1. The set $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, |\phi_4\rangle\}$ is an orthonormal basis set for a 4-dimensional Hilbert space. The states $|W\rangle$, $|\psi\rangle$, $|\theta\rangle$ are defined as

$$|W\rangle = \alpha|\phi_1\rangle + i\alpha|\phi_2\rangle - \alpha|\phi_3\rangle + \alpha|\phi_4\rangle ,$$

$$|\psi\rangle = \frac{i}{2}|\phi_1\rangle - \frac{\sqrt{3}}{2}|\phi_3\rangle , \qquad |\theta\rangle = \frac{1}{\sqrt{3}}|\phi_1\rangle - \frac{i}{\sqrt{3}}|\phi_2\rangle + \frac{i}{\sqrt{3}}|\phi_4\rangle .$$

(a) What does 'orthonormal' mean, in terms of the inner product between $|\phi_i\rangle$ and $|\phi_j\rangle$? (You can use an equation or explain in words.)

[5 marks]

(b) Find the constant α such that $|W\rangle$ is normalized. **[6 marks]**

(c) Calculate the inner product $\langle\psi|\theta\rangle$. **[6 marks]**

(d) Using the representation

$$|\phi_1\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad |\phi_2\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad |\phi_3\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |\phi_4\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

express $|\psi\rangle$ and $|\theta\rangle$ as column vectors. Also write down $\langle\psi|$ and $\langle\theta|$ as row vectors.

[6 marks]

(e) Using the same representation, express the operator $\hat{A} = |\psi\rangle\langle\theta|$ as a matrix. Determine whether or not $\hat{A}$ is hermitian. **[12 marks]**

(f) Calculate the expectation value of $\hat{B} = |\phi_1\rangle\langle\phi_1|$ in the state $|\psi\rangle$.

[15 marks]

2. (a) Prove that, if $|w_1\rangle$ and $|w_2\rangle$ are eigenstates of a hermitian operator corresponding to distinct eigenvalues, then they are orthogonal to each other.

[12 marks]

(b) The functions $\phi_1(x)$ and $\phi_2(x)$ are found to be solutions of the time-independent Schroedinger equation, $\hat{H}\psi(x) = E\psi(x)$. The corresponding energies are E_1 and E_2 respectively. Show that

$$f(x,t) = A_1\phi_1(x)e^{-iE_1t/\hbar} + A_2\phi_2(x)e^{-iE_2t/\hbar}$$

is a solution of the corresponding time-dependent Schroedinger equation.

[18 marks]

(c) Write down the time-independent Schroedinger equation for a particle of mass m confined to the x-axis, but not subject to any potential. (A free particle in one dimension.) Show that the function

$$\phi(x) = A\sin(kx)$$

is a solution of this equation. Highlight the relationship between the constant k and the energy of the particle.

[12 marks]

(d) For a single particle in a one-dimensional harmonic oscillator potential, $V(x) = \frac{1}{2}m\omega^2x^2$, what are the energy eigenvalues of the time-independent Schroedinger equation? Sketch plots of the four lowest-energy eigenstates as functions of position.

[8 marks]

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SAMPLE SOLUTIONS

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1. Question 1.

The set $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, |\phi_4\rangle\}$ is an orthonormal basis set for a 4-dimensional Hilbert space. The states $|W\rangle$, $|\psi\rangle$, $|\theta\rangle$ are defined as

$$|W\rangle \;=\; \alpha\,|\phi_1\rangle + i\alpha\,|\phi_2\rangle - \alpha\,|\phi_3\rangle + \alpha\,|\phi_4\rangle\ ,$$

$$|\psi\rangle \;=\; \frac{i}{2}\,|\phi_1\rangle - \frac{\sqrt{3}}{2}\,|\phi_3\rangle\ , \qquad |\theta\rangle \;=\; \frac{1}{\sqrt{3}}\,|\phi_1\rangle - \frac{i}{\sqrt{3}}\,|\phi_2\rangle + \frac{i}{\sqrt{3}}\,|\phi_4\rangle\ .$$

(a) What does 'orthonormal' mean, in terms of the inner product between $|\phi_i\rangle$ and $|\phi_j\rangle$? (You can use an equation or explain in words.)

[5 marks]

[Sample Answser:]

$$\langle\phi_i|\phi_j\rangle \;=\; \delta_{ij}$$

In words: the states $|\phi_i\rangle$ are each normalized and are orthogonal to each other.

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(b) Find the constant α such that $|W\rangle$ is normalized. **[6 marks]**

[Sample Answser:]

$$\langle W|W\rangle \;=\; |\alpha|^2 \;+\; |i\alpha|^2 \;+\; |-\alpha|^2 \;+\; |\alpha|^2 \;=\; 4|\alpha|^2$$

Thus

$$|\alpha|^2 \;=\; 1/4 \qquad \Longrightarrow \qquad \alpha \;=\; \frac{1}{2}\times\text{arbitrary phase factor}$$

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(c) Calculate the inner product $\langle\psi|\theta\rangle$. **[6 marks]**

[Sample Answswer:]

$$\langle\psi| \;=\; -\frac{i}{2}\langle\phi_1| - \frac{\sqrt{3}}{2}\langle\phi_3| \;, \qquad |\theta\rangle \;=\; \frac{1}{\sqrt{3}}|\phi_1\rangle - \frac{i}{\sqrt{3}}|\phi_2\rangle + \frac{i}{\sqrt{3}}|\phi_4\rangle \;.$$

Multiplying, only the term with $\langle\phi_1|\,|\phi_1\rangle$ gives a contribution; the others vanish by orthogonality. Thus

$$\langle\psi|\theta\rangle \;=\; -\frac{i}{2}\times\frac{1}{\sqrt{3}} \;=\; -\frac{i}{2\sqrt{3}}$$

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(d) Using the representation

$$|\phi_1\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad |\phi_2\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad |\phi_3\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad |\phi_4\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

express $|\psi\rangle$ and $|\theta\rangle$ as column vectors. Also write down $\langle\psi|$ and $\langle\theta|$ as row vectors.

[6 marks]

[Sample Answswer:]

$$|\psi\rangle = \begin{pmatrix} i/2 \\ 0 \\ -\sqrt{3}/2 \\ 0 \end{pmatrix}, \qquad |\theta\rangle = \begin{pmatrix} 1/\sqrt{3} \\ -i/\sqrt{3} \\ 0 \\ i/\sqrt{3} \end{pmatrix}$$

$$\langle\psi| = \begin{pmatrix} -i/2 & 0 & -\sqrt{3}/2 & 0 \end{pmatrix}, \qquad \langle\theta| = \begin{pmatrix} 1/\sqrt{3} & +i/\sqrt{3} & 0 & -i/\sqrt{3} \end{pmatrix}$$

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(e) Using the same representation, express the operator $\hat{A} = |\psi\rangle\langle\theta|$ as a matrix. Determine whether or not $\hat{A}$ is hermitian. **[12 marks]**

[Sample Answser:]

$$\hat{A} = |\psi\rangle\langle\theta| = \begin{pmatrix} i/2 \\ 0 \\ -\sqrt{3}/2 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{3}} & 0 & -\frac{i}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{i}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & 0 & \frac{+1}{2\sqrt{3}} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{i}{2} & 0 & +\frac{i}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hermitian? The matrix doesn't have the structure $A_{ij} = A_{ji}^*$, for example, $A_{12} = -\frac{1}{2\sqrt{3}}$ and $A_{21} = 0$. Hence not hermitian.

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(f) Calculate the expectation value of $\hat{B} = |\phi_1\rangle\langle\phi_1|$ in the state $|\psi\rangle$.

[15 marks]

[Sample Answser:]
The operator is

$$\hat{B} \;=\; |\phi_1\rangle\langle\phi_1| \;=\; \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix} \;=\; \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

The expectation value is

$$\langle\psi|\,\hat{B}\,|\psi\rangle \;=\; \begin{pmatrix}-i/2 & 0 & -\sqrt{3}/2 & 0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}i/2\\0\\-\sqrt{3}/2\\0\end{pmatrix}$$

$$=\; \begin{pmatrix}-i/2 & 0 & -\sqrt{3}/2 & 0\end{pmatrix}\begin{pmatrix}i/2\\0\\0\\0\end{pmatrix} \;=\; (-i/2)(i/2) \;=\; \frac{1}{4}$$

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2. Question 2.

(a) Prove that, if $|w_1\rangle$ and $|w_2\rangle$ are eigenstates of a hermitian operator corresponding to distinct eigenvalues, then they are orthogonal to each other.

[12 marks]

[Sample Answser:]
Let the operator be $\hat{A}$, and the corresponding eigenvalues be a_1 and a_2:

$$\hat{A}\,|w_1\rangle = a_1\,|w_1\rangle \qquad \hat{A}\,|w_2\rangle = a_2\,|w_2\rangle$$

Since $\hat{A}$ is hermitian, $\hat{A} = \hat{A}^\dagger$. Also, a_1 and a_2 are real. So, the first equation gives the dual relation:

$$\langle w_1|\,\hat{A}^\dagger = a_1^*\langle w_1| \qquad \Longrightarrow \qquad \langle w_1|\,\hat{A} = a_1\langle w_1|$$

Applying $\langle w_1|$ on the second eigenvalue equation:

$$\langle w_1|\,\hat{A}\,|w_2\rangle = \langle w_1|\,a_2\,|w_2\rangle \qquad \Longrightarrow \qquad a_1\langle w_1|w_2\rangle = a_2\langle w_1|w_2\rangle$$

Since $a_1 \neq a_2$, this implies $\langle w_1|w_2\rangle = 0$, i.e., that the two kets are orthogonal to each other.

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(b) The functions $\phi_1(x)$ and $\phi_2(x)$ are found to be solutions of the time-independent Schroedinger equation, $\hat{H}\psi(x) = E\psi(x)$. The corresponding energies are E_1 and E_2 respectively. Show that

$$f(x,t) \;=\; A_1\phi_1(x)e^{-iE_1t/\hbar} \;+\; A_2\phi_2(x)e^{-iE_2t/\hbar}$$

is a solution of the corresponding time-dependent Schroedinger equation.

[18 marks]

[**Sample Answser:**]
It's given that:

$$\hat{H}\phi_1(x) = E_1\phi_1(x) \qquad \hat{H}\phi_2(x) = E_1\phi_2(x)$$

The corresponding time-dependent Schroedinger equation is

$$\hat{H}\psi(x,t) \;=\; i\hbar\frac{\partial}{\partial t}\psi(x,t)$$

Trying the solution $\psi(x,t) = f(x,t)$ on both sides of this equation:

$$\begin{aligned}\text{left side} \;=\; \hat{H}f(x,t) \;&=\; \hat{H}\left[A_1\phi_1(x)e^{-iE_1t/\hbar} \;+\; A_2\phi_2(x)e^{-iE_2t/\hbar}\right]\\ &=\; A_1E_1\phi_1(x)e^{-iE_1t/\hbar} \;+\; A_2E_2\phi_2(x)e^{-iE_2t/\hbar}\end{aligned}$$

and

$$\begin{aligned}\text{right side} \;=\; i\hbar\frac{\partial}{\partial t}f(x,t) \;&=\; i\hbar\frac{\partial}{\partial t}\left[A_1\phi_1(x)e^{-iE_1t/\hbar} \;+\; A_2\phi_2(x)e^{-iE_2t/\hbar}\right]\\ &=\; A_1\phi_1(x)\left(i\hbar\frac{\partial}{\partial t}e^{-iE_1t/\hbar}\right) \;+\; A_2\phi_2(x)\left(i\hbar\frac{\partial}{\partial t}e^{-iE_2t/\hbar}\right)\\ &=\; A_1\phi_1(x)\left(i\hbar\frac{-iE_1}{\hbar}e^{-iE_1t/\hbar}\right) \;+\; A_2\phi_2(x)\left(i\hbar\frac{-iE_2}{\hbar}e^{-iE_2t/\hbar}\right)\\ &=\; A_1E_1\phi_1(x)e^{-iE_1t/\hbar} \;+\; A_2E_2\phi_2(x)e^{-iE_2t/\hbar}\end{aligned}$$

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(c) Write down the time-independent Schroedinger equation for a particle of mass m confined to the x-axis, but not subject to any potential. (A free particle in one dimension.) Show that the function

$$\phi(x) \;=\; A\sin(kx)$$

is a solution of this equation. Highlight the relationship between the constant k and the energy of the particle.

[12 marks]

[Sample Answser:]

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) \;=\; E\psi(x)$$

Trying the solution $\psi(x) = \phi(x) = A\sin(kx)$ on this equation:

$$\text{Left side} \;=\; -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}A\sin(kx) \;=\; A\frac{\hbar^2 k^2}{2m}\sin(kx)$$

$$\text{Right side} \;=\; EA\sin(kx)$$

The two sides can be equal at all x, if

$$E \;=\; \frac{\hbar^2 k^2}{2m}$$

Thus $A\sin(kx)$ is a solution, if $E = \hbar^2 k^2/2m$.

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(d) For a single particle in a one-dimensional harmonic oscillator potential, $V(x) = \frac{1}{2}m\omega^2 x^2$, what are the energy eigenvalues of the time-independent Schroedinger equation? Sketch plots of the four lowest-energy eigenstates as functions of position.

[8 marks]

[Sample Answser:]

$$E_n \;=\; \left(n + \frac{1}{2}\right)\hbar\omega$$

The plots should clearly have zero, one, two, three nodes.

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