbar}{m} \, \partial_x \theta(x)$$

i.e., the probability current is proportional to the *gradient* of the phase.

(c) **[2 pts.]** Show that there is no current in a region where the wavfunction is real.

4. Consider the time-dependent Schroedinger equation (TDSE) for a single particle on a one-dimensional line:

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) \;=\; -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x,t) \;+\; V(x)\psi(x,t)$$

The potential $V(x)$ is real-valued, i.e., $V^*(x) = V(x)$.

(a) **[6 pts.]** Using the TDSE, show that the probability density $\rho(x,t) = \psi^*\psi$ and the probability current density $j(x,t)$ satisfy the one-dimensional continuity equation

$$\frac{\partial j}{\partial x} = -\frac{\partial \rho}{\partial t}$$

(b) **[SELF]** Guess the form of the continuity equation for the probability density, for the case of a particle in three spatial dimensions.

(c) **[SELF]** What does the continuity equation imply physically?
The wikipedia page on "Continuity Equation" has a section called "quantum mechanics". You might try reading through.