1 Planck and blackbody radiation

Planck solved the mismatch between the experimental spectrum of blackbody radiation and the classical prediction for this spectrum. (The classical prediction is the Rayleigh-Jeans formula, which has an ultraviolet divergence.)

The classical idea is that the energy of an electromagnetic mode is a continuous variable. This leads to average energy per mode being kT, given that the probability density of a mode is proportional to $e^{-E/kT}$ (Boltzmann distribution).

The quantum correction is based on the idea that an electromagnetic mode of frequency f can only have energy in multiples of hf.

So, the difference boils down to the difference between the calculation of an average over a continuous variable, and the calculation of an average over a discrete variable.

2 Average of a continuous variable

Consider continuous variable $y \in [a, b]$.

If the probability density is $P(y)$, i.e., the probability of finding the variable y in the interval $[\xi, \xi + d\xi]$ is $P(\xi)d\xi$.

Then $\int_a^b P(y)dy = 1$, because a probability distribution should be normalized.

The average or expectation value of y is

$$\bar{y} = \langle y \rangle \;=\; \int_a^b dy\; y\; P(y).$$

The average or expectation value of y^2 is

$$\overline{y^2} = \langle y^2 \rangle \;=\; \int_a^b dy\; y^2\; P(y),$$

etc. The overline and the angular brackets are common ways of representing averages.

If the probability density is known to be $\propto f(y)$, i.e., up to some normalization constant, then we have to normalize first:

$$P(y) \;=\; \frac{f(y)}{\int_a^b dy\; f(y)}$$

so that

$$\overline{y} \;=\; \int_a^b dy\; y\; P(y) \;=\; \frac{\displaystyle\int_a^b dy\; y\; f(y)}{\displaystyle\int_a^b dy\; f(y)}$$

For example, in the Rayleigh-Jeans calculation, the energy E of a mode is a continuous variable between 0 and ∞, and energy E is found with probability $e^{-\beta E}$, where $\beta = 1/kT$. Therefore the average energy per mode is

$$\langle E \rangle \;=\; \frac{\displaystyle\int_0^\infty dE\; E\; e^{-\beta E}}{\displaystyle\int_0^\infty dE\; e^{-\beta E}}$$

You should be able to calculate both integrals. The result should be $\langle E \rangle = 1/\beta$.

3 Average of a discrete variable

As an example, let us consider a discrete variable $m \in \mathbb{N}$.

The values of the variable appear with probabilities $P(m)$. We must have

$$\sum_{m=1}^{\infty} P(m) \;=\; 1$$

because if $P(m)$ represents a probability, than the total probability for the variable to have some allowed value should be unity. This is called normalization.

The average value of m is

$$\overline{m} \;=\; \sum_{m=1}^{\infty} m P(m)$$

If the probability is known to be $\propto f_m$, then following the same logic as in the continuous case, we find

$$\overline{m} \;=\; \frac{\displaystyle\sum_{m=1}^{\infty} m f_m}{\displaystyle\sum_{m=1}^{\infty} f_m}$$