Quantum Mechanics in Discrete Phase-Space

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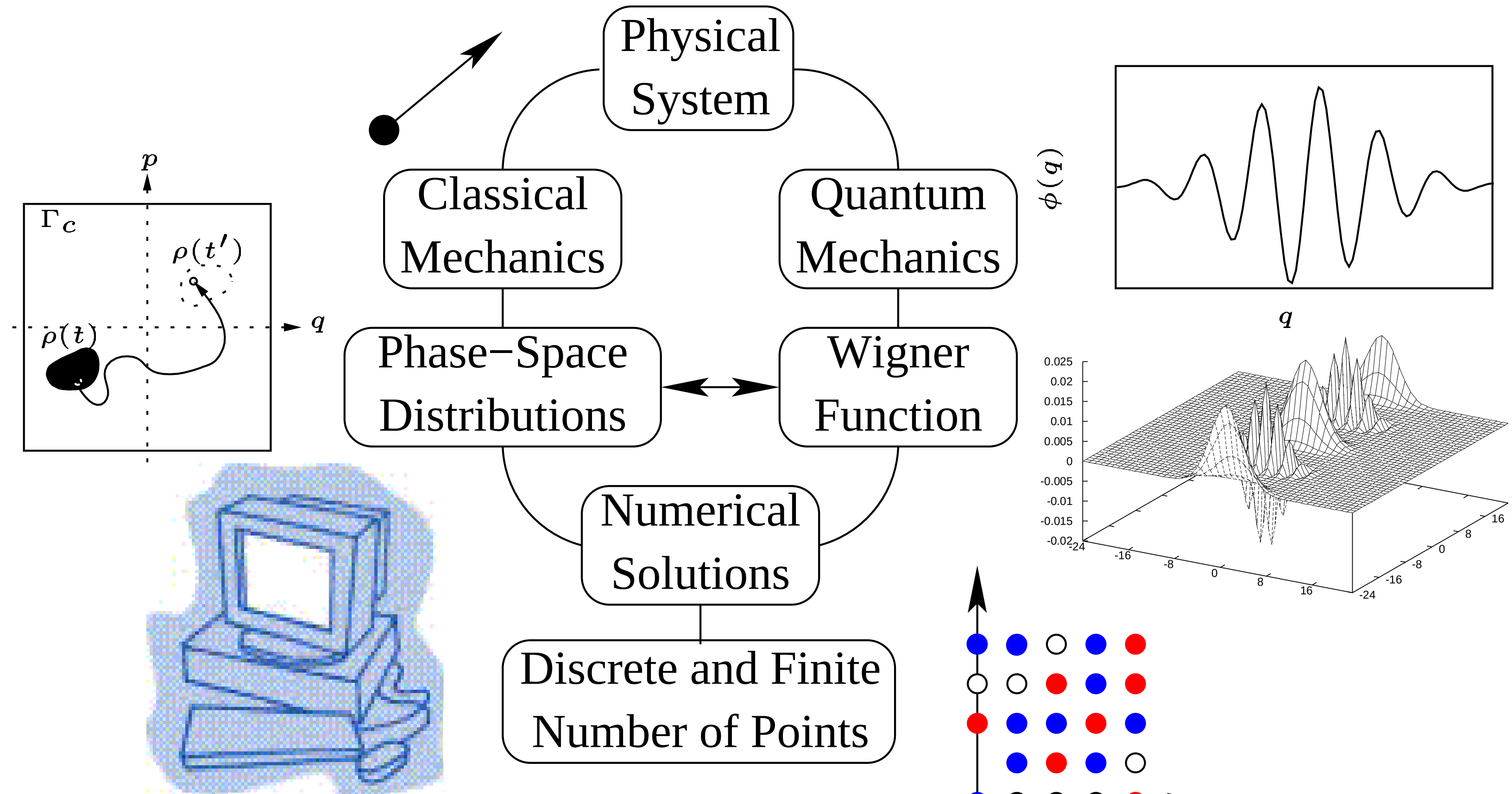
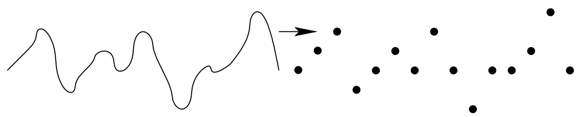
Motivation



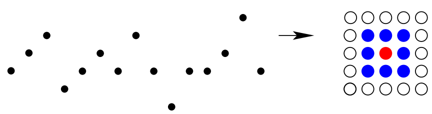
Physical System
Classical Mechanics
Quantum Mechanics
Phase−Space Distributions
Wigner Function
Numerical Solutions
Discrete and Finite Number of Points
Γ_c
p
q
$\rho(t)$
$\rho(t')$
$\phi(q)$
q

1. Objectives

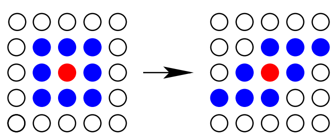
1. Analize consecuences of discretization.



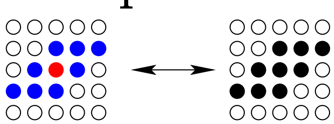
2. Construct a well-defined discrete Wigner function.



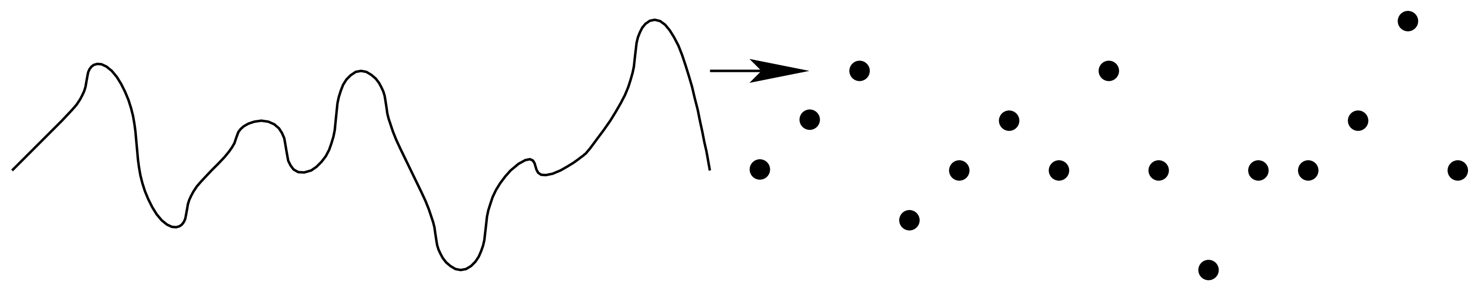
3. Use this formulation to calculate the quantum time evolution.



4. Compare classical and quantum results for several examples.



2. Periodic & Discrete



2. Periodic & Discrete

In quantum mechanics the position and momentum matrix elements transformation are given by

$$\langle p|q\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipq/\hbar}.$$

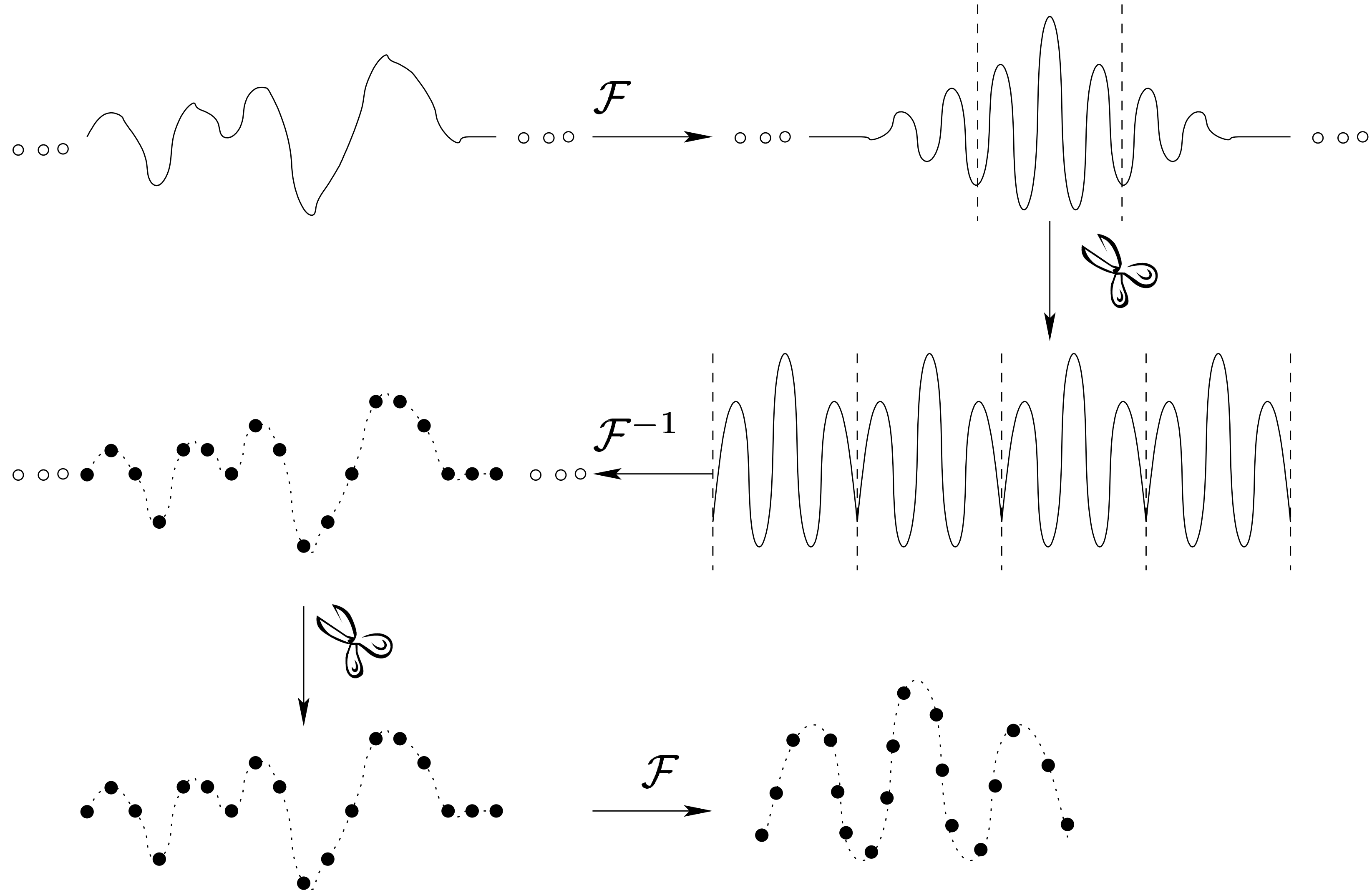
Therefore, the relation is a Fourier transformation.

The Wigner function is defined as

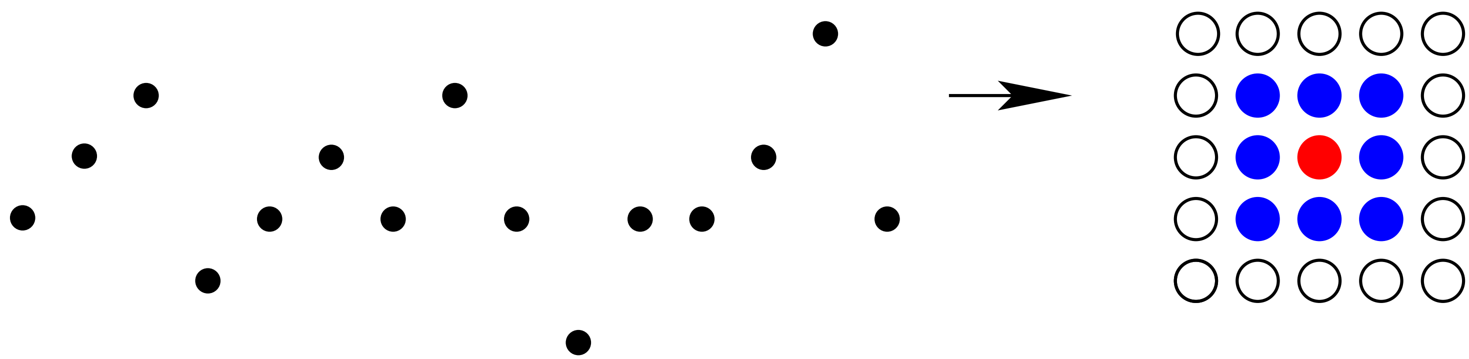
$$\rho_w(p,q) = \frac{1}{2\pi\hbar}\int dq' \langle q+q'/2|\hat{\rho}|q-q'/2\rangle \exp(-ipq'/\hbar),$$

where there appears a Fourier transformation as well.

2. Periodic & Discrete



3. The Discrete Wigner Function

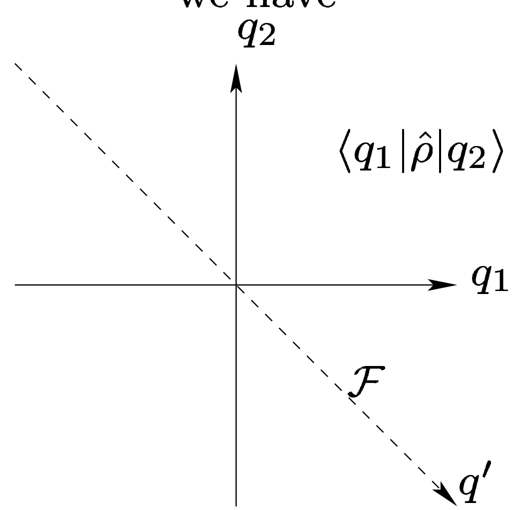


3. The Discrete Wigner Function

From the definition $\rho_w(p,q) = \frac{1}{2\pi\hbar}\int dq' \langle q+q'/2|\hat{\rho}|q-q'/2\rangle \exp(-ipq'/\hbar)$, we can see that the Wigner function is a Fourier transformation of the density operator along an inclined axe. Defining

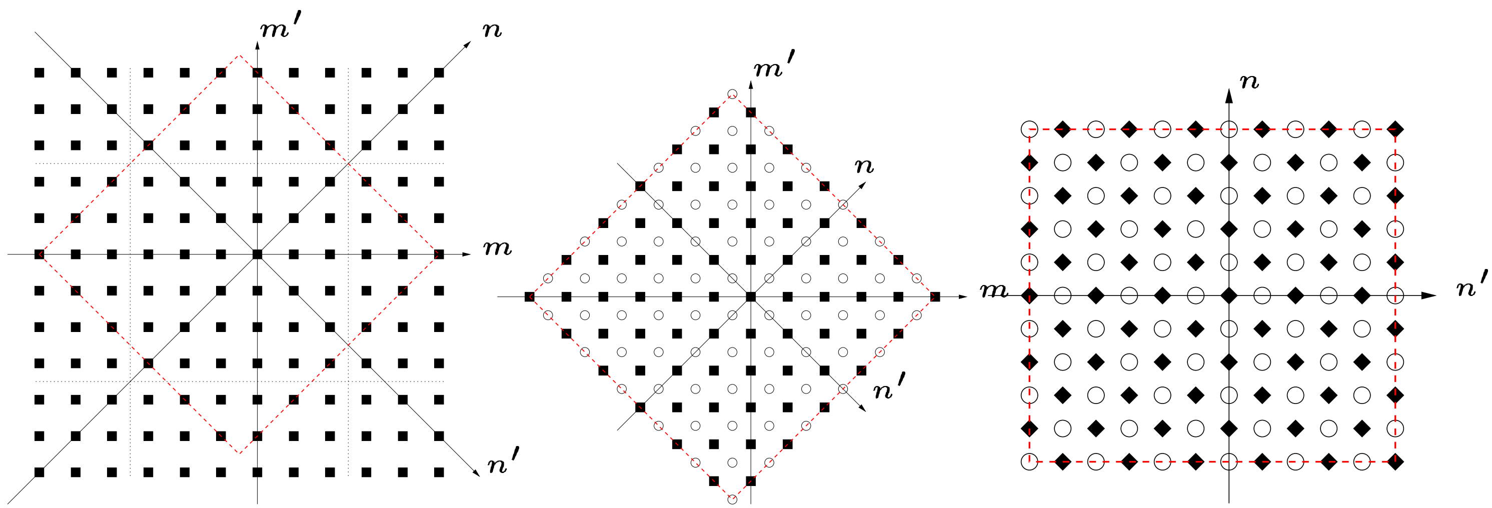
$$q_1 = q + q'/2, \quad q = (q_1+q_2)/2,$$
$$q_2 = q - q'/2, \quad q' = q_1 - q_2;$$

we have



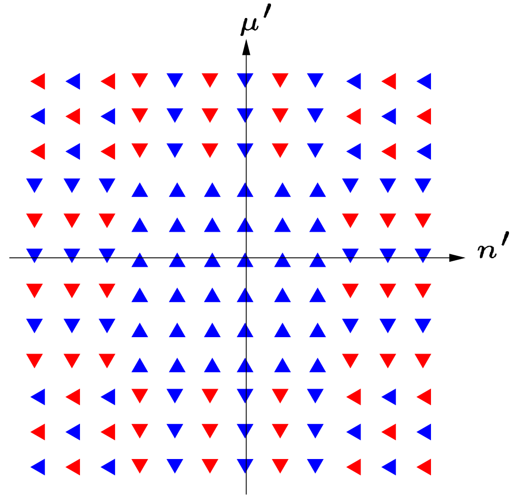
In the case of a dicrete space, we have $q_m = mQ/N$, $p_\mu = \mu P/N$ and $N = QP/2\pi\hbar$, where Q and P are the period in space and momentum respectively.

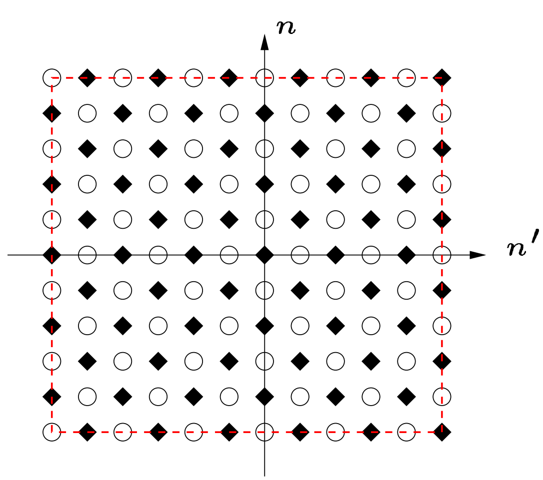
3. The Discrete Wigner Function

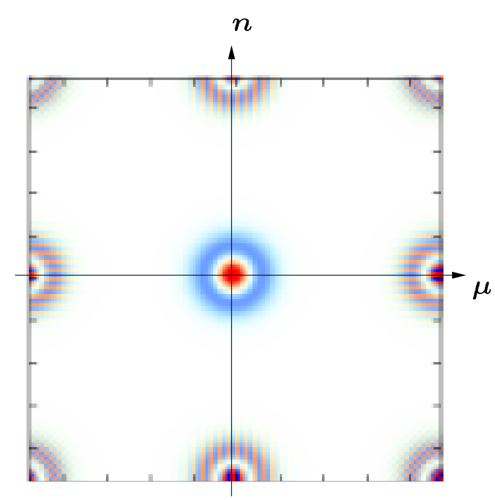


$N^2 \to 2N^2$ $\quad 2N^2 \to (2N)^2$ $\quad 4N^2$

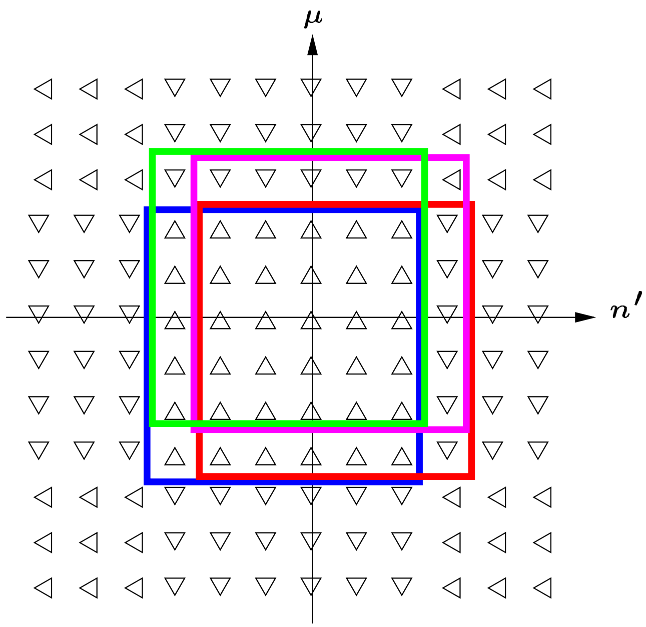
3. The Discrete Wigner Function

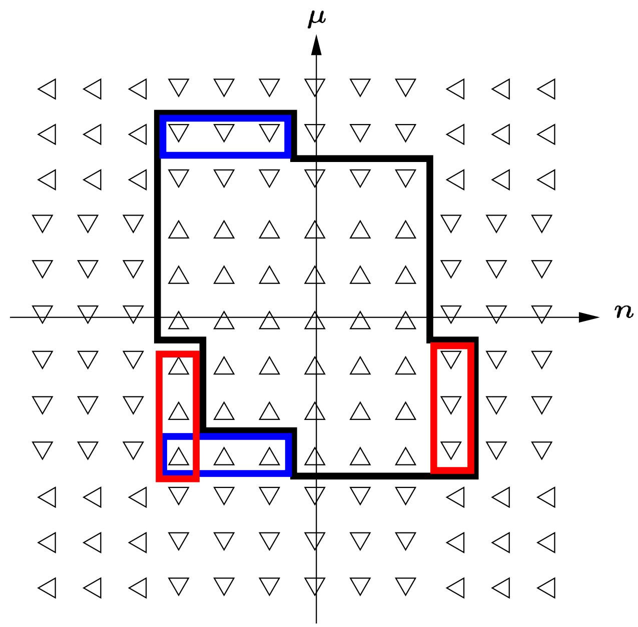






3. The Discrete Wigner Function





3. The Discrete Wigner Function

Finally, the discrete Wigner function is given by the expression

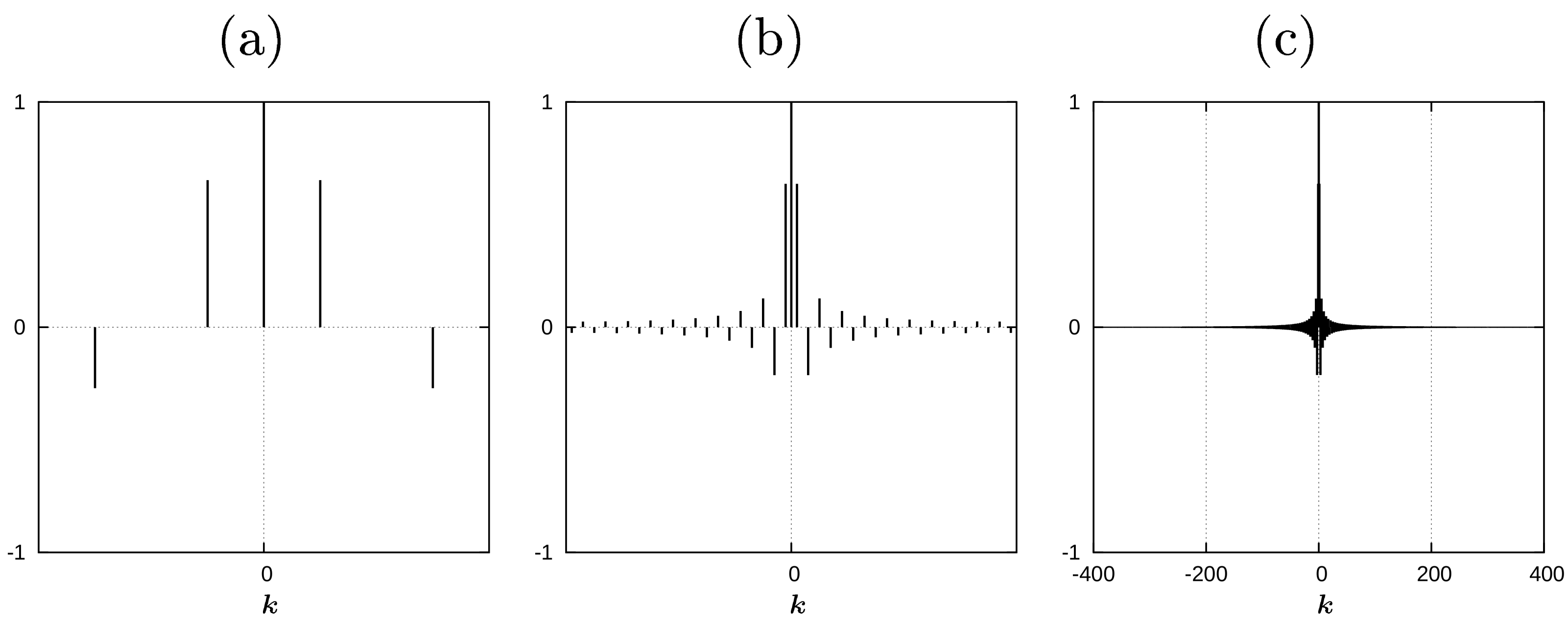
$$\boxed{\rho_w(p_\mu, q_n, t) = \frac{1}{N}\sum_{n'}\sum_{n}^{2N} \delta_n^{n'} \left\langle \frac{n+n'}{2} \middle| \hat{\rho}(t) \middle| \frac{n-n'}{2} \right\rangle \tilde{\delta}(2m-n) e^{-2i\pi \frac{n'\mu}{N}}},$$

where

$$\delta_n^{n'} = \frac{1+(-1)^{n+n'}}{2}, \tilde{\delta}(k) = \frac{1}{N}\frac{\sin(\pi k/2)}{\sin(\pi k/2N)};$$

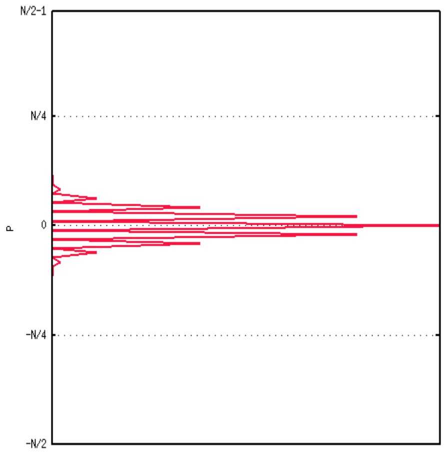
with the property $\tilde{\delta}(2k) = \delta(k)$. One can prove that

- ρ_w is real.
- $\sum_m \rho_w(\mu, m) = \langle \mu|\hat{\rho}|\mu\rangle$, $\sum_\mu \rho_w(\mu, m) = \langle m|\hat{\rho}|m\rangle$.
- $\sum_{\mu m} \rho_w(\mu, m) A_w(\mu, m) = \mathrm{Tr}[\hat{\rho}\hat{A}]$

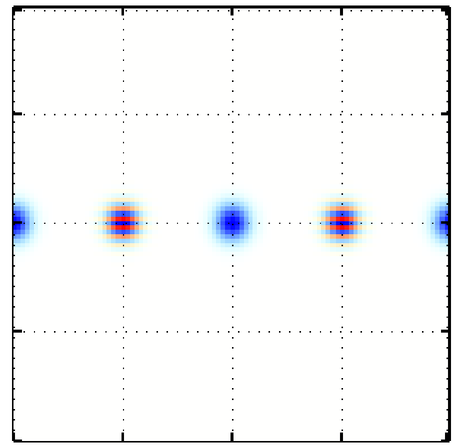


$\tilde{\delta}(k)$ plots for (a) $N = 4$, (b) $N = 40$ and (c) $N = 400$.

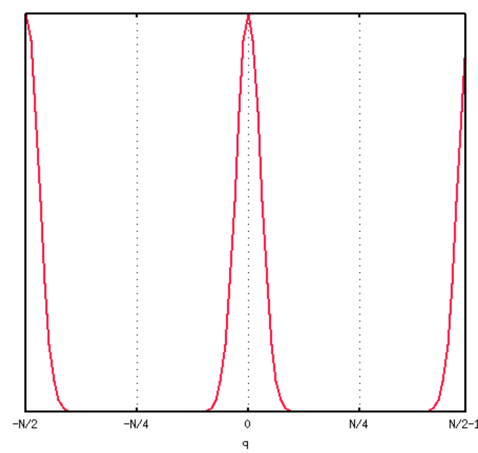
3. The Discrete Wigner Function



$\sum_q \rho_w(p_\mu, q_m)$

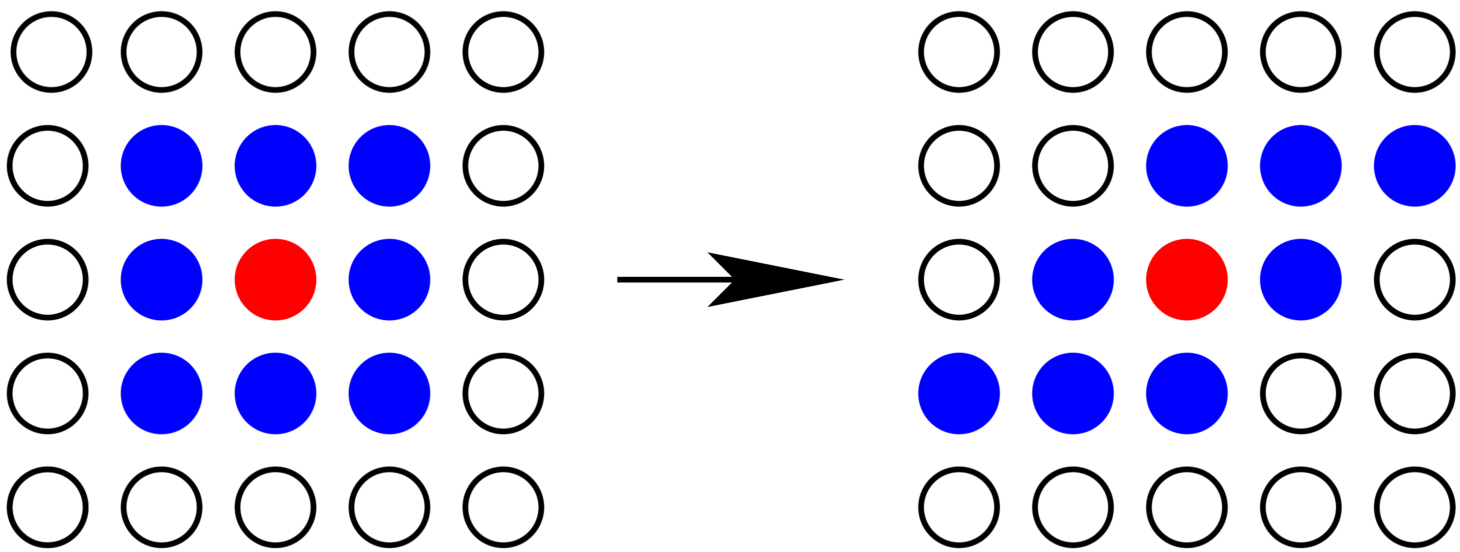


$$\rho_w = \mathcal{T}[\hat{\rho}]$$

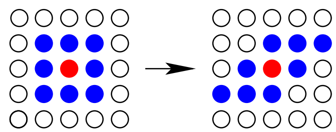


$\sum_p \rho_w(p_\mu, q_m)$

4. Time Evolution



4. Time Evolution



In order to perform the quantum time evolution:

- Take a basis on the quantum Hilbert space.
- Rewrite the Weyl transformation of the density matrix using the choosed basis.
- Use the quantum time evolution operator to calculate the density matrix at time t' from t.

4. Time Evolution

We can use the set of eigenvalues of an operator as a basis for the Hilbert space and for the space of linear operators on this Hilbert space, for example choosing the energy eigenstates $\hat{H}|\phi_a\rangle = E_a|\phi_a\rangle$; therefore we have

$$|\psi\rangle = \sum_a \psi_a|\phi_a\rangle, \hat{A} = \sum_{ab} A_{ab}|\phi_a\rangle\langle\phi_b|;$$

where $\psi_a = \langle\psi|\phi_a\rangle$ and $A_{ab} = \langle\phi_a|\hat{A}|\phi_b\rangle$.

The $\{|\phi_a\rangle\}$ form a complete and orthonormal basis,

$$\hat{I} = \sum_a |\phi_a\rangle\langle\phi_a|, \langle\phi_a|\phi_b\rangle = \delta(a-b).$$

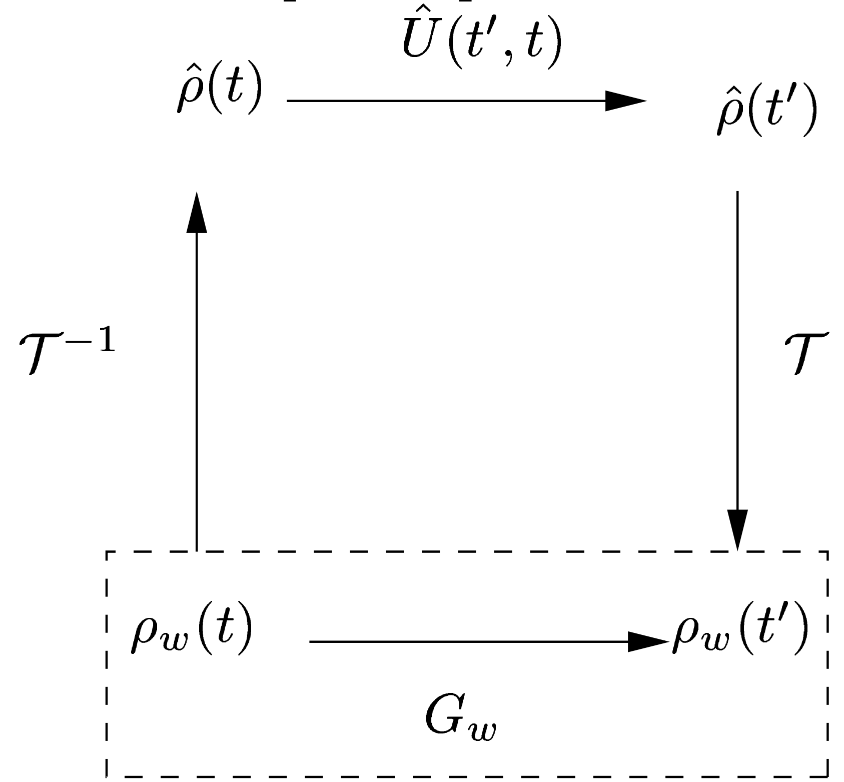
4. Time Evolution

In the case of the quantum phase-space, one can choose the Weyl transform of the operator basis $\mathcal{T}[|\phi_a\rangle\langle\phi_b|] = \Phi_{ab}(p_\mu, q_m)$, and we have

$$\begin{aligned} \sum_a \Phi_{aa}(p_\mu, q_m) &= 1, \\ \frac{1}{N}\sum_{\mu m} \Phi_{ab}(p_\mu, q_m)\Phi^\star_{cd}(p_\mu, q_m) &= \delta(a-c)\delta(b-d); \end{aligned}$$

These are the completeness and orthogonality relations in phase-space.

From the phase-space basis and



we can rewrite the Wigner function at t' using the Wigner function at another time t as

$$\rho_w(t') = \mathcal{T}[U(t',t)\mathcal{T}^{-1}[\rho_w(t)]U^\dagger(t',t)];\ t,t' \in \mathcal{R}$$

4. Time Evolution

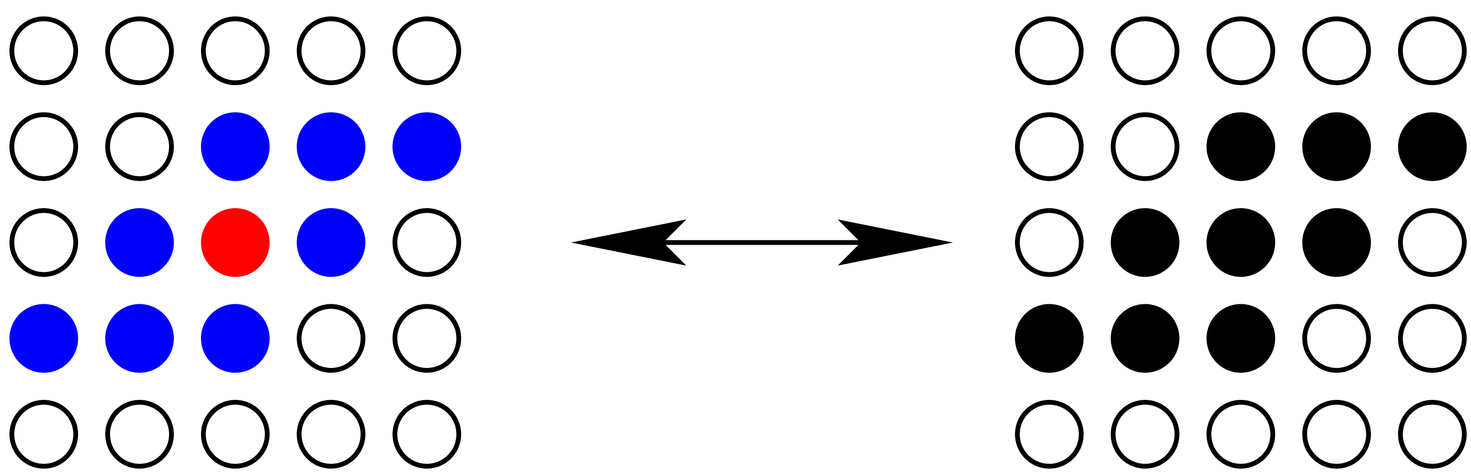
Therefore, the Wigner function propagator is given by

$$\boxed{G_w(\mathbf{r}', t'; \mathbf{r}, t) = \frac{1}{N} \sum_{ab} e^{-\frac{i}{\hbar}(E_a - E_b)(t' - t)} \Phi^{\star}_{ab}(\mathbf{r}) \Phi_{ab}(\mathbf{r}'),}$$

where $\mathbf{r} = (p_\mu, q_m)$. The properties of the propagator are

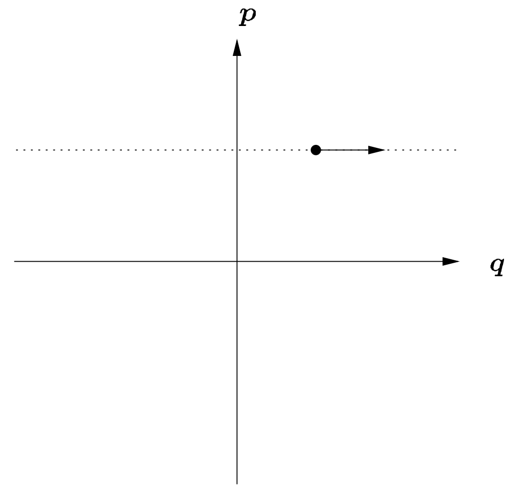
- It is real.
- $G_w(\mathbf{r}', t; \mathbf{r}, t) = \delta(\mu' - \mu)\delta(m' - m)$.
- $G_w(t'', t) = G_w(t'', t') \circ G_w(t', t)$.
- $G_w^{-1}(t', t) = G_w(t, t')$.

5. Classical and quantum comparations



5. Classical and quantum comparations

Free Particle



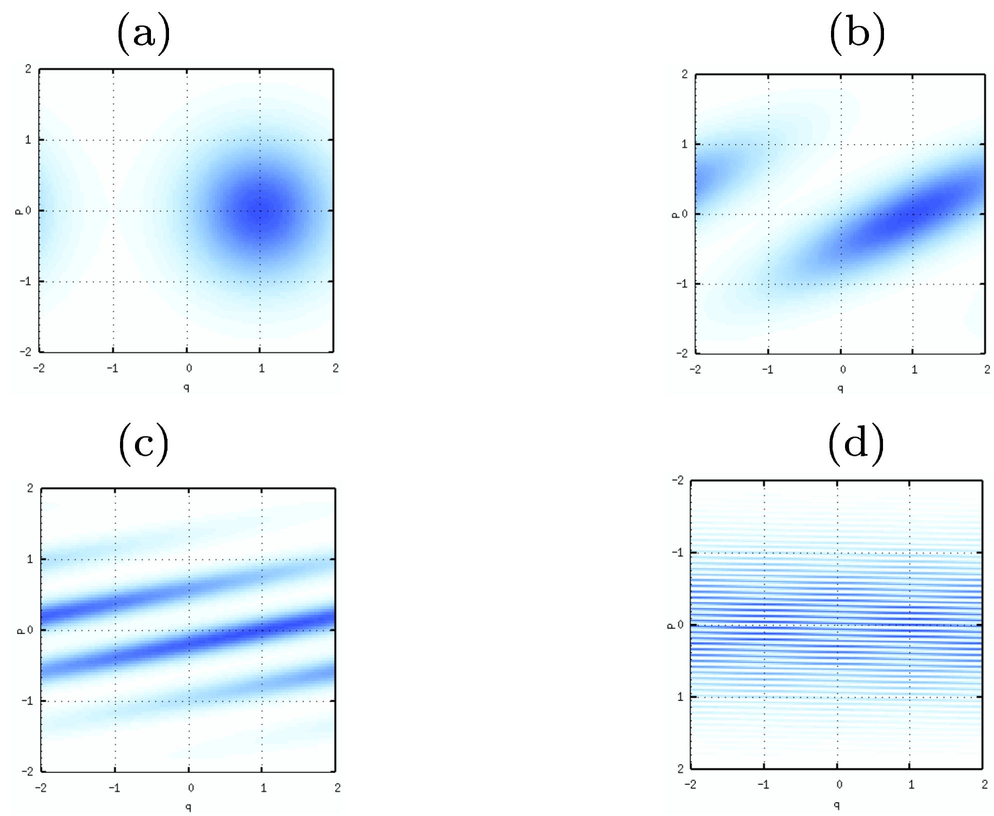
The classical propagator of a free particle is given by

$$G_c(p', q', t'; p, q, t) = \delta(p' - p)\delta(q' - q + p(t' - t)/m).$$

Thus, for a periodic system the distribution in t' is

$$\rho_c(p, q, t') = \rho_c(p \mod P, (q - pt'/m) \mod Q, 0).$$

Classical Free Particle



(a) $t = 0$ (b) $t = 1$, (c) $t = 5$ y (d) $t = 50$. $m = 1$.

Quantum Free Particle

In the case of quantum mechanics, the eigenstates in coordinate representation are

$$\phi_p(q) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipq/\hbar}.$$

Imposing the periodicity with periods $P = Q = 1$, we have

$$\phi_\nu(m) = \frac{1}{\sqrt{N}} e^{2i\pi\nu m/N}.$$

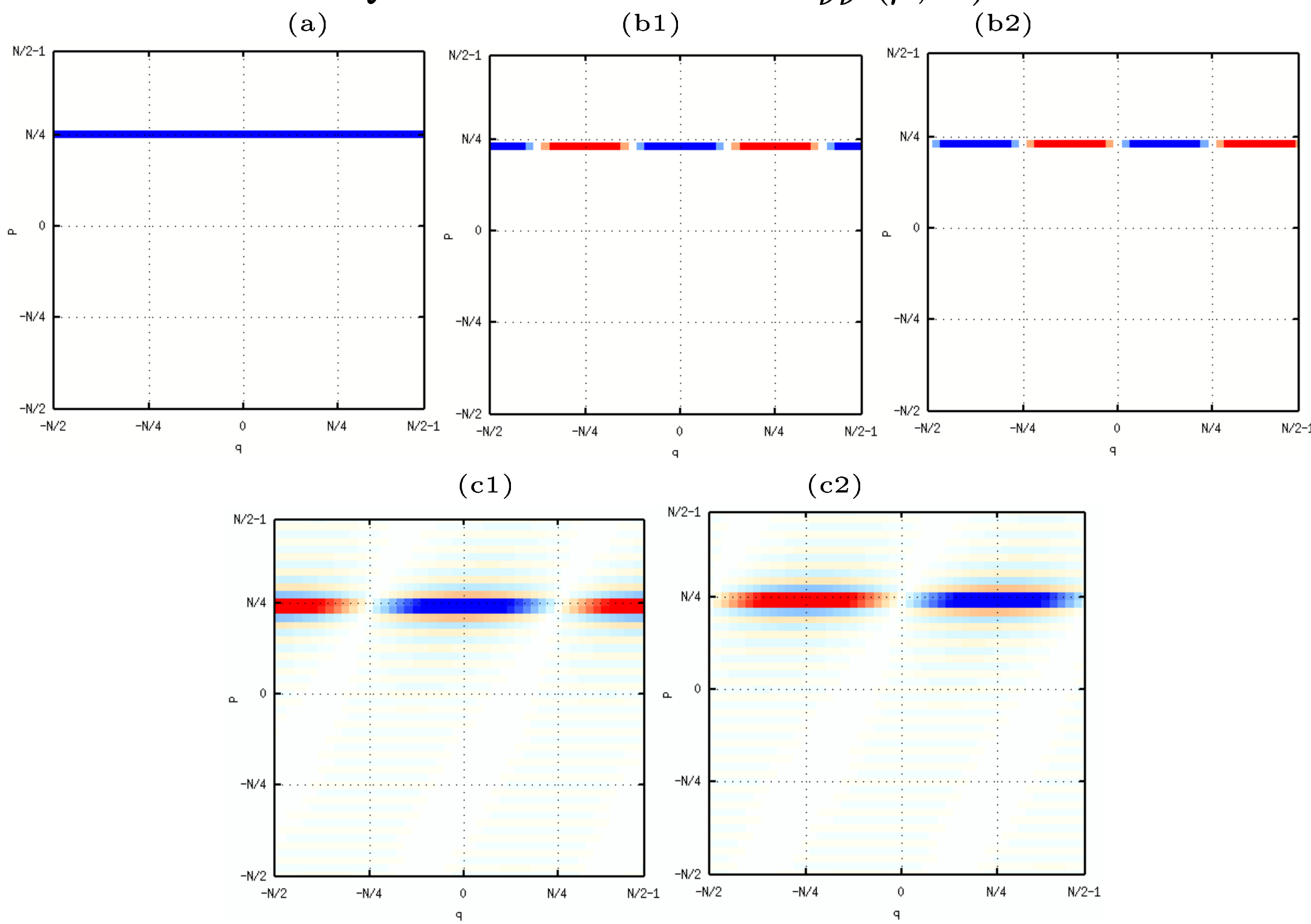
These eigenstates allow us to calculate the Weyl basis functions

$$\Phi_{\nu\nu'}(\mu, m) = \frac{1}{N} e^{2i\pi m(\nu-\nu')/N} \tilde{\delta}(\nu + \nu' - 2\mu),$$

for $\mathrm{par}(\nu) = \mathrm{par}(\nu')$, $\bar{\nu} = (\nu + \nu')/2$ and $\Delta\nu = \nu - \nu'$ it is

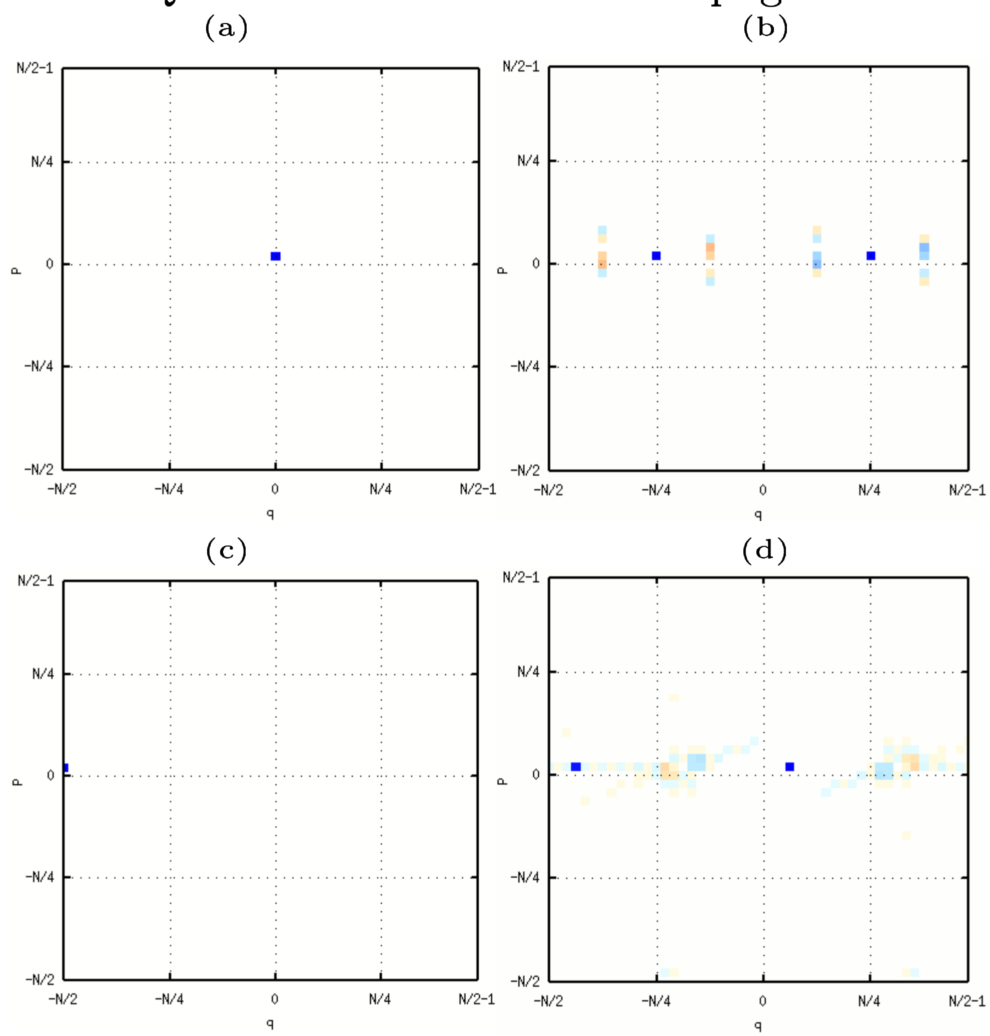
$$\Phi_{\nu\nu'}(\mu, m) = \frac{1}{N} e^{2i\pi m\Delta\nu/N} \delta(\bar{\nu} - \mu),$$

Quantum Free Particle-$\Phi_{\nu\nu'}(\mu, m)$



(a)$\nu = \nu' = N/4$ (b) $\Phi_{12,10}$ (c) $\Phi_{12,11}$ (1) real part (2) imaginary part. N=48

Quantum Free Particle-Propagator

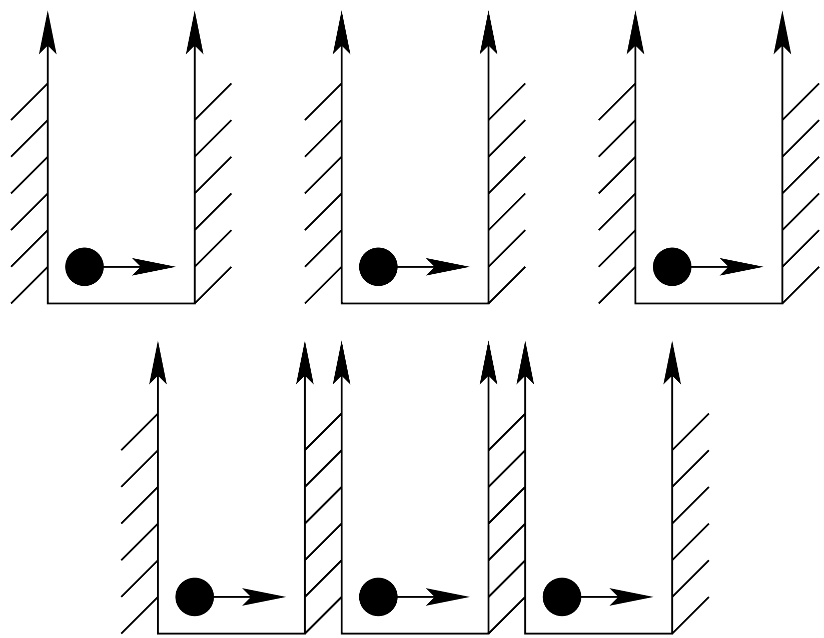


Wigner Function Propagator for $\mu = 1$, $m = 0$, $t = 0$, $N = 48$ and (a) $t' = 0$, (b) $t' = N/4$, (c) $t' = N/2$ y (d) $t' = 52$.

5. Classical and quantum comparations

Delta spike model

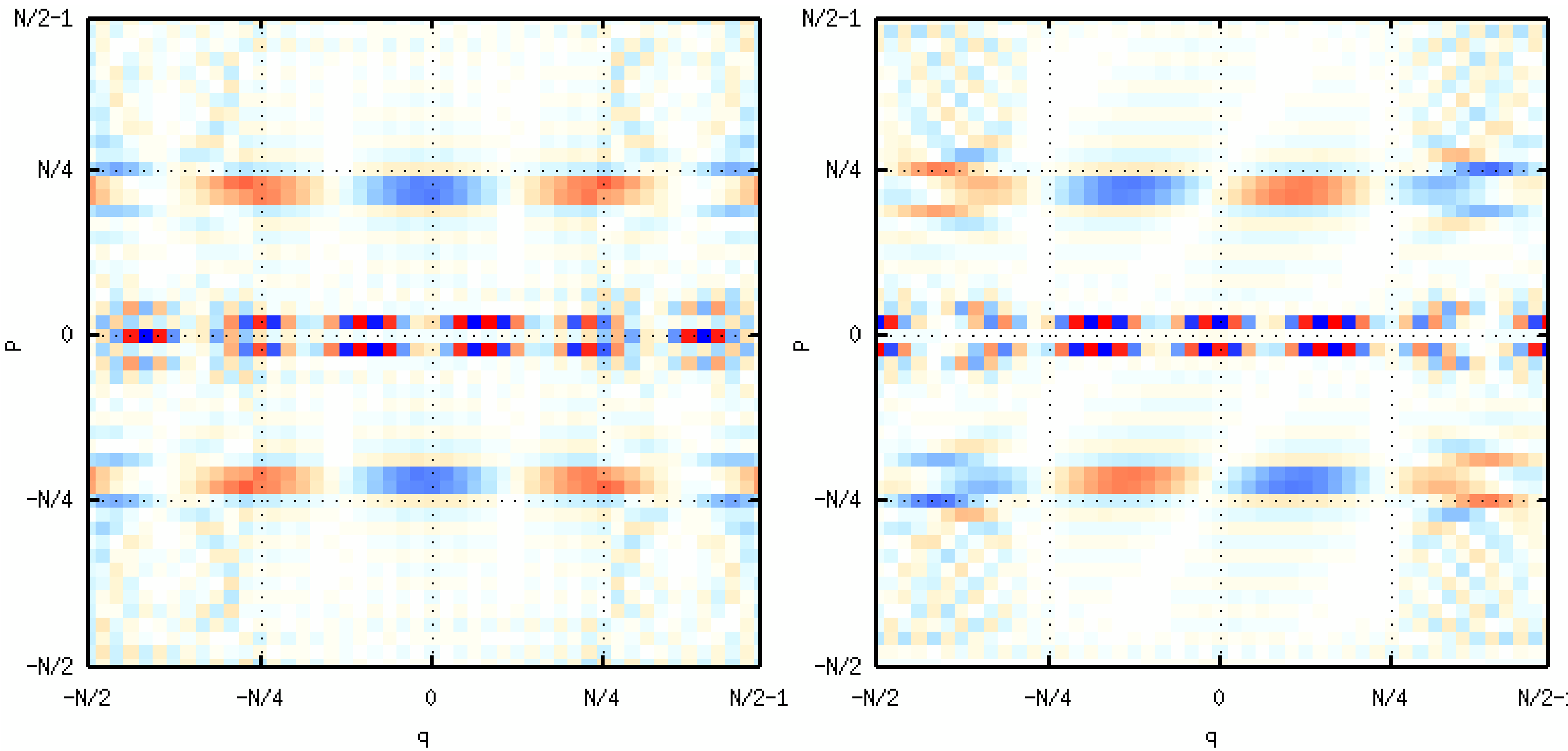
The scheme of the system is



Taking the discretization into account, the eigenfunctions are

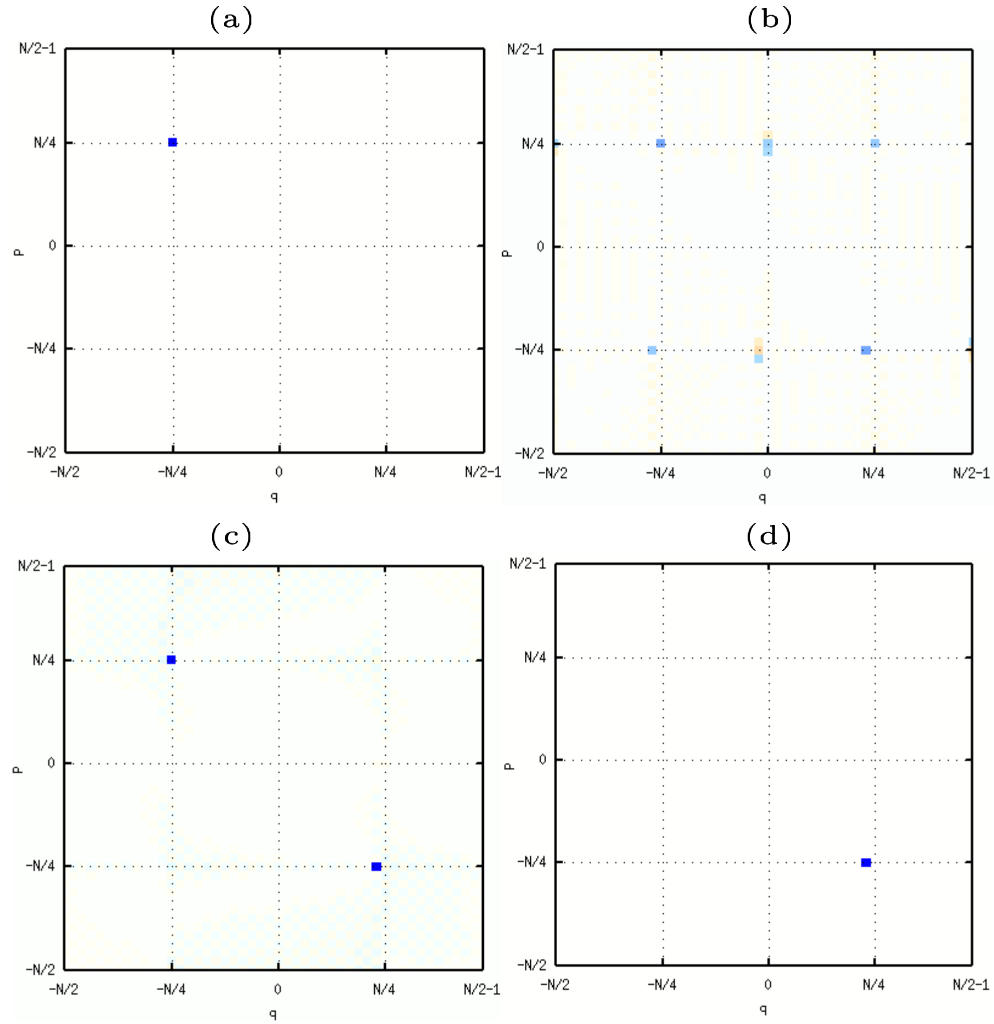
$$\Phi_\nu(n) = \begin{cases} \sqrt{\frac{2}{N}}\sin(\pi\nu(n-1/2)/N) & \text{for} \quad \nu = 1,\ldots,N-1, \\ \sqrt{\frac{1}{N}}\sin(\pi(n-1/2)) & \text{for} \quad \nu = N. \end{cases}$$

The analytical calculation of phase-space basis is more complex than the previous case. Then we use the numerical tool in order to calculate the propagator.



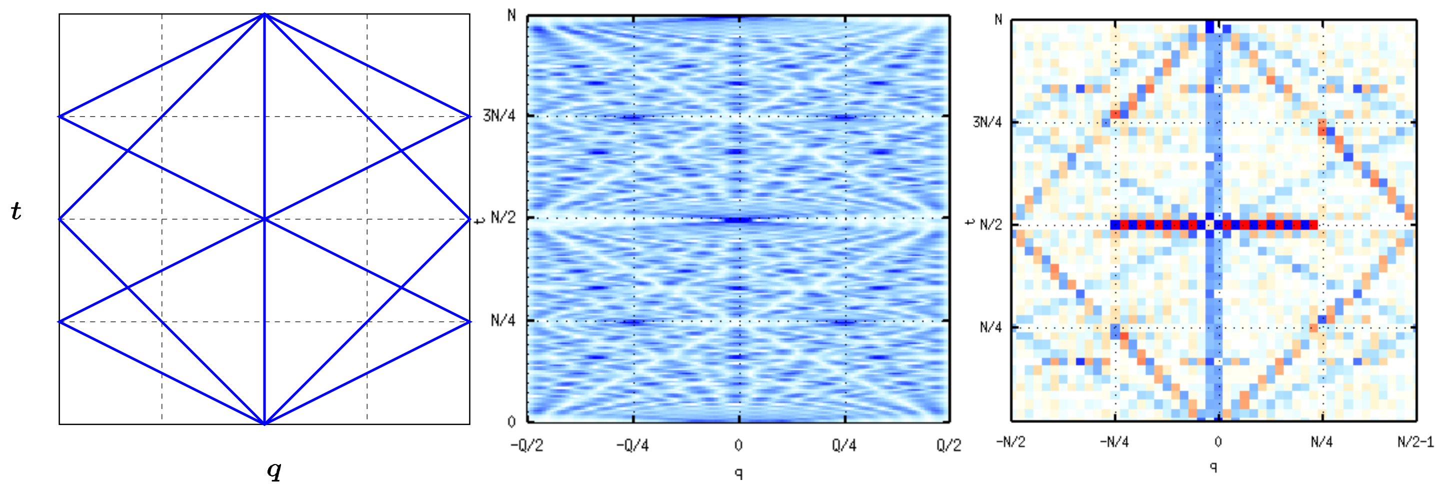
$\Phi_{29-25}(\mu, m)$

Delta spike model-Propagator



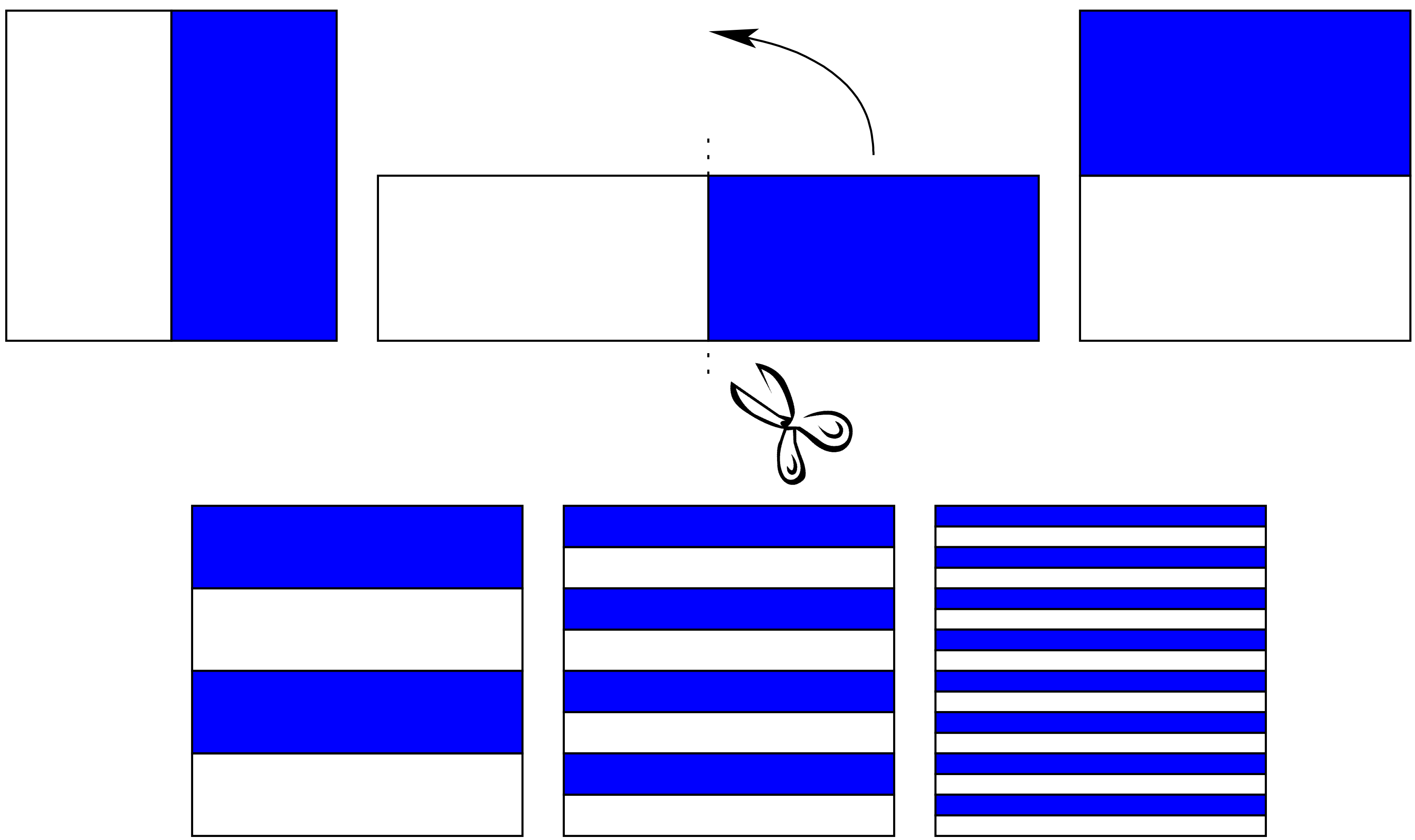
Wigner Function Propagator for $\mu = 1$, $m = 0$, $t = 0$, $N = 48$ and (a) $t' = 0$, (b) $t' = N/4$, (c) $t' = N/2$ y (d) $t' = N$.

Delta Spike model - Quantum Carpets



5. Classical and quantum comparations

Baker map



Baker map

The classical map is given by

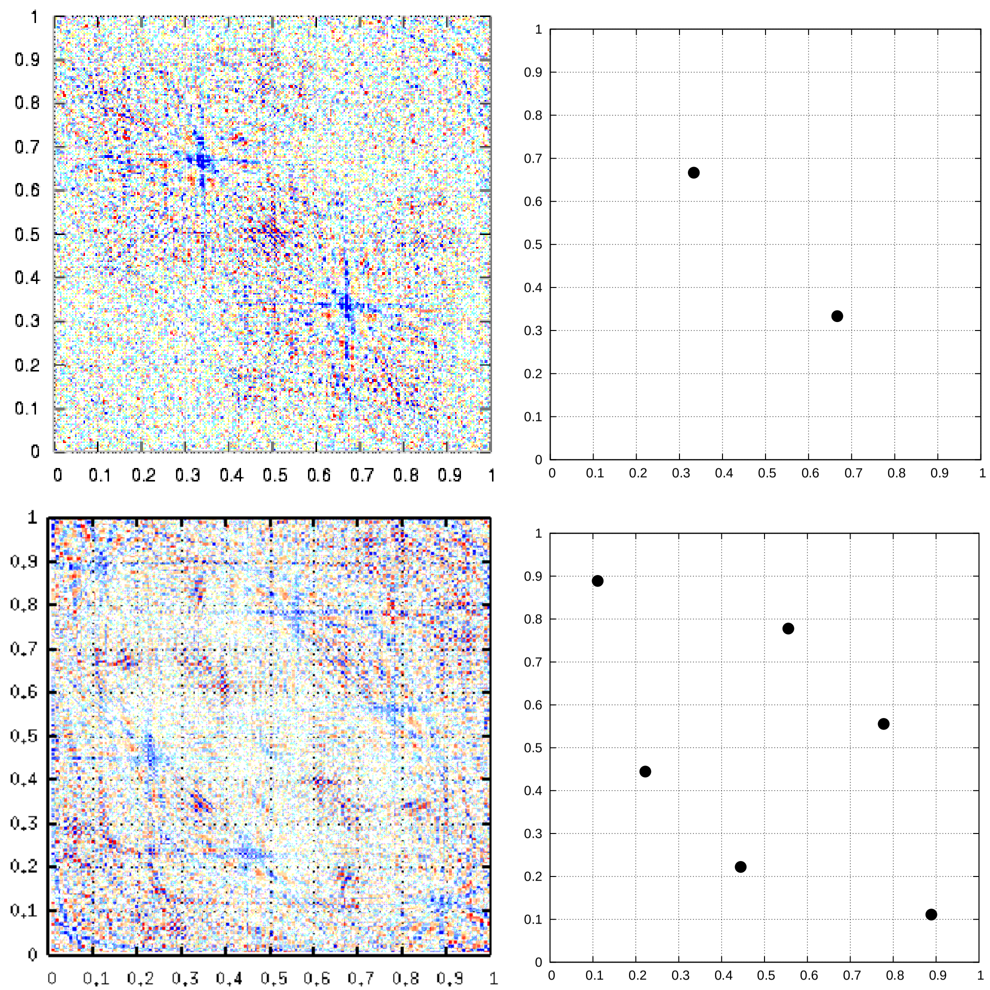
$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} 2q \\ p/2 \end{pmatrix} \qquad 0 \leq q < Q/2,$$

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} 2q - Q \\ (p+P)/2 \end{pmatrix} \qquad Q/2 \leq q < Q.$$

The quantum counterpart is given by

$$\begin{pmatrix} \Psi'_R(q) \\ \Psi'_L(q) \end{pmatrix} = \mathcal{F}_N^{-1} \begin{pmatrix} \mathcal{F}_{N/2} & 0 \\ 0 & \mathcal{F}_{N/2} \end{pmatrix} \begin{pmatrix} \Psi_R(q) \\ \Psi_L(q) \end{pmatrix}.$$

Baker map



6. Conclusions

- The proposed discrete Wigner function holds with the properties of continuous version.
- Our Discrete Wigner Function allows us to define a propagator unlike the redundant definition.
- The numerical tool let us to analyze as time-independed (eigenfunctions) as time-depended (evolution) discrete systems.
- The discrete Wigner function also shows correspondence between classical and quantum mechanics.

7. Perspectives

- Analyze smooth periodic potentials like pendulum potential.
- Optimize the numerical computation to perform larger calculations.
- Analize the relation between off-diagonal phase-space basis functions and the Ruelle-Pollicott resonances.