In these sheets we look at partial differential equations.

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1. Consider the diffusion-like equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (1 - x^2)$$

   (a) Discretize this equation, using the Forward Time Centered Space (FTCS) scheme. Write down the resulting difference equations.

   (b) Using von Neumann stability analysis, find out the condition for your scheme to be stable.

   (c) Now discretize the equation using the Backward Time Centered Space (BTCS) scheme, and write down the resulting difference equations.

   (d) Explain why you need to solve a linear system of equations *at every step* in order to evolve the BTCS scheme.

   Imagine that you want to use Gauss-Seidel iteration to solve the linear system of equations. Find out whether the coefficient matrix is *diagonally dominant*. What does diagonal dominance tell us about Gauss-Seidel iteration?

   (e) Perform a von Neumann stability analysis on the BTCS equations, to find out whether or not the scheme is stable.

   (f) Explain how you can combine the FTCS and BTCS schemes to obtain a discretization that has second-order accuracy in time.

2. Fourier transforms.

   (a) What is the shape of the Fourier transform of a gaussian function of time? How does the shape in frequency space change change if the gaussian in time domain is made narrower?

   (b) Consider the function $f(x) = \sin(x) + 0.5\cos(3x)$. Sketch the Fourier spectrum of this function.

   (c) Consider the ODE

$$\frac{d^2}{dx^2}\phi(x) + A\frac{d}{dx}\phi(x) = \lambda(x).$$

   Show how this is easy to solve in Fourier space.