1

Relativity

Resolves inconsistency of classical mech. with E&M

GENERAL RELATIVITY $\approx$ gravity

small speeds $v < c$

CLASSICAL (NEWTONIAN) Mechanics

(incomplete/approximate)

- inconsistent with E&M
- invalid for microscopic phenomena

large masses

QUANTUM MECHANICS (NONRELATIVISTIC)

small speeds

RELATIVISTIC Q.M.

$\approx$ Quantum Field Theory

Quantum Electrodynamics, Quantum chromodynamics

"Standard Model"

Still incomplete &

Does not incorporate gravity

* QM is less intuitive than Classical Mech, E&M
* Many new concepts, some confusing ["Nobody understands QM"]
* First few ($\simeq 6$) classes: broad overview, initial intro to several concepts/eqs., historical introduction, etc.

* Fundamental equation/law of Newtonian Mechanics: (one particle)

$$F = ma \quad \text{or} \quad -\frac{\partial V}{\partial x} = m\frac{d^2x}{dt^2} \quad 1D$$

$$\vec{F} = m\vec{a} \quad \text{or} \quad -\vec{\nabla}V = m\frac{d^2\vec{r}}{dt^2} \quad 3D$$

1D means: the particle is constrained to move along the x-direction.

→ If you know $x(0)$, $\dot{x}(0)$, you can calculate $\left\{\begin{array}{l} x(t) \\ \dot{x}(t) \end{array}\right.$

→ can be generalized:

* to $N \geq 1$ particles; one eq. for each pcle
* to constrained motion. (Lagrangian mechanics)

→ Impossible to prove Newton's laws; we believe it because it keeps explaining physical phenomena, in its regime of applicability (macroscopic objects, "slow" speeds)

③

* Fundamental equation of QM : $\boxed{\text{Schrödinger Eq.}}$

~~Most~~ General form: $\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$

Here ~~#~~ $\hbar = \frac{h}{2\pi}$ and $h \approx 6.63 \times 10^{-34}$ J.s
is PLANCK'S CONSTANT

Ψ is the wavefunction or state vector,
describes the state of the system.
⟶ SOMETIMES A FUNCTION, SOMETIMES A "VECTOR" !!

$\hat{H}$ is the Hamiltonian operator.

* Example: single particle in 1D:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t) = i\hbar \frac{\partial}{\partial t}\Psi(x,t)$$

- In this case, $\hat{H} = \underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}}_{=\hat{T}} + \underbrace{V(x)}_{=\hat{V}}$

$\hat{T}$ (kinetic energy operator)

$\hat{V}$ (potential energy operator)

- Ψ has been written as function of x and t

- The SE doesn't give x as a function of t, unlike in classical mechanics.

- Instead, $|\Psi(x,t)|^2$ gives the probability density for finding the particle at position x at time t.

- In QM, knowledge of a state provides <u>probabilistic</u> knowledge of physical variables.

* One important class of solutions to SE are <u>stationary</u> solutions.

For 1 pcle in 1D,
$$\Psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$$

Substituting is SE gives

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

$\longrightarrow$ Time-independent Schrödinger equation.

* $\hat{H}\Psi(t) = i\hbar \frac{\partial}{\partial t}\Psi(t)$ Time-dependent Schr. Eq. (TDSE)

Using $\Psi(t) = \Psi e^{-iEt/\hbar}$,

$\hat{H}\Psi = E\Psi$ Time-independent Schr. Eq.

* We'll study both in detail.

* Example: Spin-$\frac{1}{2}$ object (electron/proton/singly-charged ion) fixed in space

---

Ψ has no spatial dependence.

Can be expressed as a two-component "vector"

$$\Psi(t) = u(t)\,|\uparrow\rangle + v(t)\,|\downarrow\rangle$$

$$\Psi(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

u, v can be complex

$|\uparrow\rangle$ = state where spin points in +z direction

$|\downarrow\rangle$ = " " " " " −z "

In this case, Hamiltonian operator is 2X2 matrix, TDSE $(\hat{H}\psi = i\hbar \frac{\partial}{\partial t}\psi)$ has the form:

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = i\hbar \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix}$$

TIME-INDEP form from p. 11

* Looks very different from the case of particle on the x-axis ?

* The difference is in the dimension of HILBERT SPACE.

* The wavefunctions (state vectors, wavevectors) are members of a vector space called Hilbert space. * If Hilbert space is D dimensional, wavevector has D complex components.

different from Euclidean vectors.

* Two-state (spin-$\frac{1}{2}$) system :

$$\Psi(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

TDSE has the form: $\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} i\hbar \dot{u}(t) \\ i\hbar \dot{v}(t) \end{pmatrix}$

An important class of solutions: STATIONARY SOLUTIONS

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} u_0 e^{-iEt/\hbar} \\ v_0 e^{-iEt/\hbar} \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} e^{-iEt/\hbar}$$

Then SE becomes time-independent :

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = E \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

Stationary states are determined by an EIGENVALUE/EIGENVECTOR problem

* Spin-$\frac{1}{2}$ system: Hilbert space is 2-dimensional.

$$\psi = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad c_1, c_2 \text{ complex}$$

* Particle in 1D: $\psi(x)$

Has an <u>infinite</u> # of complex components

One at every x. !

Hilbert space is infinite-dimensional.

* Particle in 3D: $\psi(\vec{r})$

One complex component at every $\vec{r}$

Hilbert space is ∞-dimensional

* An <u>INNER PRODUCT</u> is defined in Hilbert space, analogous to dot product.

For a spin-$\frac{1}{2}$ system, if

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \text{ then}$$

$$\langle\varphi|\psi\rangle = \varphi^\dagger\psi = \begin{pmatrix}\varphi_1^* & \varphi_2^*\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$= \varphi_1^*\psi_1 + \varphi_2^*\psi_2$$

For system with D-dimensional Hilbert space: if

$$\varphi = \begin{pmatrix}\varphi_1\\ \vdots\\ \varphi_D\end{pmatrix} \qquad \psi = \begin{pmatrix}\psi_1\\ \vdots\\ \psi_D\end{pmatrix}$$

Then

$$\langle\varphi|\psi\rangle = \varphi_1^*\psi_1 + \varphi_2^*\psi_2 + \cdots \varphi_D^*\psi_D$$

* For pcle in 1 spatial dimension (infinite-dimensional Hilbert space)

Inner prod. of $\varphi(x)$ and $\psi(x)$ is

$$\int \varphi^*(x)\,\psi(x)\,dx$$

* Inner product is called "overlap" of wavefunctions. When overlap $= 0$, the wavefunctions are "orthogonal" to each other

* NORM and NORMALIZATION

For Euclidean vector, magnitude (norm) is square root of inner product with itself:

$$\vec{A} \cdot \vec{A} = A^2 \quad \Rightarrow \quad A = \sqrt{\vec{A} \cdot \vec{A}}$$

For wavevector, norm is (square root of inner product with itself: $\mathcal{N} = \langle \psi | \psi \rangle$

The norm is $\sqrt{\mathcal{N}}$.

Usually, we deal with wavevectors of unit norm:

$$\langle \psi | \psi \rangle = 1$$

Then the wavefunction is said to be

NORMALIZED

Multiplying a wavefunction by appropriate constant so that it becomes normalized $\longrightarrow$ NORMALIZING the wavefunction

9

* A BRIEF HISTORY of QM

1900 (Planck): Light is emitted in quantized lumps, having energy

$$E = hf = \hbar\omega \quad [f \text{ or } \nu]$$

$\omega = 2\pi f$ "angular frequency"

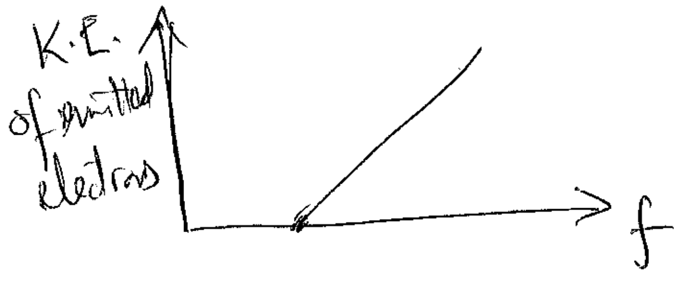
often called "frequency"

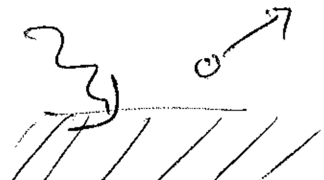
CORRECT description of BLACKBODY RADIATION

[Note: for everyday purposes, the packets ("photons") are so small that light can be regarded as continuous]

1905 (Einstein): Light also propagates and is absorbed in lumps. i.e., light is "inherently" quantized.

→ Explained the photoelectric effect.





For a light wave, $E = pc$

$$E = hf = \frac{hc}{\lambda} \Rightarrow \boxed{p = \frac{h}{\lambda}}$$

Momentum of photon.

Also $\frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$ is called the wavenumber:

$$\boxed{k = \frac{\omega}{c} = \frac{2\pi}{\lambda}}$$

Photon momentum $\boxed{p = \frac{h}{\lambda} = \hbar k}$

1913 (Bohr): Electrons in atoms have wavelike properties; hence only certain orbits are allowed. ⇒ "Explains" quantization of energy of atoms.

(Known from discrete atomic spectra.)

\# 1924 (de Broglie) All particles are associated with waves, with wavelength $\lambda = \frac{h}{p}$

"WAVE-PARTICLE DUALITY"

\# 1925 (Heisenberg) Matrix formulation of QM

\# 1925 (Schrödinger) Wave function formulation [Schr. equation

\# 1926 (Born) Probabilistic interpretation

\# 1926 (Dirac) Unification of viewpoints, relativistic QM

~~INTERLUDE for p. 6~~

~~One class of solutions:~~ $\cancel{\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} e^{-iEt/\hbar}}$

~~Then SE becomes time-indep.:~~

$$\cancel{\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}}$$

~~Stationary states determined by an EIGENVALUE problem.~~

Feynman Lectures Vol. 1, Ch. 41

Vol. 3 Sec. 4-5

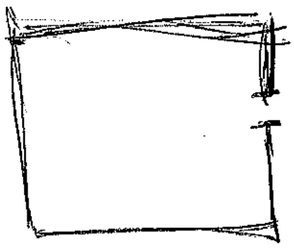
* BLACK-BODY RADIATION

"Black" body" → object that absorbs all EM. waves (light etc.) incident on it.

→ An idealization.

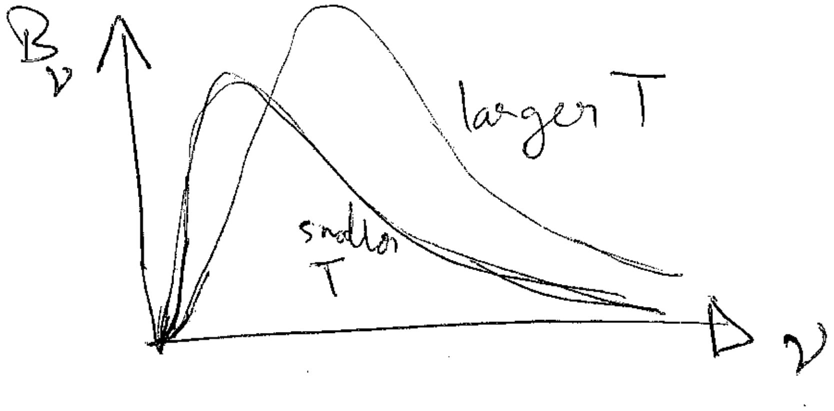
Experimental approximation: - black-coated surface,

- Better: large cavity with small aperture;



* When hot, emits EM. radiation, at all frequencies.

* Experiments performed, 19th century.



B_ν = spectral radiance per unit frequency

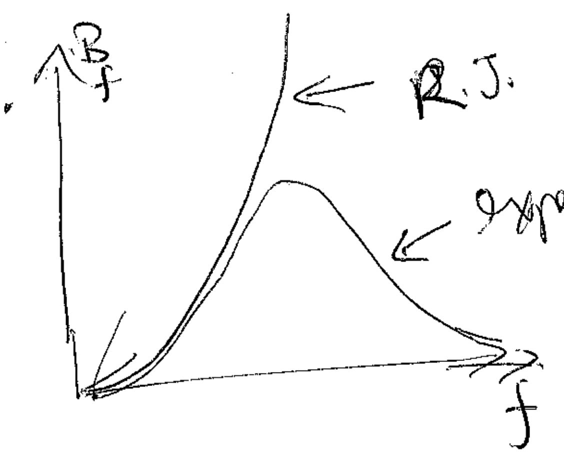
* Theory of B_ν: calculate # modes per unit frequency in cavity, multiply by angle, c etc. factors and by ENERGY PER MODE

WE OMIT these steps

(13) At temp. T, each mode carries energy ?

Classical: $E = k_B T$ per mode.

$$\Rightarrow \boxed{B_f = \frac{2 k_B T}{c^2} f^2}$$ Rayleigh Jeans formulas



WRONG!

at large f, $B_f \to \infty$!!

ULTRAVIOLET CATASTROPHE

* Correction to "mode per energy" required.

Classical: probability distribⁿ of energy:

$$P(E) \propto e^{-E/k_B T} \quad \text{with } E \text{ continuous}$$

$$\langle E \rangle = \frac{\int_0^\infty E e^{-E/k_B T}}{\int_0^\infty e^{-E/k_B T}} = k_B T \quad \text{SHOW}$$

Quantum: $P(E) = e^{-nhf/k_B T}$ discrete

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} e^{-nhf/k_B T} nhf}{\sum_{n=0}^{\infty} e^{-nhf/k_B T}} = \frac{hf}{e^{hf/k_B T} - 1}$$

$\sum x^n = 1/(1-x)$
$\sum n x^n = x/(1-x)^2$

$$\boxed{\begin{array}{l}\text{Use} \\ \sum x^n = \frac{1}{1-x} \\ \sum n x^n = \frac{x}{(1-x)^2}\end{array}}$$

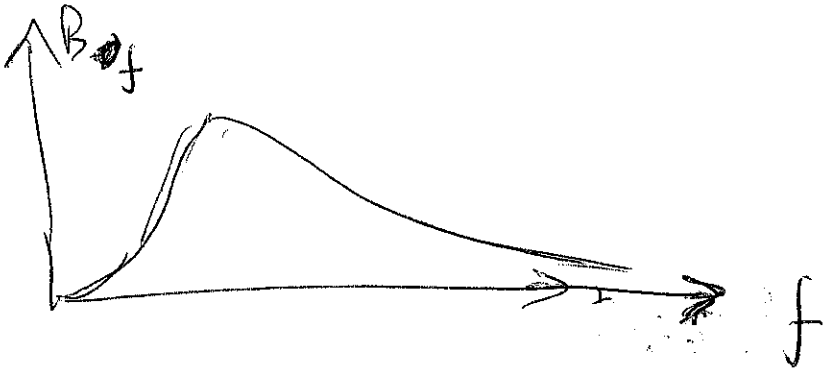
Replacing $k_B T \longrightarrow \dfrac{hf}{e^{hf/k_BT}-1}$ in R-J.

formula, Planck obtained the correct relation:

$$\boxed{B_f = \frac{2h}{c^2}\,\frac{f^3}{e^{hf/k_BT}-1}}$$

Planck distribution.

Matches expt.:



→ Lesson: radiation in cavity/emitter exists in packets of energy $E=hf$.

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* ~~QUANTIZATION OF ENERGY~~

~~Observed in 19th energy:~~

Feynman Vol. III, sec. 2-5

~~Frequencies emitted by atoms are DISCRETE.~~

~~→ Fixed values for every atomic species~~

NOTE ON: CONTINUOS PROBABILITY DISTRIBUTIONS VS. DISCRETE PROBABILITIES

* Average of a continuous variable.

$y \in [a,b]$; probability $P(y)$; i.e. $\int_a^b P(y)\,dy = 1$

Average (expectation value) $= \int_a^b dy\; y\, P(y)$

If Probability $\propto f(y)$? Have to normalize first:

$$P(y) = \frac{f(y)}{\int_a^b dy\, f(y)}$$

So $\bar{y} = \int dy\; y\, P(y) = \dfrac{\int dy\; y\, f(y)}{\int dy\, f(y)}$

* Average of a discrete variable.

$m \in \mathbb{N}$, probabilities $P(m)$; $\sum_{m=1}^{\infty} P(m) = 1$

Average $\bar{m} = \sum_{m=1}^{\infty} m\, P(m)$

If probability $\propto f_m$,

$$\bar{m} = \frac{\sum_m m\, f_m}{\sum_{m}^{\infty} f_m}$$

* QUANTIZATION OF ENERGY

Feynman Vol. III, Sec. 2-5

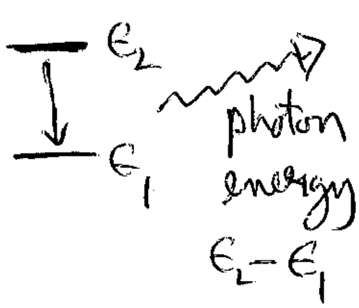
Observed in late 19th century:

- FREQUENCIES EMITTED BY ATOMS ARE DISCRETE
- FIXED VALUES for EVERY ATOMIC SPECIES

Quantum explanation:

① Energies of bound stationary states / orbits of electrons are DISCRETE. [Quantization of energy.]

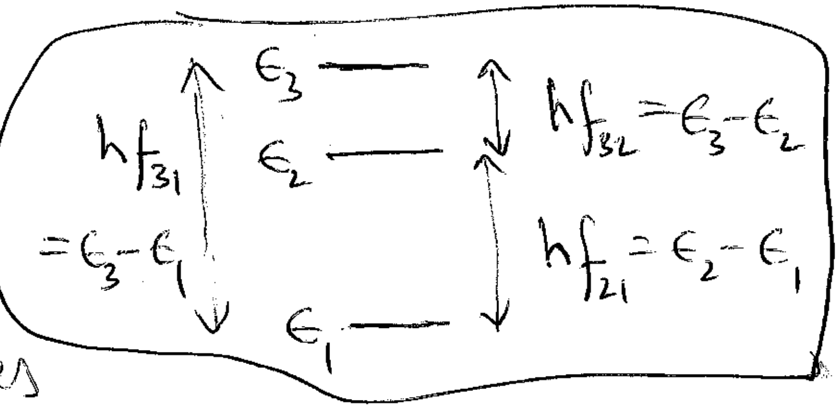
② Emission involves transition from higher → lower energy level.



③ Photon frequency $hf = \epsilon_2 - \epsilon_1$

An implication:

$$f_{31} = f_{32} + f_{21}$$



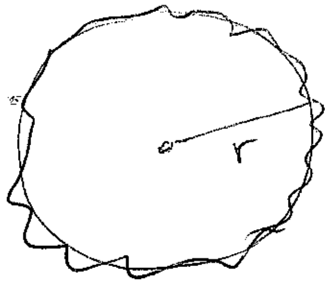
The discrete frequencies in atomic spectra often come in triplets $\{f_A, f_B, f_A + f_B\}$

→ Was noticed before QM. Explained by QM.

WHY SHOULD ENERGIES BE QUANTIZED?

- Quantization from the de Broglie "wave" picture

① Orbits of an electron



Constructive interference necessary for stationary solution: $2\pi r = n\lambda = \frac{nh}{p}$

[de Broglie w.length $\lambda = \frac{h}{p}$]

⇒ only some radii are allowed

⇒ only some energies are allowed ($\frac{1}{2}mv^2$)

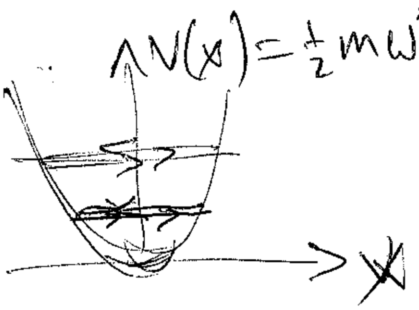
$$2\pi r = \frac{nh}{m_e v} \Rightarrow mvr = n\frac{h}{2\pi} = n\hbar$$

Angular momentum $L = mvr = n\hbar$

⇒ Bohr's model of the hydrogen atom is based on this quantization condition for angular momentum.

(17)

② Harmonic oscillator ~~[illegible]~~

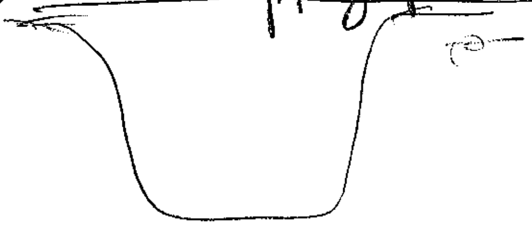


Constructive interference only for some path lengths $\Rightarrow$ only for some energies.

Energies turns out to be $E_n=\left(n+\frac{1}{2}\right)\hbar\omega$

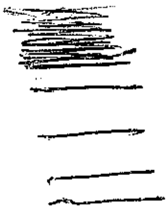
ω is sometimes called the "trapping frequency"

③ Other trapping potential



Energies of bound states are discrete

Energies of non-bound states ("scattering" states) are continuous (all energies possible)



For non-bound states, no "interference".

* Quantization from the SE

Stationary solutions are given by eigenvalue equations $H\psi=E\psi$

For finite-dimensional Hilbert spaces,

$$\begin{pmatrix} & & \\ & H & \\ & & \end{pmatrix}\begin{pmatrix} \\ \psi \\ \\ \end{pmatrix} = E\begin{pmatrix} \\ \psi \\ \\ \end{pmatrix}$$

only D eigenvalues for D-dimensional problem

$\Rightarrow$ obviously discrete energies.

For ∞-dim. Hilbert spaces, e.g., a pcle in 3D: $\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]\psi(\vec{r}) = E\psi(\vec{r})$

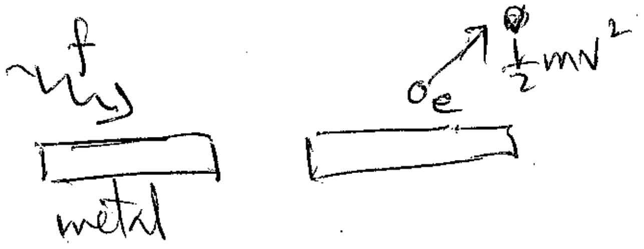
also an eigenvalue problem, with infinite # of solutions. E can be discrete or continuous. E.g., for $V(\vec{r})=0$, the allowed energy values are continuous.

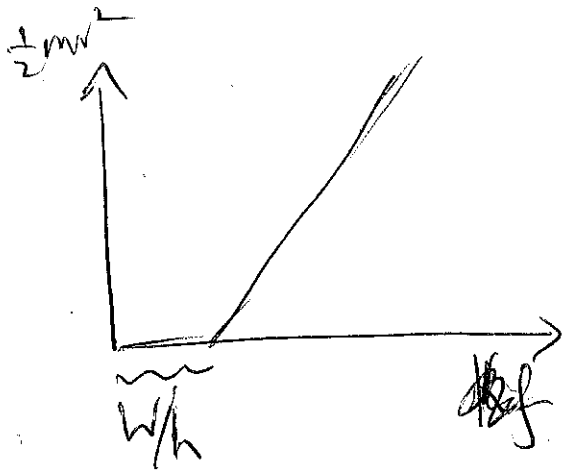
For $V(\vec{r}) = \frac{1}{2}m\omega^2(x^2+y^2+z^2)$, the energies are discrete.

In 1D: $\quad V(x)=0 \quad$ continuous spectrum $E=\frac{\hbar^2k^2}{2m}$

$V(x) = \frac{1}{2}m\omega^2x^2 \quad$ discrete spectrum $E_n = (n+\frac{1}{2})\hbar\omega$

* THE PHOTOELECTRIC EFFECT





Why this behavior?

If electron receives energy E from light, $\frac{1}{2}mv^2 = E - W$

W = energy required to overcome binding to metal. ("work function")

New idea (Einstein 1905): energy $E = hf$, light comes in packets (photons).

$$\frac{1}{2}mv^2 = hf - W$$

Linear dependence with frequency!
Experimental behavior explained using photon picture.