Due Wednesday December 9th, 2020.

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1. **Jacobi and Gauss-Seidel**

We want to find the solution $\vec{x}^*$ of the matrix equation $A\vec{x} = \vec{b}$. This could be attempted iteratively:

$$\vec{x}^{(m+1)} = B^{-1}(B - A)\vec{x}^{(m)} + B^{-1}\vec{b}.$$

Of course, this procedure may or may not converge.

(a) Show that, if you set B to be the diagonal matrix D with the same diagonal elements as A, you obtain Jacobi iteration.

(b) Let's rewrite the iteration equation as

$$\vec{x}^{(m+1)} = H\vec{x}^{(m)} + \vec{c}$$

If $\vec{e}^{(m)} = \vec{x}^{(m)} - \vec{x}^*$ is the error in the m-th step, then find an iteration equation for $\vec{e}^{(m)}$.

(c) Expand $\vec{e}^{(0)}$ in terms of the eigenvectors of H.

Using the notation you just introduced, express $\vec{e}^{(m)}$ in terms of the eigenvalues and eigenvectors of H.

What property should the eigenvalues of H have, so that the iteration converges?

(d) Look up the **diagonal dominance** condition for convergence of Jacobi and Gauss-Seidel iteration. Explain why, for the following set of equations, these iterations are unlikely to converge, and what rearrangement would fix the problem:

$$\begin{aligned} 5x_1 - x_2 + 3x_3 &= -1 \\ -x_1 + 2x_2 - 7x_3 &= -1 \\ 3x_1 + 11x_2 - x_3 &= 2 \end{aligned}$$

2. **Discretization of initial-value PDE: stability.**

Consider the first-order PDE for $u(x,t)$

$$\frac{\partial u}{\partial t} = -M\frac{\partial u}{\partial x}.$$

We want to solve this, i.e., evolve forward in time, given initial conditions $u(x,0)$. We try a forward-time, centered-space scheme:

$$\frac{u_j^{m+1} - u_j^m}{\Delta t} = -M\,\frac{u_{j+1}^m - u_{j-1}^m}{2a}. \tag{1}$$

As in lecture, the subscripts are spatial lattice indices and superscripts are timestep indices, and $a = \Delta x$ is the lattice spacing.

(a) Show using von Neumann stability analysis that this scheme is **never** stable.

This will involve decomposing into Fourier modes

$$u_j = \sum_k u_k e^{ikx} = \sum_k u_k e^{ikja},$$

and considering the evolution of a single mode, i.e., using $u_j = u_k e^{ikja}$, in the discrete evolution equation. If one defines $u_k^{m+1} = \xi_k u_k^m$, then the magnitude of ξ_k tells us whether or not small errors will blow up or get damped. This argument is described in more detail in the wikipedia page on von Neumann stability analysis, in the lecture notes, and in many other places.

(b) One trick is to replace

$$u_j^m \to \frac{1}{2}\left(u_{j-1}^m + u_{j+1}^m\right)$$

in the left side of the FTCS iteration equation (1). This seems reasonable, since $u(x)$ should be differentiable and hence smooth.

Find the stability condition for the resulting scheme.