Due on Friday, December 11th

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1. Consider a single particle on a one-dimensional line.

   (a) **[1 pt.]** If the position operator $\hat{x}$ is applied to a wavefunction $\phi(x)$, what is the result?

   (b) **[2 pts.]** If the momentum operator $\hat{p}$ is applied to a wavefunction $\phi(x)$, what is the result?

   (c) **[5 pts.]** Apply the commutator $[\hat{x}, \hat{p}]$ on an arbitrary wavefunction (arbitrary function of x). Hence evaluate the operator $[\hat{x}, \hat{p}]$.

   (d) **[5 pts.]** Define the operator $\hat{M} \equiv \dfrac{d^2}{dx^2} \equiv \partial_x^2$. (As usual I am sloppy about distinguishing full and partial derivatives.) By applying on an arbitrary function of x, evaluate the commutator $[\hat{x}, \hat{M}]$.

2. The Hamiltonian of a system has energy eigenvalues ϵ_j and the corresponding orthornormal eigenstates are $|\phi_j\rangle$. The system is prepared in the initial state

$$|\psi(0)\rangle \;=\; \frac{1}{2}\,|\phi_2\rangle \;+\; \frac{\sqrt{3}}{2}\,|\phi_5\rangle$$

   Here $|\psi(t)\rangle$ represents the wavefunction at time t.

   Note: The system is defined completely generally. Do not assume a particular dimension of the Hilbert space. The system could be any quantum system - one particle in three dimensions, a spin-1/2 object, a collection of fifty spin-1/2 objects, twenty particles in two dimensions,.... etc. It should not be necessary or even useful to assume a particular type of system. E.g., do not assume inner products to involve integrals over a single spatial coordinate or that states/kets are vectors of dimension 3.

   (a) **[2 pts.]** If the energy of the system is measured, what are the possible results? With what probabilities will these values be found?

   (b) **[2 pts.]** Assume in the following that no measurement is performed at $t = 0$. Instead, the system is allowed to evolve starting from the state $|\psi(0)\rangle$ given above.

   What is the state of the system, $|\psi(t)\rangle$, at a later time t?

(c) **[7 pts.]** The operator $\hat{B}$ has zero expectation value in each of the eigenstates, i.e., $\langle \phi_j | \hat{B} | \phi_j \rangle = 0$ for all j. In addition, the so-called off-diagonal matrix elements are all equal and real:

$$\langle \phi_i | \hat{B} | \phi_j \rangle \;=\; \beta \quad \text{whenever} \quad i \neq j.$$

Calculate $\langle \hat{B} \rangle$ as a function of time in the state $|\psi(t)\rangle$, i.e., calculate

$$\langle \hat{B}(t) \rangle \;=\; \langle \psi(t) | \hat{B} | \psi(t) \rangle$$

Describe in a sentence the time-dependence of the expectation value of the observable B. If there is a frequency involved, what is it?

(d) **[SELF]** Why are $\langle \phi_i | \hat{B} | \phi_j \rangle$ called the off-diagonal matrix elelments whenever $i \neq j$?

(e) **[4 pts.]** A measurement of energy is performed at time $t = t_0$ and yields the value ϵ_5. What is the state of the system immediately after this measurement? What is the state of the system at a later time $t > t_0$

(f) **[3 pts.]** What will be the behaviour of $\langle \hat{B} \rangle$ after the measurement, at times $t > t_0$? Justify/explain your answer.

(g) **[3 pts.]** Sketch a possible plot of $\langle \hat{B} \rangle$ versus time, running from $t = 0$ to $t = 2t_0$.

3. **[5 pts.]** Show that the eigenvalues of a unitary operator have unit modulus, i.e., that any eigenvalue of a unitary operator is a pure phase $e^{i\lambda}$, where λ is a real number.

4. Recall for a spin-1/2 system:

$$S_x \;=\; \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y \;=\; \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z \;=\; \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We will consider the time-dependent wavefunction

$$|\psi(t)\rangle \;=\; \begin{pmatrix} \alpha e^{-i\beta t} \\ \sqrt{3}\alpha \end{pmatrix}$$

where α and β are positive real constants.

(a) **[2 pts.]** Calculate α so that $|\psi(t)\rangle$ is normalized.

(b) **[4 pts.]** Calculate $\langle \hat{S}_z \rangle$ as a function of time in the state $|\psi(t)\rangle$, i.e., calculate

$$\langle \hat{S}_z(t) \rangle \;=\; \langle \psi(t)| \, \hat{S}_z \, |\psi(t)\rangle \; .$$

Plot the expectation value of S_z (in the state $|\psi(t)\rangle$) as a function of time.

(c) **[5 pts.]** Calculate $\langle \hat{S}_x \rangle$ as a function of time in the state $|\psi(t)\rangle$, i.e., calculate $\langle \hat{S}_x(t) \rangle \;=\; \langle \psi(t)| \, \hat{S}_x \, |\psi(t)\rangle$. Plot the expectation value of S_x, in the state $|\psi(t)\rangle$, as a function of time.