Due on Friday, December 18th

– – – – – – – – – – – – – – ⋆ – – – – – – – – – – – – –

1. **[4 pts.]** Show that an operator, when expressed as a matrix in the basis of its own eigenstates, is diagonal.

   Hint: Let's make this more specific: let $\{|w_n\rangle\}$ be the orthonormalized eigenstates of the operator $\hat{Y}$. Use the set $\{|w_n\rangle\}$ as basis set and express $\hat{Y}$ as a matrix in this representation. Show that this matrix is diagonal.

2. Consider a system of two spin-1/2 objects. (E.g., the spins of two ions trapped next to each other, or the electron and proton in a hydrogen atom, where we ignore spatial motion and concentrate on their spins.) The Hilbert space is 4-dimensional and is spanned by the basis

   $$|\uparrow\uparrow\rangle \qquad |\uparrow\downarrow\rangle \qquad |\downarrow\uparrow\rangle \qquad |\downarrow\downarrow\rangle$$

   with the obvious meanings: e.g., $|\uparrow\downarrow\rangle$ represents the state where the first spin has z-component up $(+\hbar/2)$ and the second spin has z-component down $(-\hbar/2)$.

   Using these four states as the basis set, a general state can be written as

   $$|\psi\rangle \;=\; \alpha\,|\uparrow\uparrow\rangle \;+\; \beta\,|\uparrow\downarrow\rangle \;+\; \gamma\,|\downarrow\uparrow\rangle \;+\; \delta\,|\downarrow\downarrow\rangle \;=\; \begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix}$$

   (a) **[2 pts.]** In the state $|\psi\rangle$, what is the probability of finding the first spin to have z-component up?

   (b) **[2 pts.]** What is the probability of finding the second spin to have z-component down?

   (c) **[2 pts.]** In a simultaneous measurement of the z-components of both the spins, one finds the first spin to be spin-up, and the second to be spin-down. What is the state of the system immediately after the measurement?

   (d) **[2 pts.]** Imagine instead a measurement of only the second spin. The result is found to be spin-up. What is the state of the system immediately after the measurement?

3. (a) **[4 pts.]** Consider the operator $\hat{A} = \frac{d}{dx}$. Which of the following functions are eigenfunctions of $\hat{A}$? (Show your calculation/argument.)

$$f_1(x) = e^{ikx} \qquad f_2(x) = ax \qquad f_3(x) = \cos(kx) \qquad f_4(x) = e^{-ax^2}$$

Here k and a are real constants. For those functions that are eigenfunctions, give the eigenvalue.

(b) **[5 pts.]** Consider the operator $\hat{B} = \frac{d^2}{dx^2}$. Which of the above functions are eigenfunctions of $\hat{B}$? For those functions that are eigenfunctions, give the eigenvalue.

4. The quantum harmonic oscillator.

We denote the orthonormalized energy eigenstates of the harmonic oscillator by $|n\rangle$, on which the creation/annihilation operators act as follows:

$$\hat{a}\,|n\rangle \;=\; \sqrt{n}\,|n-1\rangle \qquad \hat{a}^\dagger\,|n\rangle \;=\; \sqrt{n+1}\,|n+1\rangle$$

(a) **[7 pts.]** Express $\hat{x}^2$ in terms of $\hat{a}$ and $\hat{a}^\dagger$. Hence show that the uncertainty of the position in the state $|n\rangle$ is

$$\Delta x \;=\; \sigma\sqrt{n+\frac{1}{2}}$$

(b) **[7 pts.]** Express $\hat{p}^2$ in terms of $\hat{a}$ and $\hat{a}^\dagger$. Hence calculate the uncertainty of the momentum, Δp, in the state $|n\rangle$.

(c) **[3 pts.]** Find the uncertainty product $\Delta x \Delta p$ in the state $|n\rangle$. Also express this quantity as a function of the eigenenergy E_n.

(d) **[4 pts.]** If the system wavefunction at time $t=0$ is

$$|\psi(0)\rangle \;=\; \frac{\sqrt{3}}{2}\,|2\rangle - \frac{i}{2}\,|3\rangle$$

then what is the wavefunction at a later time $t=T$? Your answer should contain the oscillator frequency ω.

(e) **[8 pts.]** Given that the initial system wavefunction is

$$|\psi(0)\rangle \;=\; \frac{\sqrt{3}}{2}\,|2\rangle - \frac{i}{2}\,|3\rangle,$$

calculate the expectation value of position in the state $|\psi(t)\rangle$ as a function of time. Sketch a plot of $\langle\psi(t)|\,\hat{x}\,|\psi(t)\rangle$ against time.

5. (a) **[SELF]** Show that for any three operators $\hat{A}$, $\hat{B}$, and $\hat{C}$,

$$[\hat{A}\hat{B},\hat{C}] \;=\; \hat{A}[\hat{B},\hat{C}]+[\hat{A},\hat{C}]\hat{B}$$

(b) **[SELF]** Use the relation $[\hat{x},\hat{p}] = i\hbar$ and the result above to prove by mathematical induction the identity

$$[\hat{x}^n,\hat{p}] = in\hbar\hat{x}^{n-1}$$

for any positive integer n.

(If unfamiliar, please look up "mathematical induction", a standard technique for proving statements for all positive integers n.)