1 Equivalent terminology

hermitian conjugate $\equiv$ adjoint
hermitian operator $\equiv$ self-adjoint operator

2 In this note

In this note, we motivate the definition of the hermitian conjugate or adjoint of an operator:

> The operator $\hat{N}$ is the adjoint (or the hermitian conjugate) of the operator $\hat{M}$ if
>
> $$\left\langle (\hat{M}\chi) \middle| \phi \right\rangle = \left\langle \chi \middle| (\hat{N}\phi) \right\rangle \qquad \text{for all } |\phi\rangle \text{ and } |\chi\rangle.$$
>
> The adjoint of $\hat{M}$ is represented by $\hat{M}^\dagger$. Thus by definition
>
> $$\left\langle (\hat{M}\chi) \middle| \phi \right\rangle = \left\langle \chi \middle| (\hat{M}^\dagger\phi) \right\rangle \qquad \text{for all } |\phi\rangle \text{ and } |\chi\rangle.$$
>
> Here the notation $\left\langle (\hat{A}\chi) \right|$ means the dual of $\left| (\hat{A}\chi) \right\rangle = \hat{A}\,|\chi\rangle$.

This definition admittedly seems quite arbitrary, and the connection to the matrix definition is not obvious. The matrix defintion is:

$$A^\dagger = (A^{\mathrm{T}})^*$$

where the superscript T represents the transpose and the star represents complex conjugation of every element of A. If A_{ij} are the elements of the matrix A, then the matrix $A^\dagger$ has elements $(A^\dagger)_{ij} = A^*_{ji}$.

Why do we need the abstract definition? For infinite Hilbert spaces, we may have to deal with operators that can't be written as a finite matrix, e.g., the momentum operator $\hat{p} = -i\hbar\partial_x$. The abstract defintion is then essential.

Let's try to motivate the abstract definition, starting from the matrix definition.

3 Duals of vectors

$$\text{The dual of the vector} \quad \begin{pmatrix} w_1 \\ w_2 \\ \vdots \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} w_1^* & w_2^* & \ldots \end{pmatrix}.$$

In abstract notation, the dual of the ket $|\phi\rangle$ is written as the bra $\langle\phi|$.

Now let's examine the dual of $\hat{A}\,|\phi\rangle$. Is it just $\langle\phi|\,\hat{A}$? To answer this, let's work it out for a simple matrix example. Take

$$B = \begin{pmatrix} 1 & i \\ -1 & 2 \end{pmatrix} , \quad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} , \qquad \text{so that} \quad B\vec{w} = \begin{pmatrix} w_1 + iw_2 \\ -w_1 + 2w_2 \end{pmatrix} .$$

whose dual is

$$\big(B\vec{w}\big)^\dagger \;=\; \big(w_1 - iw_2 \quad -w_1 + 2w_2\big) .$$

On the other hand,

$$\vec{w}^\dagger B \;=\; \big(w_1 \quad w_2\big) \begin{pmatrix} 1 & i \\ -1 & 2 \end{pmatrix} \;=\; \big(w_1 - w_2 \quad iw_1 + 2w_2\big) ,$$

so $\big(B\vec{w}\big)^\dagger$ is definitely not $\vec{w}^\dagger B$. What if we try $\vec{w}^\dagger B^\dagger$ instead of just $\vec{w}^\dagger B$?

$$\vec{w}^\dagger B^\dagger \;=\; \big(w_1 \quad w_2\big) \begin{pmatrix} 1 & -1 \\ -i & 2 \end{pmatrix} \;=\; \big(w_1 - iw_2 \quad -w_1 + 2w_2\big) .$$

So the dual of $B\vec{w}$ is actually $\vec{w}^\dagger B^\dagger$. We've shown this for a simple 2×2 example, but one can of course formalize it for a general N-dimensional system with $N \times N$ matrices.

4 To the abstract case

Generalizing for the abstract case, we therefore want to define $\hat{A}^\dagger$ such that the dual of $\hat{A}\,|\phi\rangle$ is $\langle\phi|\,\hat{A}^\dagger$. However, this defines $\hat{A}^\dagger$ only as acting leftward on bra vectors, i.e., only acting on the dual space. Surely we want to define $\hat{A}^\dagger$ to act on the original ket vectors as well.

Thinking further with matrices, the associativity of matrix multiplication means

$$\Big(\langle\phi|\,\hat{A}^\dagger\Big)\,|\psi\rangle \;=\; \langle\phi|\,\Big(\hat{A}^\dagger\,|\psi\rangle\Big)$$

for any vectors ϕ and ψ. We would like this to hold for the abstract case as well. Since $\langle\phi|\,\hat{A}^\dagger$ is the dual of $\hat{A}\,|\phi\rangle$, we can write it as $\Big\langle(\hat{A}\phi)\Big|$. Hence we want the hermitian conjugates in the abstract case to satisfy

$$\Big\langle(\hat{A}\phi)\Big|\psi\Big\rangle \;=\; \Big\langle\phi\Big|(\hat{A}^\dagger\psi)\Big\rangle$$

for every $|\phi\rangle$ and $|\psi\rangle$.

Thus, starting from the matrix definition, we have motivated the abstract definition.

5 Hermitian operators

We define hermitian operators as those which are equal to their hermitian conjugate. (Since they are equal to their adjoint, such operators are also called self-adjoint.)
Thus an operator $\hat{A}$ is hermitian or self-adjoint if

$$\left\langle (\hat{A}\phi) \middle| \psi \right\rangle \;=\; \left\langle \phi \middle| (\hat{A}\psi) \right\rangle$$

for every $|\phi\rangle$ and $|\psi\rangle$.
With this definition, we can try checking whether operators are Hermitian, even for those opeartors that can't easily be expressed as matrices. For example, the following operators act on the Hilbert space of normalizable complex functions of one variable (e.g., wavefunctions of a single particle on a line):

$$\hat{A}_1 = \frac{d}{dx}\,, \quad \hat{A}_2 = \frac{d^2}{dx^2}\,, \quad \hat{A}_3 = i\frac{d}{dx}\,, \quad \hat{A}_4 = ix\,.$$

Show in each case whether or not the operator is Hermitian.