1. Consider a particle of mass m on an infinite one-dimensional line.

   (a) Write down the operators for momentum and for kinetic energy.

   [4 marks]

   (b) Find out if the functions e^{ikx} and $\cos(kx)$ are eigenfunctions of the momentum operator. If so, find the corresponding eigenvalues.

   [6 marks]

   (c) Find out if the functions e^{ikx} and $\cos(kx)$ are eigenfunctions of the kinetic energy operator. If so, find the corresponding eigenvalues.

   [8 marks]

   (d) Find the commutator $[\hat{x}, \hat{p}]$ between the position and momentum operators.

   [10 marks]

   (e) Consider a normalized wavefunction $\psi_0(x)$. The expectation value of momentum in this state is p_0. Find the expectation value of momentum in the state

   $$\psi_{\text{new}}(x) \;=\; e^{iqx/\hbar}\psi_0(x)\;.$$

   Based on your result, explain the physical significance of multiplying a wavefunction by the phase factor $e^{iqx/\hbar}$.

   [12 marks]

   (f) If $f(x)$ and $g(x)$ are eigenfunctions of a hermitian operator $\hat{A}$ corresponding to *different* eigenvalues, find out whether any linear combination $C_1 f(x) + C_2 g(x)$ is also an eigenfunction of $\hat{A}$.

   [10 marks]

2. Consider a spin-1/2 system. The components of the spin are described by the operators

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Consider the state $|\phi\rangle = \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$.

Given that the state is normalized, determine the most general expression for α.

[7 marks]

(b) Calculate the expectation value of S_z in the state $|\phi\rangle$.

[9 marks]

(c) Calculate the uncertainty of S_z in the state $|\phi\rangle$.

[12 marks]

(d) What are the possible values that can be obtained in a measurement of S_y? Derive the possible values using the matrix $\hat{S}_y$.

[10 marks]

(e) A measurement of S_y yields a negative value. What is the state of the system immediately after the measurement?

[12 marks]

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SAMPLE SOLUTIONS

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1. Question 1.

   Consider a particle of mass m on an infinite one-dimensional line.

   (a) Question 1(a)

   Write down the operators for momentum and for kinetic energy.

   [4 marks]

   [Sample Answser:]

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} = -i\hbar\partial_x \qquad \hat{T} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} = -\frac{\hbar^2}{2m}\partial_x^2$$

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   (b) Question 1(b)

   Find out if the functions e^{ikx} and $\cos(kx)$ are eigenfunctions of the momentum operator. If so, find the corresponding eigenvalues.

   [6 marks]

   [Sample Answser:]

   e^{ikx} an eigenfunction??

$$\hat{p}e^{ikx} = -i\hbar\partial_x e^{ikx} = -i\hbar(ik)e^{ikx} = \hbar k e^{ikx}$$

$$\implies \text{Yes, eigenfunction, with eigenvalue } \hbar k.$$

   $\cos(kx)$ an eigenfunction??

$$\hat{p}\cos(kx) = -i\hbar\partial_x\cos(kx) = -i\hbar(-k)\sin(kx) = i\hbar k\sin(kx)$$

$$\implies \text{Not an eigenfunction.}$$

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(c) Question 1(c)

Find out if the functions e^{ikx} and $\cos(kx)$ are eigenfunctions of the kinetic energy operator. If so, find the corresponding eigenvalues.

[8 marks]

[Sample Answser:]

e^{ikx} an eigenfunction??

$$\hat{T}e^{ikx} = -\frac{\hbar^2}{2m}\partial_x^2 e^{ikx} = -\frac{\hbar^2}{2m}(ik)^2 e^{ikx} = \frac{\hbar^2 k^2}{2m}e^{ikx}$$

$$\Longrightarrow \text{ Yes, eigenfunction, with eigenvalue } \frac{\hbar^2 k^2}{2m}.$$

$\cos(kx)$ an eigenfunction??

$$\hat{T}\cos(kx) = -\frac{\hbar^2}{2m}\partial_x^2 \cos(kx) = -\frac{\hbar^2}{2m}(-k)\partial_x \sin(kx)$$

$$= -\frac{\hbar^2}{2m}(-k)(k)\cos(kx) = \frac{\hbar^2 k^2}{2m}\cos(kx)$$

$$\Longrightarrow \text{ Yes, eigenfunction, with eigenvalue } \frac{\hbar^2 k^2}{2m}.$$

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(d) Question 1(d)

Find the commutator $[\hat{x}, \hat{p}]$ between the position and momentum operators.

[10 marks]

[Sample Answser:]
Applying the operator $[\hat{x}, \hat{p}]$ on an arbitrary (wave)function $f(x)$, we obtain

$$[\hat{x}, \hat{p}]f(x) = (\hat{x}\hat{p} - \hat{p}\hat{x}) f(x) = \hat{x}\hat{p}f(x) - \hat{p}\hat{x}f(x)$$
$$= x\left(-i\hbar\frac{d}{dx}\right) f(x) - \left(-i\hbar\frac{d}{dx}\right) xf(x) = -i\hbar xf'(x) + i\hbar\frac{d}{dx}\left[xf(x)\right]$$
$$= -i\hbar xf'(x) + i\hbar\left[f(x) + xf'(x)\right]$$

Thus
$$[\hat{x}, \hat{p}]f(x) = i\hbar f(x) \qquad \Longrightarrow \qquad [\hat{x}, \hat{p}] = i\hbar$$
i.e., operating with $[\hat{x}, \hat{p}]$ on a function involves multiplying the function by the constant $i\hbar$.

(To look more consistent, one could have operators on both sides of the equation; sometimes this is written as

$$[\hat{x}, \hat{p}] = i\hbar\hat{1}$$

where $\hat{1}$ is the unit operator which leaves a function unchanged.)

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(e) Question 1(e)
Consider a normalized wavefunction $\psi_0(x)$. The expectation value of momentum in this state is p_0. Find the expectation value of momentum in the state

$$\psi_{\rm new}(x) = e^{iqx/\hbar}\psi_0(x) .$$

Based on your result, explain the physical significance of multiplying a wavefunction by the phase factor $e^{iqx/\hbar}$.

[12 marks]

[Sample Answser:]
We are given
$$\langle\hat{p}\rangle_{\psi_0} = \langle\psi_0|\hat{p}|\psi_0\rangle = p_0$$

We want to find the expectation value in the state $\psi_{\rm new}(x)$:

$$\langle\hat{p}\rangle_{\psi_{new}} = \langle\psi_{new}|\hat{p}|\psi_{new}\rangle = \int_{-\infty}^{\infty}\psi_{new}(x)^*\hat{p}\psi_{new}(x)dx$$
$$= -i\hbar\int_{-\infty}^{\infty}\psi_{new}(x)^*\frac{d}{dx}\psi_{new}(x)dx$$

Now

$$\frac{d}{dx}\psi_{new}(x) = \frac{d}{dx}\left[e^{iqx/\hbar}\,\psi_0(x)\right] = \frac{iq}{\hbar}e^{iqx/\hbar}\,\psi_0(x) + e^{iqx/\hbar}\frac{d}{dx}\psi_0(x)$$

so that

$$\psi_{new}(x)^*\frac{d}{dx}\psi_{new}(x)$$
$$= \left[e^{-iqx/\hbar}\,\psi_0(x)^*\right]\left(\frac{iq}{\hbar}e^{iqx/\hbar}\,\psi_0(x) + e^{iqx/\hbar}\frac{d}{dx}\psi_0(x)\right)$$
$$= \frac{iq}{\hbar}\left|\psi_0(x)\right|^2 + \psi_0(x)^*\frac{d}{dx}\psi_0(x)$$

Thus

$$\langle\hat{p}\rangle_{\psi_{new}} = -i\hbar\frac{iq}{\hbar}\int_{-\infty}^{\infty}\left|\psi_0(x)\right|^2dx - i\hbar\int_{-\infty}^{\infty}\psi_0(x)^*\frac{d}{dx}\psi_0(x)dx$$
$$= q + \int_{-\infty}^{\infty}\psi_0(x)^*\hat{p}\psi_0(x)dx = q + \langle\hat{p}\rangle_{\psi_0} = q + p_0$$

Multiplying by this phase factor $e^{iqx/\hbar}$ shifts the momentum by the quantity q, i.e., the wavefunction is 'boosted' by momentum q.

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(f) Question 1(f)

If $f(x)$ and $g(x)$ are eigenfunctions of an operator $\hat{A}$ corresponding to *different* eigenvalues, find out whether any linear combination $C_1f(x)+C_2g(x)$ is also an eigenfunction of $\hat{A}$.

[10 marks]

[Sample Answser:]
We are given

$$\hat{A}f(x) \;=\; a_1 f(x)\;, \quad \hat{A}g(x) \;=\; a_2 g(x)\;, \quad \text{with } a_1 \neq a_2.$$

Is the linear combination $C_1 f(x)+C_2 g(x)$ an eigenfunction? Let's try:

$$\hat{A}\Big[C_1 f(x)+C_2 g(x)\Big] \;=\; C_1\hat{A}f(x))+C_2\hat{A}g(x) \;=\; C_1 a_1 f(x)+C_2 a_2 g(x)$$

which is not a constant multiple of $C_1 f(x) + C_2 g(x)$, because $a_1 \neq a_2$. Hence $C_1 f(x) + C_2 g(x)$ is NOT an eigenfunction of $\hat{A}$ in general.

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2. Question 2.

   Consider a spin-1/2 system. The components of the spin are described by the operators

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

   (a) Question 2(a)

   Consider the state $|\phi\rangle = \begin{pmatrix}\alpha\\ -\alpha\end{pmatrix}$.

   Calculate α so that the state is normalized. Keep your answer general: do not assume α to be real.

   [7 marks]

   [Sample Answser:]

$$\langle\phi|\phi\rangle = \begin{pmatrix}\alpha^* & -\alpha^*\end{pmatrix}\begin{pmatrix}\alpha\\ -\alpha\end{pmatrix} = |\alpha|^2 + |\alpha|^2 = 2|\alpha|^2$$

   Normalization means $\langle\phi|\phi\rangle = 1$; hence

$$2|\alpha|^2 = 1 \qquad \Longrightarrow \qquad \alpha = \frac{1}{\sqrt{2}}e^{i\chi}$$

   The phase factor $e^{i\chi}$ is required becuase we can only determine the magnitude from the noramlization condition, if we are not allowed to assume that α is real. Here χ is an arbitrary real number.

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   (b) Question 2(b)

   Calculate the expectation value of S_z in the state $|\phi\rangle$.

   [9 marks]

[Sample Answser:]

$$
\begin{aligned}
\langle S_z \rangle \;=\; \langle \phi | \, S_z \, | \phi \rangle \;&=\; \begin{pmatrix} \alpha^* & -\alpha^* \end{pmatrix} \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix} \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix} \\
&=\; \begin{pmatrix} \alpha^* & -\alpha^* \end{pmatrix} \begin{pmatrix} \alpha\hbar/2 \\ \alpha\hbar/2 \end{pmatrix} \;=\; |\alpha|^2 \frac{\hbar}{2} - |\alpha|^2 \frac{\hbar}{2} \;=\; 0
\end{aligned}
$$

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(c) Question 2(c)
Calculate the uncertainty of S_z in the state $|\phi\rangle$.

[12 marks]

[**Sample Answser:**]
We need to calculate

$$\Delta S_z \;=\; \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2}$$

Note that $S_z^2 \;=\; \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \;=\; \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Thus

$$\langle S_z^2\rangle \;=\; \begin{pmatrix}\alpha^* & -\alpha^*\end{pmatrix}\frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha\\ -\alpha\end{pmatrix} \;=\; \frac{\hbar^2}{4}\begin{pmatrix}\alpha^* & -\alpha^*\end{pmatrix}\begin{pmatrix}\alpha\\ -\alpha\end{pmatrix} \;=\; \frac{\hbar^2}{4}$$

since the state is normalized.
The uncertainty is therefore

$$\Delta S_z \;=\; \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} \;=\; \sqrt{\frac{\hbar^2}{4} - 0^2} \;=\; \frac{\hbar}{2}$$

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(d) Question 2(d)

What are the possible values that can possibly be obtained in a measurement of S_y?

[10 marks]

[**Sample Answser:**]

The possible values that can be obtained are the eigenvalues of the operator. Thus we need to find the eigenvalues of

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Of course, this can be done in couple of different ways. If done correctly, the eigenvalues will be $\pm\hbar/2$. These are the possible values that can be obtained in a measurement of S_y.

A possible calculation: an eigenvalue λ of $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ must satisfy

$$\left| \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \lambda I \right| = 0$$

$$\implies \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0$$

$$\implies \lambda^2 - 1 = 0 \qquad \implies \lambda = \pm 1$$

$$\implies \text{The eigenvalues of } \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ are } \pm 1$$

$$\implies \text{The eigenvalues of } \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ are } \pm\hbar/2.$$

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(e) Question 2(e)

A measurement of S_y yields a negative value. What is the state of the system immediately after the measurement?

[12 marks]

[Sample Answser:]

The state is the eigenstate corresponding to the eigenvalue $-\hbar/2$, since the system will collapse to this eigenstate if the result of a measurement is $-\hbar/2$. So, this problem involves calculating the eigenstate corresponding to the eigenvalue $-\hbar/2$.

Calling this eigenstate $\begin{pmatrix} \beta \\ \gamma \end{pmatrix}$, we obtain

$$\begin{aligned} \hat{S}_y \begin{pmatrix} \beta \\ \gamma \end{pmatrix} &= -\frac{\hbar}{2} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \\ \implies \quad \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} &= -\frac{\hbar}{2} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \\ \implies \quad \begin{pmatrix} -i\gamma \\ i\beta \end{pmatrix} &= -\begin{pmatrix} \beta \\ \gamma \end{pmatrix} \end{aligned}$$

which gives $\gamma = -i\beta$, so that the eigenstate is

$$\begin{pmatrix} \beta \\ -i\beta \end{pmatrix}$$

The value of β can be calculated by normalization:

$$\beta \;=\; \frac{1}{\sqrt{2}} e^{i\chi} \; ; \qquad \chi \text{ is a real number}$$

Thus the state after the measurement is

$$\frac{1}{\sqrt{2}} e^{i\chi} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

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