Due Wednesday December 2nd, 2020.

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1. **Gram-Schmidt orthogonalization.**

   A very basic technique, used widely in numerical linear algebra.

   (a) $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$, are N-dimensional vectors, with $N \gg 3$. These three vectors form a 3-dimensional subspace of the full N-dimensional vector space.

   Construct an **orthonormal** basis for 3-dimensional subspace, using the Gram-Schmidt process.

   (b) The QR decomposition of a $N \times N$ matrix A can be done using Gram-Schmidt process.

   Explain how the Q matrix is obtained from the columns of the A matrix. (The R matrix is obtained as a byproduct of this calculation; no need to describe that.)

   (c) A is a $N \times N$ matrix and $\vec{b}$ is a $N \times 1$ vector. Construct a 4-dimensional **Krylov subspace** using matrix A and vector $\vec{b}$.

   Then construct an orthonormal basis for this subspace using Gram-Schmidt.

2. **Boundary Value Problem**

   Consider the ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{y}{x^2+1} = x^2$$

   with the boundary values $y(-2) = 1$, $y(2) = 2$, defined on the domain $x \in [-2, 2]$.

   (a) By dividing the domain into 5 equal pieces, discretize the differential equation and write it as a set of linear equations. Explain very carefully how many variables and how many equations you have.

   (b) Write the set of linear equations as a **matrix** equation.

   (c) Generalize to dividing the domain into N equal pieces. Describe the structure of the matrix. (Sparse? Banded? etc.) State/explain very carefully the size of the matrix.

(d) Imagine that the boundary condition on the left has been changed to a Neumann-type condition: $y'(-2) = -2$. Explain how you would change your equations, using a **second-order** forward difference approximation for the first derivative.

(e) It might be less cumbersome to use a **first-order** approximation for the first derivative. Explain why this is a bad idea.