Below are solutions/hints on some of the questions. There is no guarantee of completeness or correctness; please watch out for typos.

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1. Consider a single particle in one dimension.

(a) If the position operator $\hat{x}$ is applied to a wavefunction $\phi(x)$, what is the result?

Hint/Solution →
The operator $\hat{x}$ simply multiplies the function by x:

$$\hat{x}\phi(x) \;=\; x\phi(x)$$

For example, operating on the harmonic oscillator wavefunction $A_0 e^{-x^2/2\sigma^2}$ gives

$$\hat{x}A_0 e^{-x^2/2\sigma^2} \;=\; A_0 x e^{-x^2/2\sigma^2}$$

(b) If the momentum operator $\hat{p}$ is applied to a wavefunction $\phi(x)$, what is the result?

Hint/Solution →
The momentum operator $\hat{p} = -i\hbar\partial_x = -i\hbar\frac{d}{dx}$ operating on $\phi(x)$ gives

$$\hat{p}\phi(x) \;=\; -i\hbar\partial_x\phi(x) \;=\; -i\hbar\frac{d}{dx}\phi(x) \;=\; -i\hbar\phi'(x)$$

For example, operating on the harmonic oscillator wavefunction $A_0 e^{-x^2/2\sigma^2}$ gives

$$\begin{aligned}\hat{p}A_0 e^{-x^2/2\sigma^2} &= -i\hbar\frac{d}{dx}\left(A_0 e^{-x^2/2\sigma^2}\right)\\ &= -i\hbar A_0\left(-\frac{2x}{2\sigma^2}\right)e^{-x^2/2\sigma^2} \;=\; \frac{i\hbar A_0}{\sigma^2}\,x\,e^{-x^2/2\sigma^2}\end{aligned}$$

(c) Apply the commutator $[\hat{x}, \hat{p}]$ on an arbitrary wavefunction (arbitrary function of x). Hence evaluate the operator $[\hat{x}, \hat{p}]$.

Hint/Solution →

$$
\begin{aligned}
[\hat{x}, \hat{p}] f(x) &= (\hat{x}\hat{p} - \hat{p}\hat{x}) f(x) = \hat{x}\hat{p} f(x) - \hat{p}\hat{x} f(x) \\
&= x \left(-i\hbar \frac{d}{dx} \right) f(x) - \left(-i\hbar \frac{d}{dx} \right) x f(x) = -i\hbar x f'(x) + i\hbar \frac{d}{dx} [x f(x)] \\
&= -i\hbar x f'(x) + i\hbar [f(x) + x f'(x)]
\end{aligned}
$$

Thus

$$
[\hat{x}, \hat{p}] f(x) = i\hbar f(x) \qquad \Longrightarrow \qquad [\hat{x}, \hat{p}] = i\hbar
$$

i.e., the operating with $[\hat{x}, \hat{p}]$ on a function involves multiplying the function by the constant $i\hbar$.

(d) Define the operator $\hat{M} \equiv \frac{d^2}{dx^2} \equiv \partial_x^2$. (As usual I am sloppy about distinguishing full and partial derivatives.) By applying on an arbitrary function of x, evaluate the commutator $[\hat{x}, \hat{M}]$.

Hint/Solution →

$$
\begin{aligned}
[\hat{x}, \hat{M}] f(x) &= \hat{x}\hat{M} f(x) - \hat{M}\hat{x} f(x) \\
&= x\partial_x^2 f(x) - \partial_x^2 [x f(x)] = x f''(x) - \partial_x [f(x) + x f'(x)] \\
&= x f''(x) - [f'(x) + f'(x) + x f''(x)] = -2 f'(x)
\end{aligned}
$$

Thus

$$
[\hat{x}, \hat{M}] = -2\frac{d}{dx} = -2\partial_x
$$

2. The Hamiltonian of a system has energy eigenvalues ϵ_j and the corresponding orthornormal eigenstates are $|\phi_j\rangle$. The system is prepared in the initial state

$$|\psi(0)\rangle = \frac{1}{2}|\phi_2\rangle + \frac{\sqrt{3}}{2}|\phi_5\rangle$$

Here $|\psi(t)\rangle$ represents the wavefunction at time t.

(a) If the energy of the system is measured, what are the possible results? With what probabilities will these values be found?

Hint/Solution $\rightarrow$

This question is about measurement, Review of measurement in QM:

A measurement of observable Y necessarily yields one of the eigenvalues of the corresponding operator $\hat{Y}$. If $\hat{Y}|w_n\rangle = y_n|w_n\rangle$, the only possible outcomes of a measurement are y_n. The probability of finding the outcome y_n is

$$p(y_n) = \left|\langle w_n|\psi\rangle\right|^2$$

where $|\psi\rangle$ is the state of the system.

If $|\psi\rangle$ is expanded in the set $\{|w_n\rangle\}$ as $|\psi\rangle = \sum_n c_n|w_n\rangle$, then the probability of outcome y_j is $\left|\langle w_j|\psi\rangle\right|^2 = |c_j|^2$.

What if only a few of the c_j's are nonzero? Then only the corresponding eigenvalues, y_j, have nonzero probabilities to appear as the result of a measurement of Y.

Now to the present probelm.

The measured observable is energy, whose corresponding operator is the Hamiltonian. Luckily, the wavefunction is given as an expansion in the eigenstates of the Hamiltonian, with only two terms, i.e., only two nonzero coefficients in the expansion.

Therefore, the only possible values of the energy that can be found in a measurement on this state are ϵ_2 and ϵ_5. These values occur in a measurement with probabilities

$$\left|\frac{1}{2}\right|^2 = \frac{1}{4} \qquad \text{and} \qquad \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}$$

(b) Assume in the following that no measurement is performed at $t = 0$. Instead, the system is allowed to evolve starting from the state $|\psi(0)\rangle$ given above.

What is the state of the system, $|\psi(t)\rangle$, at a later time t?

Hint/Solution $\rightarrow$

Each eigenstate evolves with it's characteristic phase factor, so the time-dependence can be obtained by including phase factors with each eigenstate.

$$|\psi(t)\rangle \;=\; \frac{1}{2}\,|\phi_2\rangle e^{-i\epsilon_2 t/\hbar} \;+\; \frac{\sqrt{3}}{2}\,|\phi_5\rangle e^{-i\epsilon_5 t/\hbar}$$

Formally this can be derived as follows:

$$|\psi(t)\rangle \;=\; U(0,t)\,|\psi(0)\rangle \;=\; e^{-\frac{it}{\hbar}\hat{H}}\,|\psi(0)\rangle$$

$$= e^{-\frac{it}{\hbar}\hat{H}}\left(\frac{1}{2}\,|\phi_2\rangle \;+\; \frac{\sqrt{3}}{2}\,|\phi_5\rangle\right) \;=\; \frac{1}{2}e^{-\frac{it}{\hbar}\epsilon_2}\,|\phi_2\rangle \;+\; \frac{\sqrt{3}}{2}e^{-\frac{it}{\hbar}\epsilon_5}\,|\phi_5\rangle$$

as claimed above. In the last step we have used

$$\hat{H}\,|\phi_i\rangle = \epsilon_i\,|\phi_i\rangle \qquad \Longrightarrow \qquad e^{-\frac{it}{\hbar}\hat{H}}\,|\phi_i\rangle = e^{-\frac{it}{\hbar}\epsilon_i}\,|\phi_i\rangle$$

(c) The operator $\hat{B}$ has zero expectation value in each of the eigenstates, i.e., $\langle \phi_j | \hat{B} | \phi_j \rangle = 0$ for all j. In addition, the so-called off-diagonal matrix elements are all equal and real:

$$\langle \phi_i | \hat{B} | \phi_j \rangle = \beta \quad \text{whenever} \quad i \neq j.$$

Calculate $\langle \hat{B} \rangle$ as a function of time in the state $|\psi(t)\rangle$, i.e., calculate $\langle \hat{B}(t) \rangle = \langle \psi(t) | \hat{B} | \psi(t) \rangle$. Describe in a sentence the time-dependence of the expectation value of the observable B. If there is a frequency involved, what is it?

Hint/Solution →

The dual/bra vector corresponding to $|\psi(t)\rangle$ is

$$\langle \psi(t) | = \frac{e^{+i\epsilon_2 t/\hbar}}{2} \langle \phi_2 | + \frac{\sqrt{3}}{2} e^{+i\epsilon_5 t/\hbar} \langle \phi_5 |$$

$$\begin{aligned}
\langle \hat{B}(t) \rangle &= \langle \psi(t) | \hat{B} | \psi(t) \rangle \\
&= \left(\frac{e^{+i\epsilon_2 t/\hbar}}{2} \langle \phi_2 | + \frac{\sqrt{3}}{2} e^{+i\epsilon_5 t/\hbar} \langle \phi_5 | \right) \hat{B} \left(\frac{e^{-i\epsilon_2 t/\hbar}}{2} | \phi_2 \rangle + \frac{\sqrt{3}}{2} e^{-i\epsilon_5 t/\hbar} | \phi_5 \rangle \right) \\
&= \frac{1}{4} \langle \phi_2 | \hat{B} | \phi_2 \rangle + \frac{3}{4} \langle \phi_5 | \hat{B} | \phi_5 \rangle \\
&\qquad + \frac{\sqrt{3}}{4} \left(\langle \phi_2 | \hat{B} | \phi_5 \rangle e^{i(\epsilon_2 - \epsilon_5)t/\hbar} + \langle \phi_5 | \hat{B} | \phi_2 \rangle e^{i(\epsilon_5 - \epsilon_2)t/\hbar} \right) \\
&= \frac{1}{4} \cdot 0 + \frac{3}{4} \cdot 0 + \frac{\sqrt{3}\beta}{4} \times 2 \cos\left(\frac{\epsilon_5 - \epsilon_2}{\hbar} t \right) \\
&= \frac{\sqrt{3}\beta}{2} \cos\left(\frac{\epsilon_5 - \epsilon_2}{\hbar} t \right)
\end{aligned}$$

The expectation value of B oscillates around zero, with frequency

$$\frac{\epsilon_5 - \epsilon_2}{\hbar}$$

(d) Why are $\langle \phi_i | \hat{B} | \phi_j \rangle$ called the off-diagonal matrix elelments whenever $i \neq j$?

Hint/Solution →
If the set $\{|\phi_j\rangle\}$ is used as the basis, then $\hat{B}$ will be represented by a matrix whose elements are $B_{ij} = \langle \phi_i | \hat{B} | \phi_j \rangle$. For $i = j$, we get the elements on the diagonal of the matrix. When the indices are not equal, this quantity gives the elements of the matrix which are not on the diagonal, i.e., the off-diagonal matrix elements.

(e) A measurement of energy is performed at time $t = t_0$ and yields the value ϵ_5. What is the state of the system immediately after this measurement? What is the state of the system at a later time $t > t_0$

Hint/Solution →
At time $t = t_0$, the system is in state

$$|\psi(t_0)\rangle = \frac{1}{2} |\phi_2\rangle e^{-i\epsilon_2 t_0/\hbar} + \frac{\sqrt{3}}{2} |\phi_5\rangle e^{-i\epsilon_5 t_0/\hbar}$$

If a measurement finds the energy to be ϵ_5, the state will be projected onto the corresponding eigenstate, hence will turn into

$$\left|\psi(t_0 + 0^+)\right\rangle = |\phi_5\rangle e^{-i\epsilon_5 t_0/\hbar}$$

Note that the wavefunction has been normalized again. Here 0^+ is meant to be a negligibly small positive number.

After the measurement, the system is in an eigenstate, and hence evolves with a sinlge phase factor corresponding to that eigenstate:

$$|\psi(t > t_0)\rangle = \left[|\phi_5\rangle e^{-i\epsilon_5 t_0/\hbar} \right] \exp\left[-i\frac{\epsilon_5}{\hbar}(t - t_0) \right] = |\phi_5\rangle e^{-i\epsilon_5 t/\hbar}$$

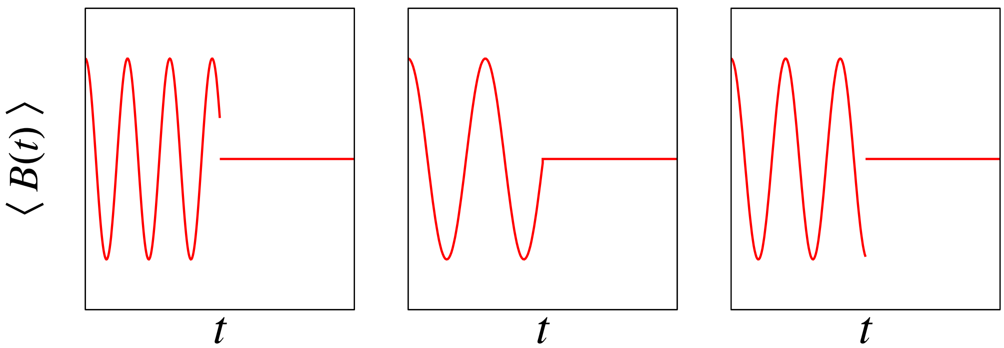
(f) What will be the behaviour of $\langle \hat{B} \rangle$ after the measurement, at times $t > t_0$? Justify/explain your answer.

Hint/Solution $\rightarrow$
After the measurement $t > t_0$, the state is in the single eigenstate $|\phi_5\rangle$ (modulo phase factor). Hence the expectation value of $\hat{B}$ is zero, since $\hat{B}$ is given to have zero expectation value in every eigenstate.

$$\langle \hat{B}(t > t_0) \rangle \;=\; 0$$

(g) Sketch a possible plot of $\langle \hat{B} \rangle$ versus time, running from $t = 0$ to $t = 2t_0$.

Hint/Solution $\rightarrow$
The curve follows a cosine until time t_0 and then changes sharply to zero. For $t > t_0$, the $\langle \hat{B} \rangle(t)$ stays at zero. Thus, unless $\langle \hat{B} \rangle(t)$ was coincidentally passing through 0 precisely at the instant $t = t_0$, there is a discontinuity (jump) at that point.
Below are possible plots, running from $t = 0$ to $t = 2t_0$.



3. Show that the eigenvalues of a unitary operator have unit modulus, i.e., that any eigenvalue of a unitary operator is a pure phase $e^{i\lambda}$.

Hint/Solution →

If $\hat{A}$ is unitary, then $\hat{A}^\dagger \hat{A} = \hat{A}\hat{A}^\dagger = \hat{1}$, where $\hat{1}$ is the identity operator. Let's take the eigenvalue equation

$$\hat{A}\,|\chi\rangle = \alpha\,|\chi\rangle$$

so that α is an eigenvalue. We have to show that $|\alpha| = 1$.

The dual (adjoint) of the above relation is

$$\langle\chi|\,\hat{A}^\dagger = \alpha^*\langle\chi|$$

Multiplying both the sides of the second (bra) equation on the two sides of the first (ket) equation gives us

$$\langle\chi|\,\hat{A}^\dagger\hat{A}\,|\chi\rangle \;=\; \alpha\alpha^*\langle\chi|\chi\rangle \quad\Longrightarrow\quad \langle\chi|\chi\rangle \;=\; |\alpha|^2\langle\chi|\chi\rangle$$
$$\Longrightarrow\quad |\alpha|^2 = 1 \quad\Longrightarrow\quad |\alpha| = 1$$

Thus α has unit modulus.

If a complex number is expressed in amplitude-phase form ($x + iy = re^{i\phi}$), then the amplitude is the modulus: $r = |x + iy|$. Hence if a complex number has unit modulus, it is a pure phase. Thus we have shown that the eigenvalue of a unitary matrix is a pure phase of the form $e^{i\lambda}$.

4. Recall for a spin-1/2 system:

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

We will consider the time-dependent wavefunction

$$|\psi(t)\rangle = \begin{pmatrix}\alpha e^{-i\beta t}\\ \sqrt{3}\alpha\end{pmatrix}$$

where α and β are positive real constants.

(a) Calculate α so that $|\psi(t)\rangle$ is normalized.

Hint/Solution →

$$\langle\psi(t)|\psi(t)\rangle = \left|\alpha e^{-i\beta t}\right|^2 + \left|\sqrt{3}\alpha\right|^2 = \left|\alpha\right|^2 + 3\left|\alpha\right|^2 = 4\left|\alpha\right|^2$$

Since β is real, we have used $\left|e^{-i\beta t}\right|^2 = 1$ above. Since α is also realy, we have $\langle\psi(t)|\psi(t)\rangle = 4\alpha^2$. Setting $\langle\psi(t)|\psi(t)\rangle = 1$ gives

$$\alpha = \frac{1}{2}$$

Thus

$$|\psi(t)\rangle = \begin{pmatrix}\frac{1}{2}e^{-i\beta t}\\ \frac{\sqrt{3}}{2}\end{pmatrix} \qquad \text{and} \qquad \langle\psi(t)| = \begin{pmatrix}\frac{1}{2}e^{+i\beta t} & \frac{\sqrt{3}}{2}\end{pmatrix}$$

(b) Calculate $\langle\hat{S}_z\rangle$ as a function of time in the state $|\psi(t)\rangle$, i.e., calculate

$$\langle\hat{S}_z(t)\rangle = \langle\psi(t)|\,\hat{S}_z\,|\psi(t)\rangle .$$

Plot the expectation value of S_z (in the state $|\psi(t)\rangle$) as a function of time.

Hint/Solution →

$$\hat{S}_z\,|\psi(t)\rangle = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{2}e^{-i\beta t}\\ \frac{\sqrt{3}}{2}\end{pmatrix} = \begin{pmatrix}\frac{\hbar}{4}e^{-i\beta t}\\ -\frac{\sqrt{3}\hbar}{4}\end{pmatrix}$$

$$\implies \langle\psi(t)|\,\hat{S}_z\,|\psi(t)\rangle \;=\; \begin{pmatrix}\frac{1}{2}e^{+i\beta t} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}\frac{\hbar}{4}e^{-i\beta t}\\ -\frac{\sqrt{3}\hbar}{4}\end{pmatrix} \;=\; \frac{\hbar}{8}-\frac{3\hbar}{8} \;=\; -\frac{\hbar}{4}$$

Plotting? This is an exercise in plotting a constant function.

(c) Calculate $\langle\hat{S}_x\rangle$ as a function of time in the state $|\psi(t)\rangle$, i.e., calculate $\langle\hat{S}_x(t)\rangle \;=\; \langle\psi(t)|\,\hat{S}_x\,|\psi(t)\rangle$. Plot the expectation value of S_x, in the state $|\psi(t)\rangle$, as a function of time.

Hint/Solution $\rightarrow$

$$\hat{S}_x\,|\psi(t)\rangle \;=\; \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2}e^{-i\beta t}\\ \frac{\sqrt{3}}{2}\end{pmatrix} \;=\; \begin{pmatrix}\frac{\sqrt{3}\hbar}{4}\\ \frac{\hbar}{4}e^{-i\beta t}\end{pmatrix}$$

$$\begin{aligned}\implies \langle\psi(t)|\,\hat{S}_x\,|\psi(t)\rangle \;&=\; \begin{pmatrix}\frac{1}{2}e^{+i\beta t} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}\frac{\sqrt{3}\hbar}{4}\\ \frac{\hbar}{4}e^{-i\beta t}\end{pmatrix}\\ &=\; \frac{\sqrt{3}\hbar}{8}e^{+i\beta t}+\frac{\sqrt{3}\hbar}{8}e^{-i\beta t}\\ &=\; \frac{\sqrt{3}\hbar}{8}\left(e^{+i\beta t}+e^{-i\beta t}\right) \;=\; \frac{\sqrt{3}\hbar}{4}\cos(\beta t)\end{aligned}$$

Plotting this oscillatory function is left as an exercise. On your plot, please indicate the **period** of oscillations, and express this period as a function of β. Also indicate the magnitude of oscillations and express this as a function of $\hbar$.