Topological Quantum Computation

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I. Braiding Operators as Universal Quantum Gates

II. Knot Theoretic Models for Anyonic Quantum Computing based on TQFT

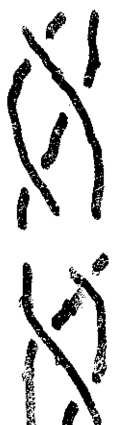
I. Braiding Operators as Universal Quantum Gates

$$V = \{\alpha|0\rangle + \beta|1\rangle \mid \alpha, \beta \in \mathbb{C}\}$$

space for single qubit.

$V \otimes V$
$\uparrow \otimes$
$V \otimes V$

For our purpose a braiding operator is a (unitary) transformation $R : V \otimes V \longrightarrow V \otimes V$ that satisfies the braid identity



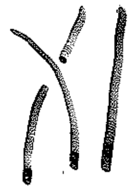
$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$$

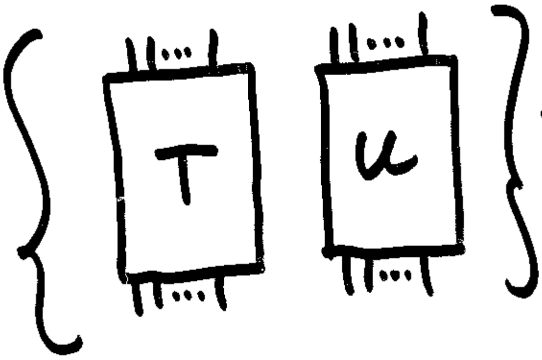
(Yang-Baxter Equation)

$$V\otimes V \xleftarrow{R} V\otimes V$$

diagram for R

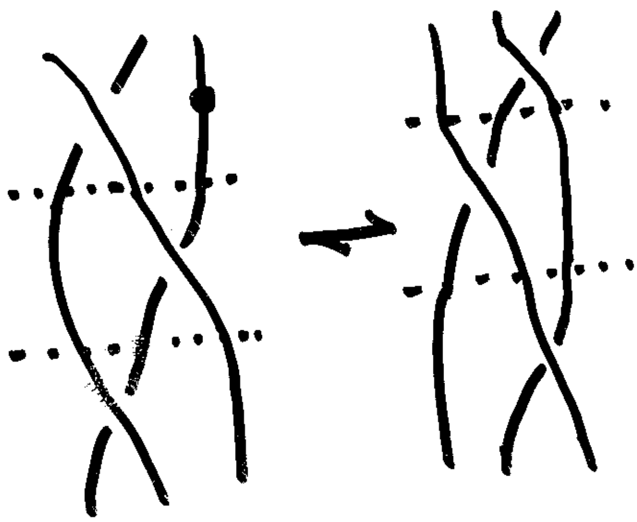
<u>Diagram for Braiding Operator</u>

 $\leftrightarrow R\otimes I$

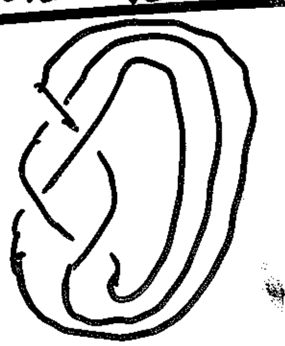


$T\otimes U$

<u>Diagram for tensor product.</u>



<u>Braid Relation</u>

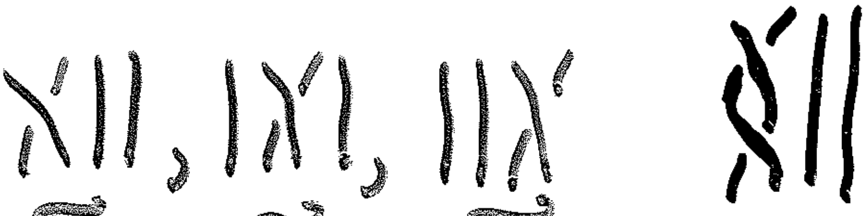


$$(R\otimes I)(I\otimes R)(R\otimes I) = (I\otimes R)(R\otimes I)(I\otimes R)$$

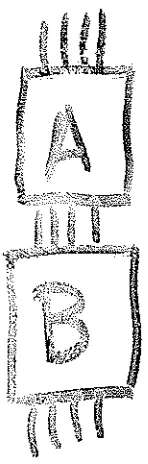
Each invertible braiding operator gives rise to a representation of the Artin Braid groups B_n (for each n).

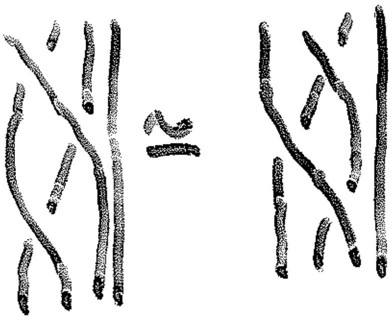
Braids and Braid Group

n=4

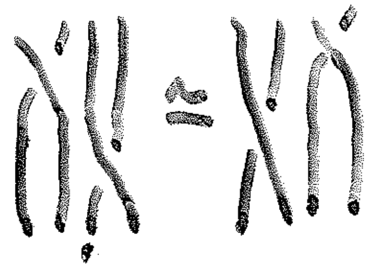


σ_1, σ_2, σ_3 $\qquad \sigma_1^2$





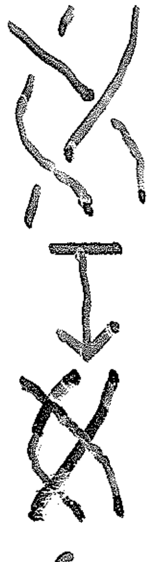
$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$



$$\sigma_i\sigma_j = \sigma_j\sigma_i$$
$$|i-j| \geq 2$$

$$B_n = \left(\sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i \\ = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ [\sigma_i, \sigma_j] = 0 \quad |i-j| \geq 2 \end{array} \right)$$

$$\downarrow$$



$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$S_n = \left(\tau_1, \dots, \tau_{n-1} \;\middle|\; \begin{array}{l} \tau_i\tau_{i+1}\tau_i \quad \tau_i^2 = 1 \\ = \tau_{i+1}\tau_i\tau_{i+1} \\ \tau_i\tau_j = \tau_j\tau_i \quad |i-j| \geq 2 \end{array} \right)$$

A well-known example of a "universal gate" is $\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

$\text{CNOT} + \left\{ \underset{V \longrightarrow V}{\text{Local Unitary Transfs}} \right\}$

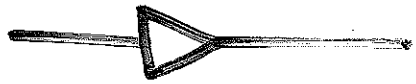
generates all unitary transfs.
(Hence generates quantum computing)

Theorem. (The Brylinskis). A gate $G : V \otimes V \to V \otimes V$ is universal (i.e. can replace CNOT above) if and only if G is <u>entangling</u>. $\frac{|01\rangle + |10\rangle}{\sqrt{2}}$

A gate G is <u>entangling</u> if $G|\psi\rangle \in V \otimes V$ is <u>not decomposable</u> as $|\lambda\rangle \otimes |\mu\rangle$ for some $|\psi\rangle$ that <u>is</u> decomposable.

(<u>Fact</u>: $a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$ is entangled $\Leftrightarrow \operatorname{Det}\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \neq 0$.)

NOT



$0 \rightsquigarrow 1$

$1 \rightsquigarrow 0$

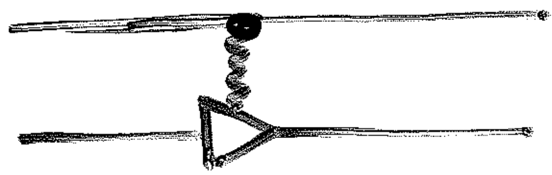
Classical Bits

$\boxed{\text{CNOT is entangling}}$

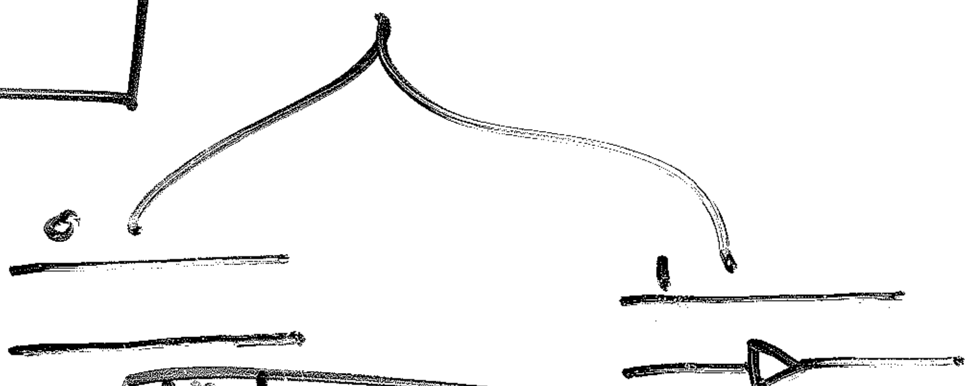
$\exists\, |\psi\rangle$ unentangled s.t. $\mathrm{CNOT}|\psi\rangle$ entangled.

CNOT – controlled NOT

Suff for Quantum Comp:
CNOT
+
local unitary transfs.
[on one Qubit]
SU(2)



unitary



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Nielsen & Chuang
"Quantum Comp"

$$[\mathrm{CNOT}] = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & 0 & 0 & 0 \\ 01 & 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & 0 & 1 \\ 11 & 0 & 0 & 1 & 0 \end{array}$$

$\boxed{|01\rangle = |0\rangle \otimes |1\rangle}$

u

$|\psi\rangle$ $\quad u|\psi\rangle$

local

$$\left.\begin{aligned} \mathrm{CNOT}|00\rangle &= |00\rangle \\ \mathrm{CNOT}|01\rangle &= |01\rangle \\ \mathrm{CNOT}|10\rangle &= |11\rangle \\ \mathrm{CNOT}|11\rangle &= |10\rangle \end{aligned}\right\}$$

Examples of universal gates that are also solutions to the Yang-Baxter-Equation:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

$|a| = |b| = 1$

$a^2 \neq b^2.$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

(Bell Basis Matrix)

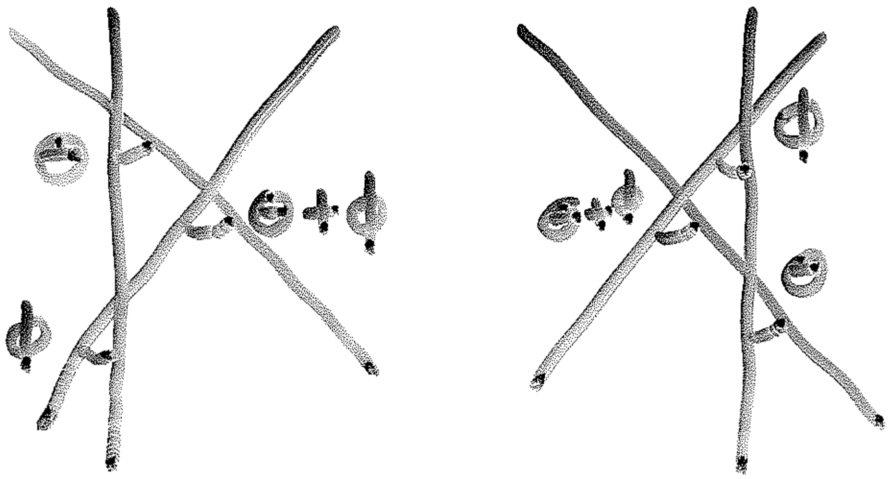
AR.FINV

See also paper by Franko, Howell, Wang

This one can detect non-trivial linking such as the Borromean Rings.

See quant-ph/0401090 ; "Braiding Operators are Univ Quantum Gates" + New J. Physics

Yang–Baxter Equation with spectral parameter

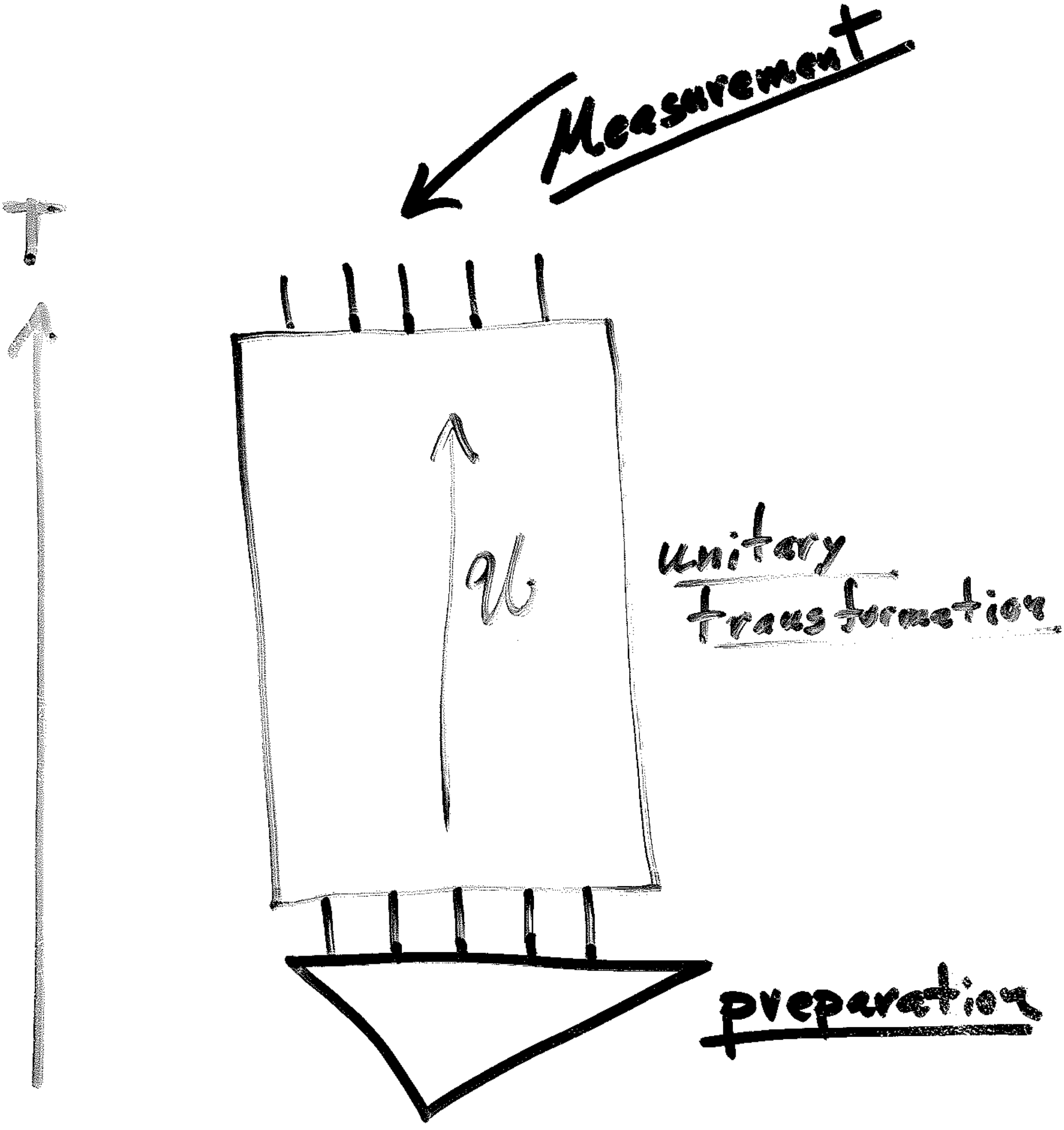


$\bar{c}$ Yong Zhang + Mo Lin Ge (see arxiv)

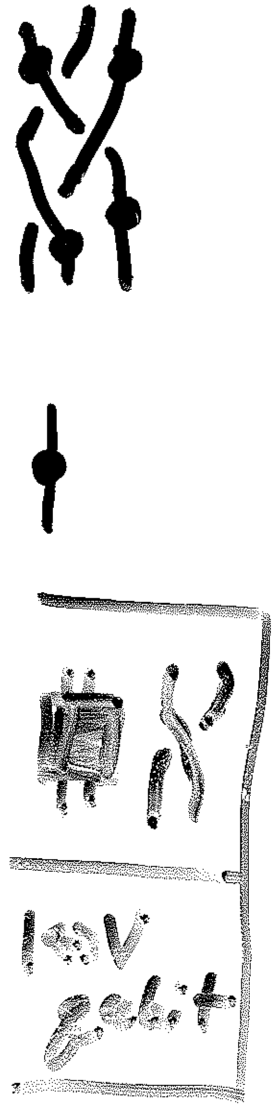
(or x, xy, y vs y, xy, x)

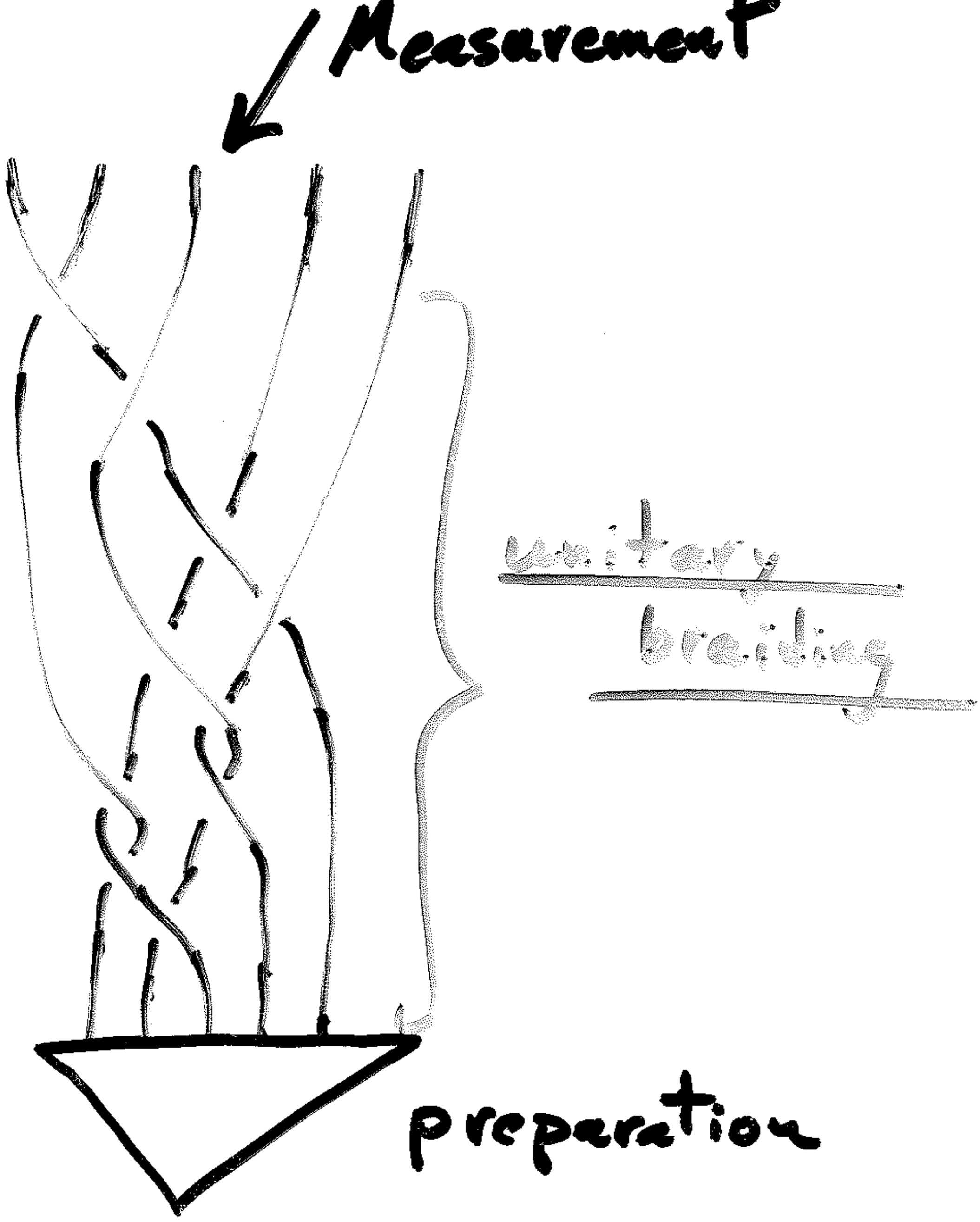
e.g.
$$\begin{pmatrix} q - x/q & & & \\ & \left(-\frac{1}{q}+q\right)x & 1-x & \\ & 1-x & -\frac{1}{q}+q & \\ & & & -\frac{1}{q}+qx \end{pmatrix}$$

$\boxed{\text{\underline{unitary} for } |x| = 1}$



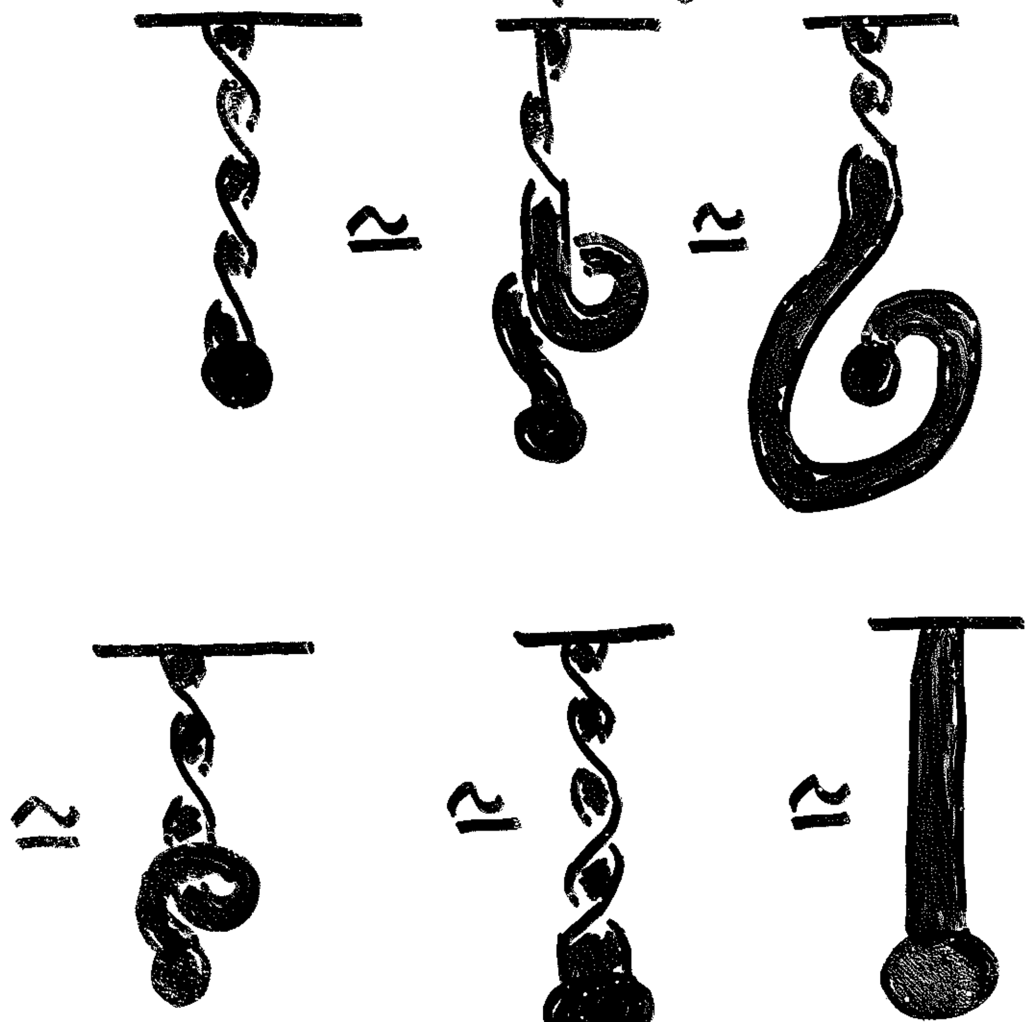
A Quantum Computer





A Topological Quantum Computer

Topology



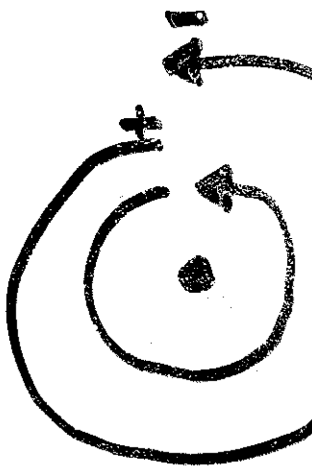
Dirac String Trick

$\longleftrightarrow$ 2-fold cover

g

$\boxed{\lambda_g : \mathbb{R}^3 \to \mathbb{R}^3, \quad \lambda_g(v) = g v g^{-1}}$

$$S^3 = SU(2) \xrightarrow{2-1} SO(3)$$



$\longleftrightarrow$ Change in phase of wavefunction for a fermion.

Quaternions & Rotations

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \in SU(2) \qquad \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

$$\longleftrightarrow \quad a+bi+(c+di)j$$

$$= \boxed{a+bi+cj+dk}$$

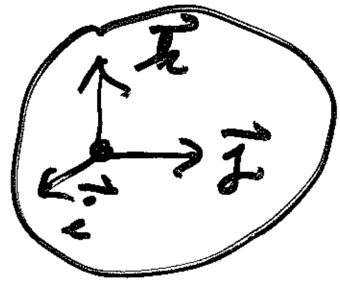
$$i^2 = j^2 = k^2 = ijk = -1$$

Quaternions can be written in form $a+b\vec{u}$ $\quad a^2+b^2=1$

$$e^{\vec{u}\theta} = a+b\vec{u}$$

$$\vec{u} = u_1 i + u_2 j + u_3 k \in S^2 \subset \mathbb{R}^3$$

$$\boxed{\vec{u}\vec{v} = -\vec{u}\cdot\vec{v} + \vec{u}\times\vec{v}}$$



$$\pi: SU(2) \longrightarrow SO(3)$$

$$\pi(g)\vec{v} = g\vec{v}\bar{g}$$

$$g = a+b\vec{u}, \quad \bar{g} = a-b\vec{u}$$

$$a = \cos(\theta/2)$$

$$\boxed{\pi(g) = \text{rotation about } \vec{u} \text{ by angle } \theta.}$$

$$g\vec{v}\bar{g} = (a^2-b^2)\vec{v} + 2ab\,\vec{v}\times\vec{u} + 2(\vec{v}\cdot\vec{u})b^2\vec{u}$$

Solving Braid Relation in $\boxed{SU(2)}$

$g, h \in SU(2)$ = unit length quaternions.

 $\leftrightarrow g$  $\leftrightarrow h$

Want $ghg = hgh$

$\Leftrightarrow \quad h^{-1}gh = ghg^{-1}$

If $g = a + bu$, $a^2 + b^2 = 1$, $u \in \mathbb{R}^3$

$h = c + dv$, $c^2 + d^2 = 1$, $v \in \mathbb{R}^3$

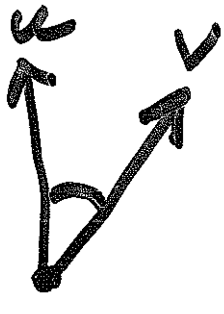
$\|u\| = \|v\| = 1$

$ghg^{-1} = a + bu^{g}$

$h^{-1}gh = c + dv^{h^{-1}}$

$\Rightarrow$ $a = c$, $d = \pm b$. Take $d = b$.

So $\boxed{\begin{array}{l} g = a + bu \\ h = a + bv \end{array}}$ $\quad u^{g} = v^{h^{-1}}$



$\longleftrightarrow$ (some calculation)

$$\boxed{u \cdot v = \frac{a^2 - b^2}{2b^2}}$$

So lots of representations

$B_3 \longrightarrow SU(2)$

(and even dense in $SU(2)$)

$$\boxed{g = a + bu \quad , \quad h = a + bv}$$

$$ghg = (a+bu)(a+bv)(a+bu)$$

$$= (a^2 + abv + abu + b^2uv)(a+bu)$$

$$\begin{aligned}
= a^3 &+ a^2bv + a^2bu + ab^2uv \\
-ab^2 &\qquad\quad + a^2bu \qquad\quad + ab^2vu \\
&\qquad\qquad\qquad + b^3uvu
\end{aligned}$$

$$\begin{aligned}
= a^3 - ab^2 &+ 2a^2bu + a^2bv \\
&+ ab^2\underbrace{(uv + vu)}_{= -2u\cdot v} \\
+ b^3 \underbrace{uvu}_{= -2(u\cdot v) + v}
\end{aligned}$$

$$\begin{aligned}
= &(a^3 - ab^2 - 2ab^2\, u\cdot v) \\
&+ (2a^2b - 2b^3\, u\cdot v)u \\
&+ (a^2b + b^3)v
\end{aligned}$$

$$ghg = hgh \iff a^2b + b^3 = 2a^2b - 2b^3(u\cdot v)$$

$$\iff b^3 = a^2b - 2b^3\, u\cdot v$$

$$2u\cdot v = \frac{a^2b - b^3}{b^3} = \frac{a^2 - b^2}{b^2}$$

$$\boxed{u\cdot v = \frac{a^2 - b^2}{2b^2}}$$

Example

$$\boxed{\begin{gathered} g = e^{i\theta} = a + bi \\ a = \cos(\theta) \\ b = \sin(\theta) \\ \\ h = a + b\left[(c^2 - s^2)i + 2cs\,k\right] \\ c^2 + s^2 = 1 \\ \text{\& require } c^2 - s^2 = \frac{a^2 - b^2}{2b^2} \end{gathered}}$$

Then $g \leftrightarrow \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = G$

$$H = FGF^*$$

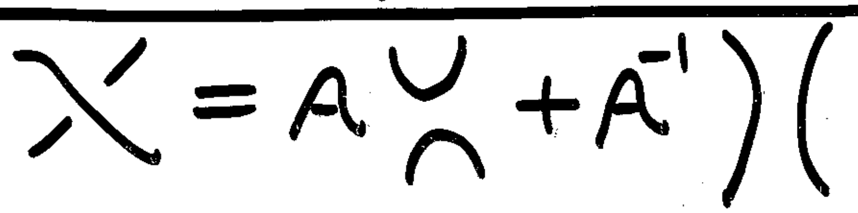
$$\boxed{F = \begin{pmatrix} ic & is \\ is & -ic \end{pmatrix}}$$

Then $GHG = HGH$

Later!

Bracket Polynomial Model

(of Jones polynomial)



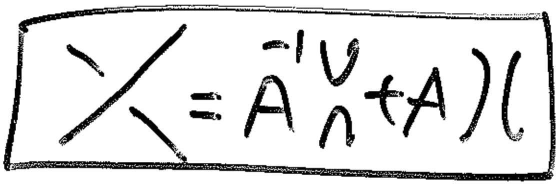
$$\bigcirc = -A^{2} - A^{-2} = \delta$$

e.g. 

$$= A\delta + A^{-1}\delta^{2}$$

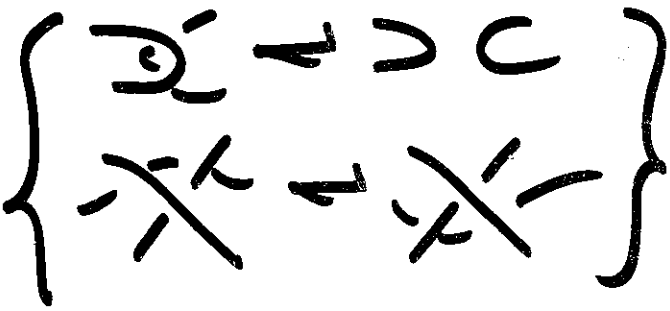
$$= (A + A^{-1}\delta)\delta$$

$$= (-A^{-3})\delta = \langle \infty \rangle$$



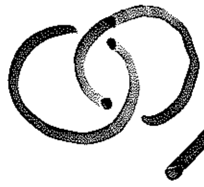
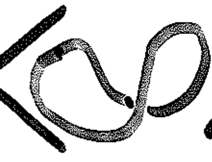
$$K \longmapsto \langle K \rangle$$

invariant under



Normalize by powers of $(-A^{3})$ to get invariance under 

$$\langle \bigcirc\bigcirc \rangle = \delta = -A^{2} - A^{-2}$$

$$\Big\langle$$  $$\Big\rangle = A \Big\langle$$  $$\Big\rangle + A^{-1}\Big\langle$$  $$\Big\rangle$$

K^{*} = mirror image $\Rightarrow$ $f_{K^{*}}(A) = f_{K}(A^{-1})$

$$= A(-A^{3}) + A^{-1}(-A^{-3})$$

$$= -A^{4} - A^{-4}$$

$\boxed{w(K) = 3}$

K

$$\Big\langle$$  $$\Big\rangle = A\Big\langle$$  $$\Big\rangle + A^{-1}\Big\langle$$  $$\Big\rangle$$

K^{*} 

$$= A(-A^{4} - A^{-4}) + A^{-1}(-A^{-3})^{2}$$

$$= -A^{5} - A^{-3} + A^{-7}$$

$$f_{K} = (-A^{3})^{-3}(-A^{5} - A^{-3} + A^{-7}) = A^{-4} + A^{-12} - A^{-16}$$

$$\langle |||\cdots|\overset{i}{\times}|\cdots| \rangle = A\langle ||\cdots|\overset{i}{\asymp}|\cdots| \rangle + A^{-1}\langle ||\cdots||||\cdots| \rangle$$

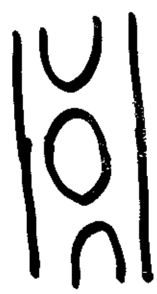
$$\langle \sigma_{i} \rangle = A\langle U_{i} \rangle + A^{-1}\langle \mathbb{1} \rangle$$

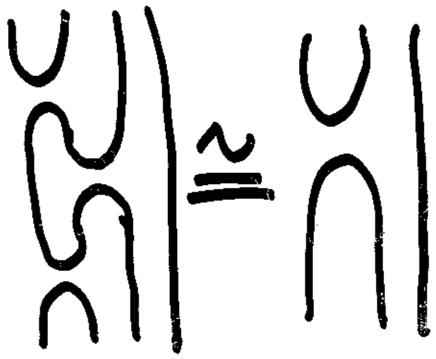
Representation of Braid Group to Temperley-Lieb Algebra.

Temperley Lieb Algebra

U_1 $\quad U_2$ $\quad U_3$ $\quad I$

 δ

 $U_i^2 = \delta U_i$



$U_i U_{i+1} U_i \cong U_i$

$$\boxed{\begin{array}{c} U_i U_j = U_j U_i \\ |i-j| \geq 2 \end{array}}$$

But Also

$P = |v\rangle\langle v|$, $Q = |w\rangle\langle w|$

$\Rightarrow PQP = k P$

Making Unitary Braid Group Representations (a)

$$\rho(\sigma) = A\,U + A^{-1} I$$

$$U \leftrightarrow \genfrac{}{}{0pt}{}{\smile}{\frown}\,, \quad I \rightarrow \|$$

If $U^{ab} = \cap_{ab} = \begin{pmatrix} 0 & iA \\ -iA^{-1} & 0 \end{pmatrix}$

then $\rho(\sigma)$ unitary just for $A = \pm i$

<u>Needed</u>: other reps of TL.

N.B. $P = \;><\;, \quad Q = \;][$

$P^2 = \Diamond P, \quad Q^2 = \square Q$

$PQP = \;>\!\lhd\!\rhd\!<\; = \lhd\!\rhd\, P$

So can use

$$P = |v\rangle\langle w|$$

$$Q = |w\rangle\langle w|$$

N.B. $\begin{pmatrix} a \\ b \end{pmatrix} (a\ b) = \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} = |v\rangle\langle w|$

$a^2 + b^2 = 1$ and use to make P's, Q's.

(b)

$$U_1 = \delta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U_2 = \delta \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$$

$$a^2 + b^2 = 1$$

$$\delta = a^{-2}$$

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = |\tbinom{1}{0}\rangle\langle\tbinom{1}{0}|$$

$$e_2 = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} = |\tbinom{a}{b}\rangle\langle\tbinom{a}{b}|$$

$$e_1^2 = e_1, \quad e_2^2 = e_2$$

$$e_1 e_2 e_1 = a^2 e_1$$

$$e_1 e_2 = \begin{pmatrix} a^2 & ab \\ a & 0 \end{pmatrix}$$

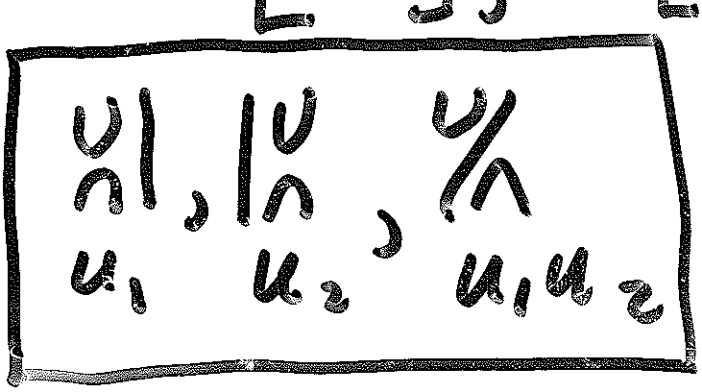
$$e_2 e_1 = \begin{pmatrix} a^2 & 0 \\ ab & 0 \end{pmatrix}$$

$$Tr(U_1) = \delta$$

$$Tr(U_2) = \delta$$

$$Tr(U_1 U_2) = 1$$

$$U_1 = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} \delta^{-1} & \sqrt{1-\delta^{-2}} \\ \sqrt{1-\delta^{-2}} & \delta - \delta^{-1} \end{bmatrix}$$



$$\rho(\asymp|) = AU_1 + A^{-1}I$$
$$\rho(|\asymp) = AU_2 + A^{-1}I$$
$$\delta = -A^2 - A^{-2},\ A = e^{i\theta}$$
$$= -2\cos(2\theta)$$

Need $\delta^2 \geq 1$: $\cos^2(2\theta) \geq \frac{1}{4}$

ρ gives unitary rep of 3-strand braids $\longrightarrow U(2)$

$B_3 \longrightarrow$

<u>and</u> bracket poly given by

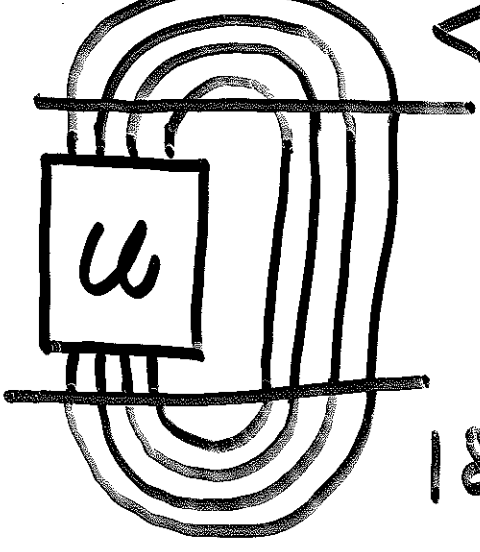
$$\langle \bar{b} \rangle = \underbrace{tr(\rho(b))}_{} + A^{w(b)}(\delta^2 - 2)$$

$w(b)$ = sum of exponents of braid

Calculate the trace by "interferometry".

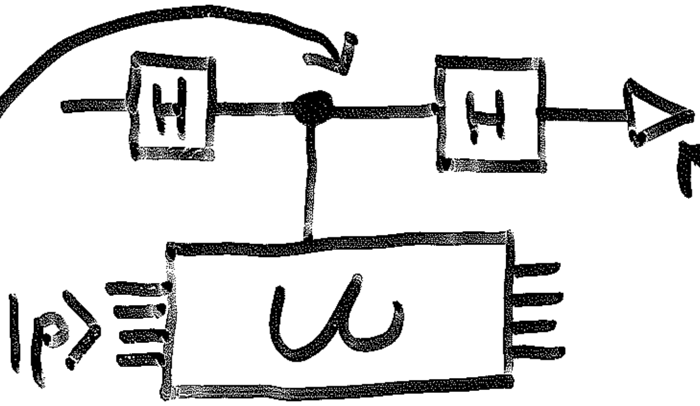
Jones Poly for B_3 via quantum computation.

Trace



$$|\delta\rangle = \sum_{\alpha} |\alpha\rangle|\alpha\rangle / \sqrt{2^n}$$

$$\langle\delta|U|\delta\rangle = |tr(U)|/2^n$$



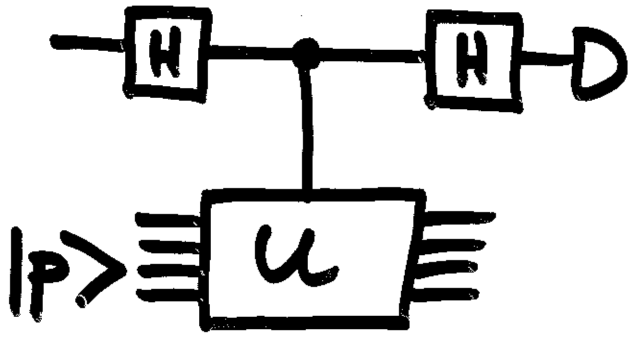
Controlled U

$$\frac{1}{2} + \frac{1}{2} Re(\langle p|U|p\rangle)$$

Expectation Value of the process

$$|1\rangle \otimes U|p\rangle + |0\rangle \otimes |p\rangle$$

Then apply H.



$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$(H|0\rangle)\otimes|p\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\otimes|p\rangle$$

$$\Big\downarrow \text{controlled } U$$

$$\frac{1}{\sqrt{2}}(|0\rangle\otimes|p\rangle + |1\rangle\otimes U|p\rangle)$$

$$\Big\downarrow H\otimes \mathrm{id}$$

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\otimes|p\rangle + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\otimes U|p\rangle\right)$$

$$\|$$

$$\frac{1}{2}\Big(|0\rangle\otimes(|p\rangle - U|p\rangle) + |1\rangle\otimes(|p\rangle - U|p\rangle)\Big)$$

$$\Big\downarrow \text{Measure}$$

$$|0\rangle : \frac{\big|\,|p\rangle + U|p\rangle\big|^2}{4} = \frac{\langle p|U^*|p\rangle + \langle p|p\rangle + \langle p|p\rangle + \langle p|U|p\rangle}{4}$$

$$= \frac{1}{2} + 2\,\mathrm{Re}\langle p|U|p\rangle$$

Braiding Operators still need arbitrary elts of $U(2)$ for local unitary transfs.
(but see previous remark)

Freedman, Kitaev, Larsen, Wang suggest using topological quantum field theory to make deeper models.
These theories can be used to produce unitary braid group representations.

A. Schwarz

E. Witten $\longleftrightarrow$ M. Atiyah

TQFT

II. Fibonacci Anyons and Knot Theoretic Models (For Fibonacci Anyons, see notes of Kitaev and notes of Preskill)

Very simple particle theory.

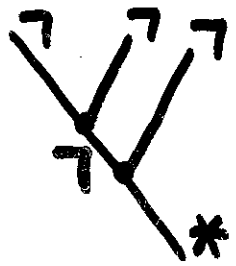
Two particles: $\neg$ "marked"

$*$ "unmarked"

$*$ acts as an empty word.

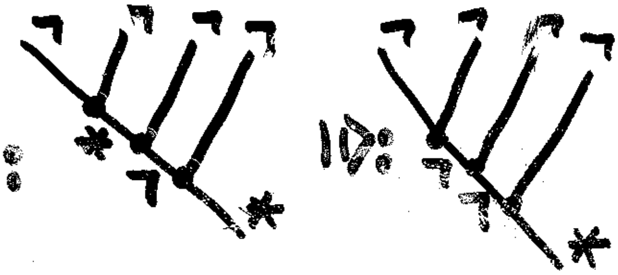
$\neg$ can interact with itself to produce $*$ or $\neg$.

Consider spaces of multiple interactions:



$$\dim V_{*}^{\neg\neg\neg} = 1$$

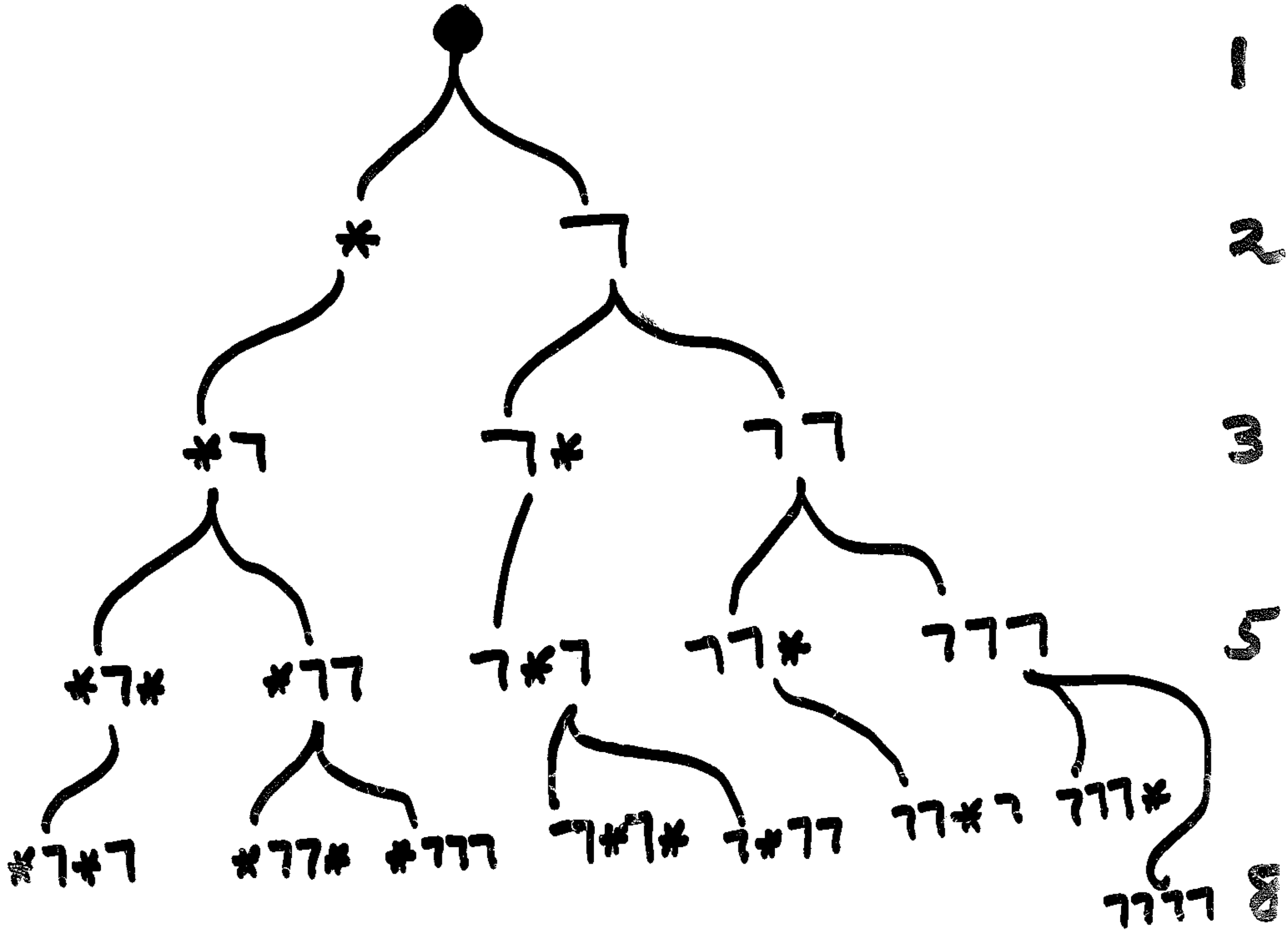
$|0\rangle$: $|1\rangle$:

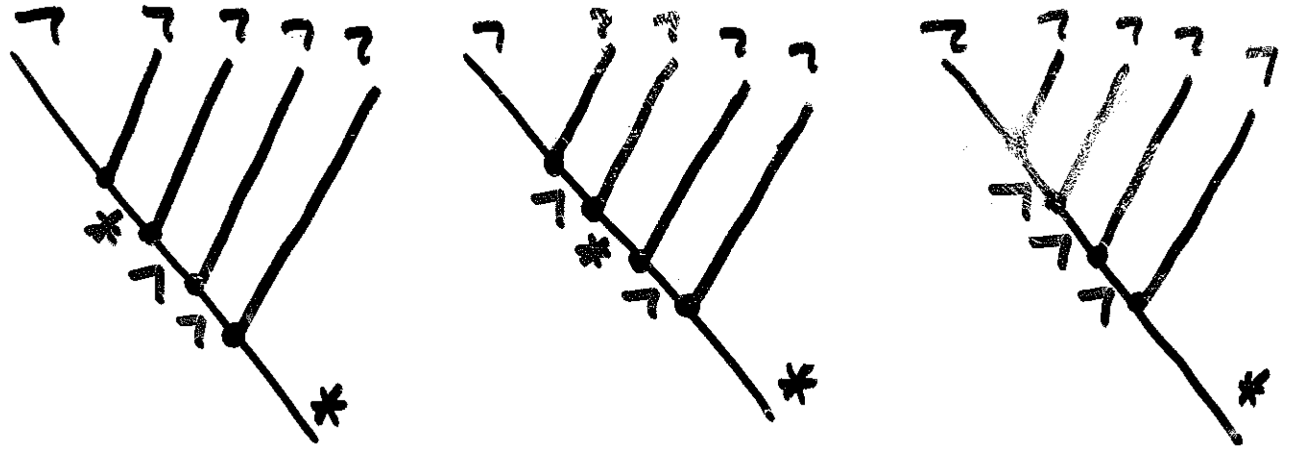


$$\dim V_{*}^{\neg\neg\neg\neg} = 2$$

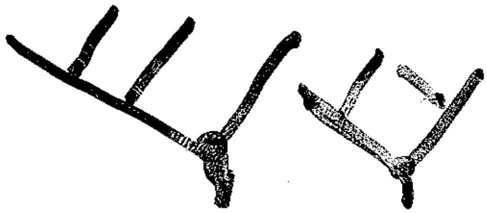
Seq in ¬, *

$\boxed{\text{** forbidden}}$





$$\dim V_{*}^{\tau^5} = 3$$

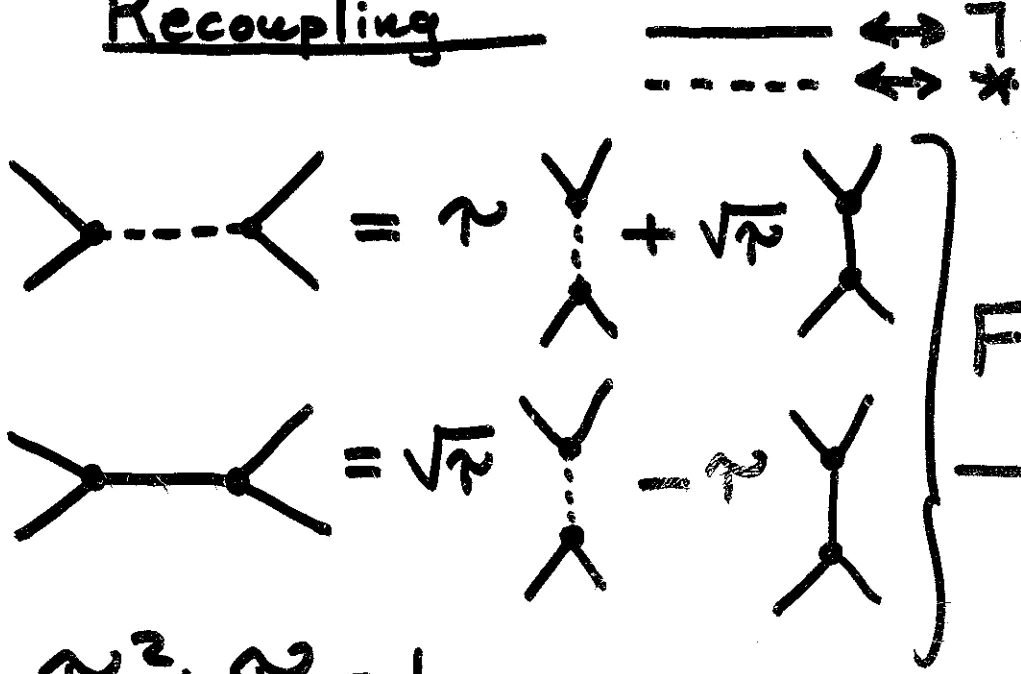


Generally: $\dim V_{*}^{(n)}$:

n	3	4	5	6	7
dim	1	2	3	5	8

Fibonacci numbers

Recoupling

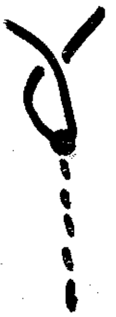


$$F = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}$$

$$\tau^2 + \tau = 1$$
$$\left(\tau = \frac{-1+\sqrt{5}}{2}\right)$$

Braiding

 $= -e^{i 2\pi/5}$ 

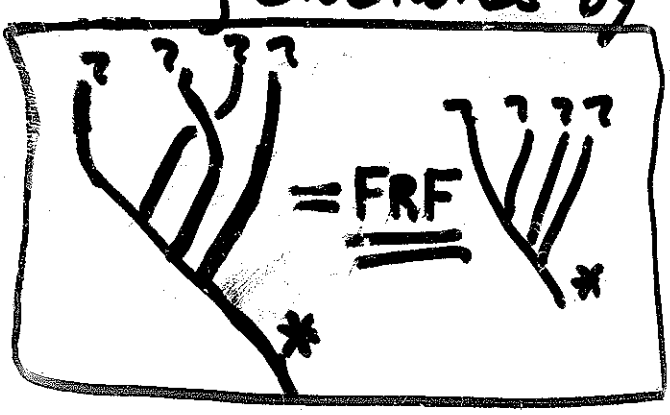
 $= e^{i 4\pi/5}$ 

$$R = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix}$$

$V = V_*^{\tau\tau\tau\tau}$ is acted on by B_3 = 3 strand braids

generated by  $\equiv S_1 \leftrightarrow R$

 $\equiv S_2 \leftrightarrow FRF$



In the Fibonacci Model, braids act on the single qubit space <u>and</u>

these representations of the braid groups

$$\rho : B_{f_n+1} \longrightarrow \mathrm{Aut}\left(V_*^{n+2}\right)$$

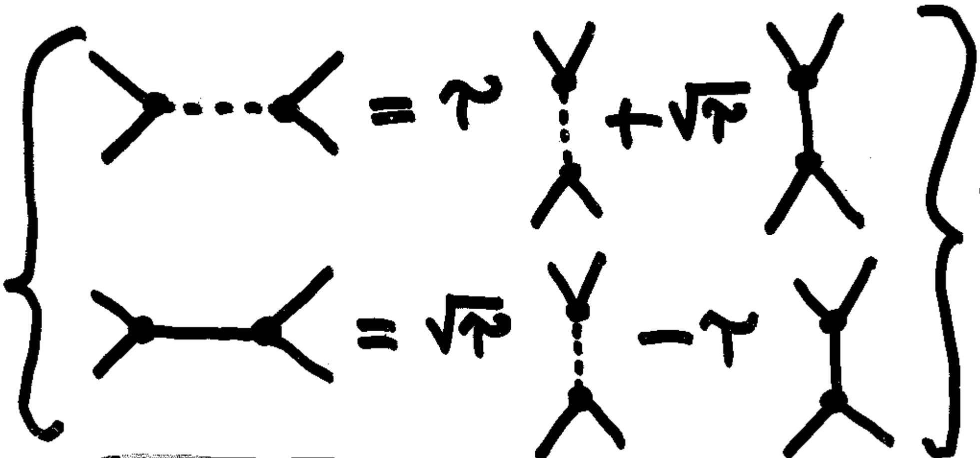
1	1	2	3	5	8	13	…
‖	‖	‖	‖	‖	‖	‖	
f_0	f_1	f_2	f_3	f_4	f_5	f_6	…

are <u>dense</u> in the corresponding unitary groups.

$$\boxed{\rho\left(B_{f_n+1}\right) \text{ dense in } U(f_n)}$$

This provides an in-principle representation of quantum computing in terms of braid groups.

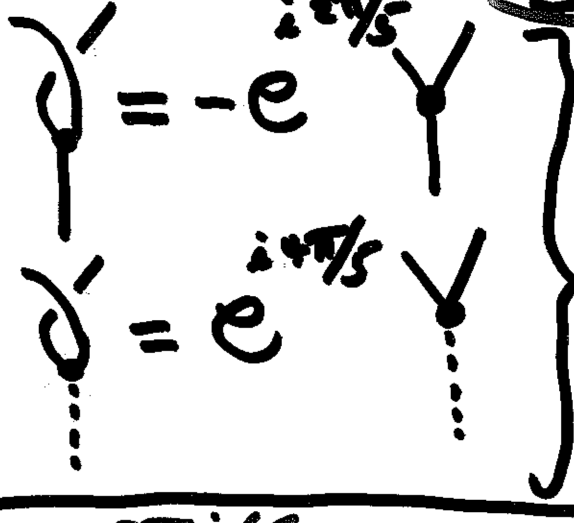
Fibonacci Model in a Nutshell



$R + FRF$ gen rep of B_3

$$F = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}$$

$$\tau^2 + \tau = 1, \quad \tau = \frac{1}{\Delta}, \quad \Delta^2 = 1 + \Delta$$
$$\Delta = (1+\sqrt{5})/2.$$



$$R = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix}$$

$$A = e^{3\pi i/5}, \quad \delta = -A^2 - A^{-2} = -2\cos(6\pi/5) = \frac{1+\sqrt{5}}{2}$$

$V_\tau^{\tau\tau\tau} = V$ gives

rep: $B_3 \longrightarrow U(2)$

$\left.\begin{array}{l} \text{rep}(\sigma_1) = R \\ \text{rep}(\sigma_2) = FRF \end{array}\right\}$ gens dense subset of $U(2)$.

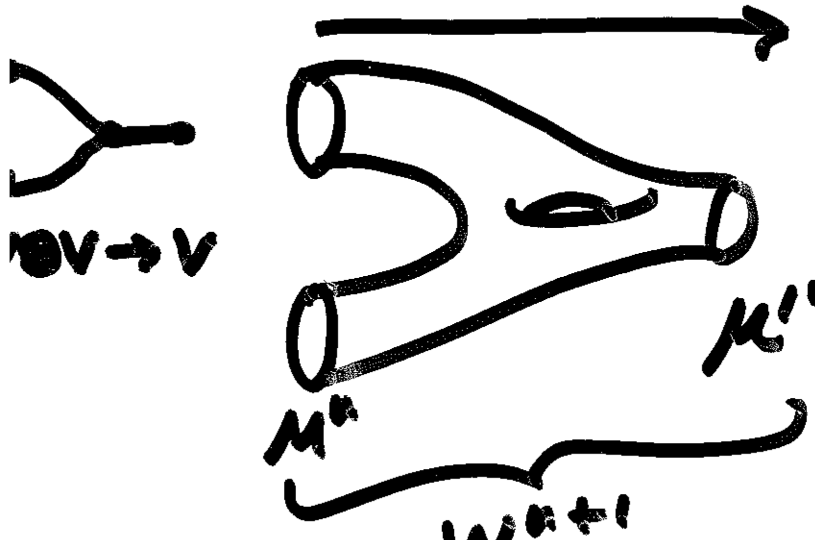
TQFT

Spin NetWorks

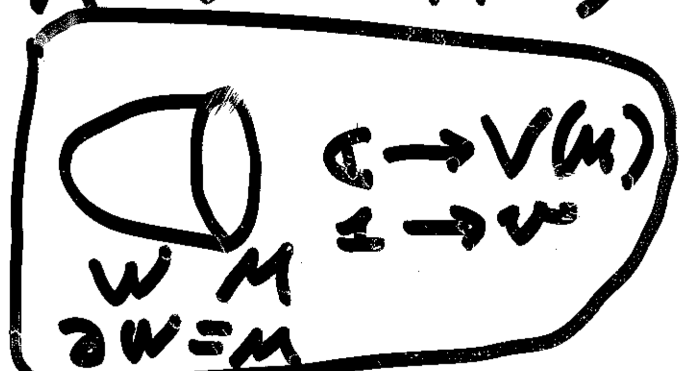
and

...

__Idea__: Category of cobordisms of manifolds M^n



$V(M^n) \longrightarrow V(M'^n)$



Functor from __Cobordisms__ to __Vect__.

For 3-manifolds, one thinks of

$M^3 = M^3_- \cup M^3_+$

$M^3_- \cap M^3_+ = S$, a surface

$V(S)$ = vector space assoc with S.

$M^3_+ : V(S) \to \mathbb{C} = \langle M^3_+|$

$M^3_- : \mathbb{C} \longrightarrow V(S) = |M^3_-\rangle$

$I(M^3) = \langle M^3_+ | M^3_- \rangle$ __the invariant__.

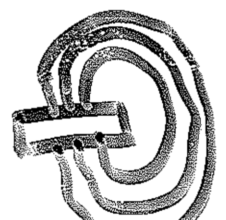
3-Manifold Invariants

$$\int \mathcal{D}a \, e^{\frac{ik}{4\pi}\int_{M^3} \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)}$$

$$\Big\updownarrow \qquad M^3 = M^3(K)$$

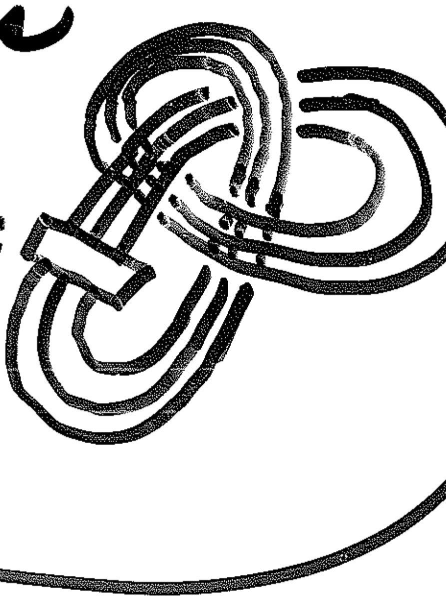
surgery along K

$$\sum_{a \in \text{admissibles}} \Delta_a \langle K^{(a)} \rangle$$



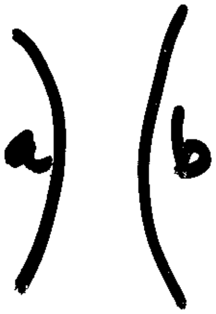
Δ_3

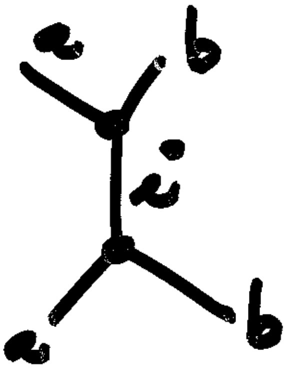
$^{(3)} =$



un-normalized 3 mfld inver

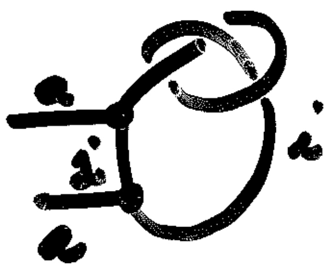
Question: Direct Heuristics??



$$= \sum_i \frac{\Delta_i}{\Theta(a,b,i)}$$


$$= \sum_i \Delta_i$$

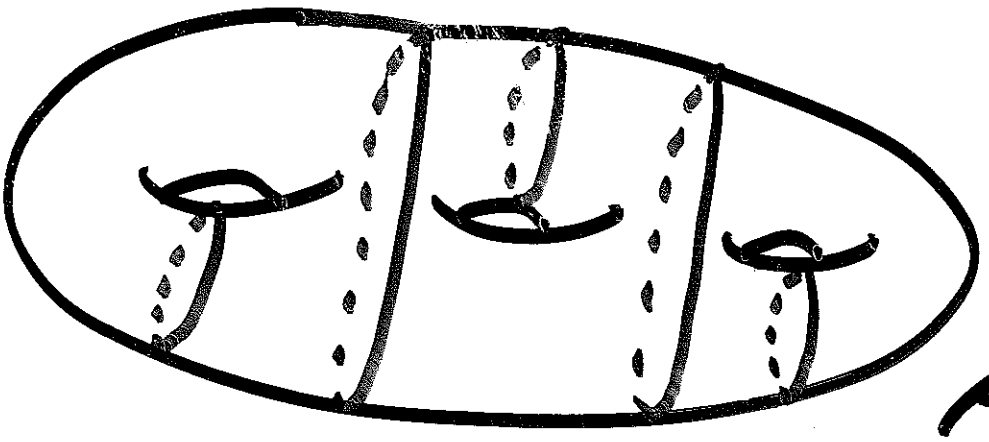

$$= \sum_i \Delta_i$$


$$= \sum_{i,j} \frac{\Delta_i \Delta_j}{\Theta(a,i,j)}$$


$$= \sum_{i,j} \frac{\Delta_i \Delta_j}{\Theta(a,i,j)}$$

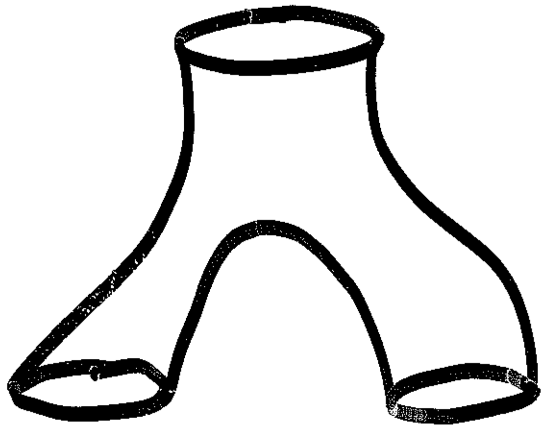

$$= \sum_j \Delta_j$$

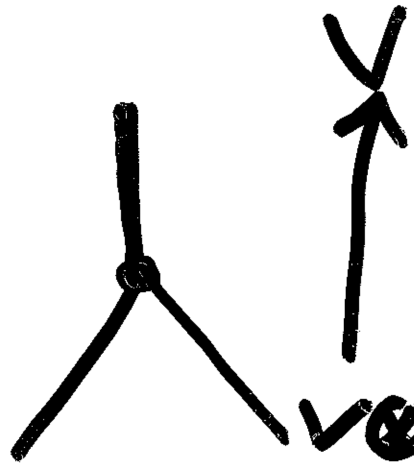

$$=$$

S

Surfaces can be decomposed into "pairs of pants"

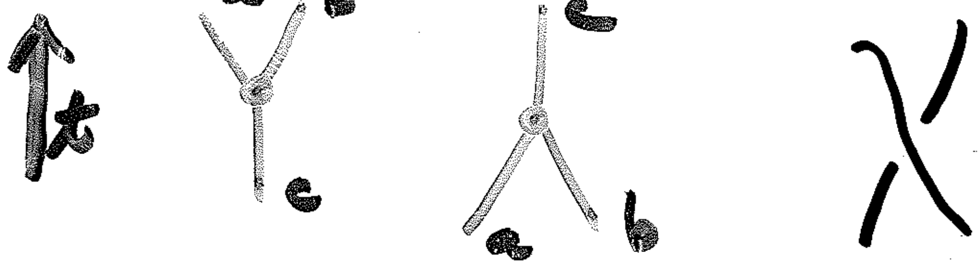




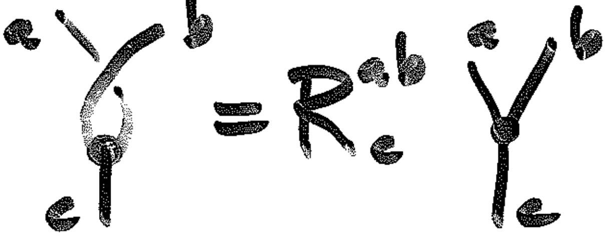
$V \otimes V$

schematic

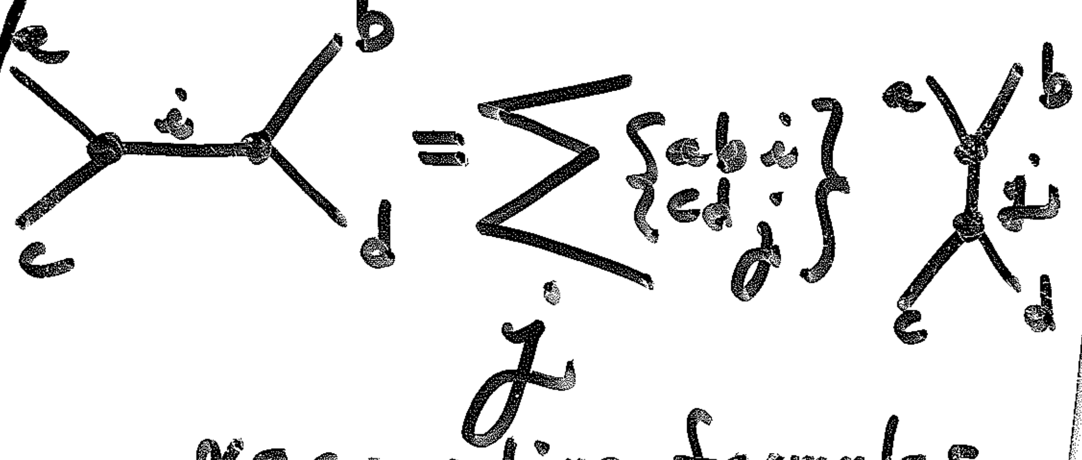
Underlying the structure of $V(S)$ is a theory of "particle interactions" with fusion and creation vertices

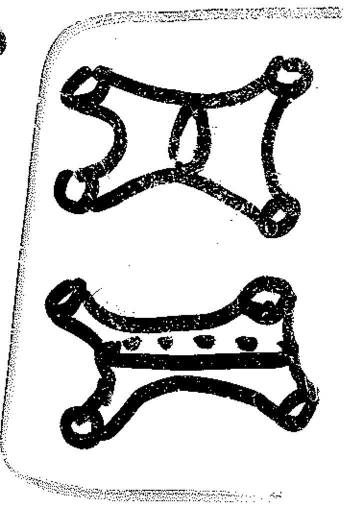


These "particle interactions" need to satisfy certain compatibility conditions to make the TQFT work:



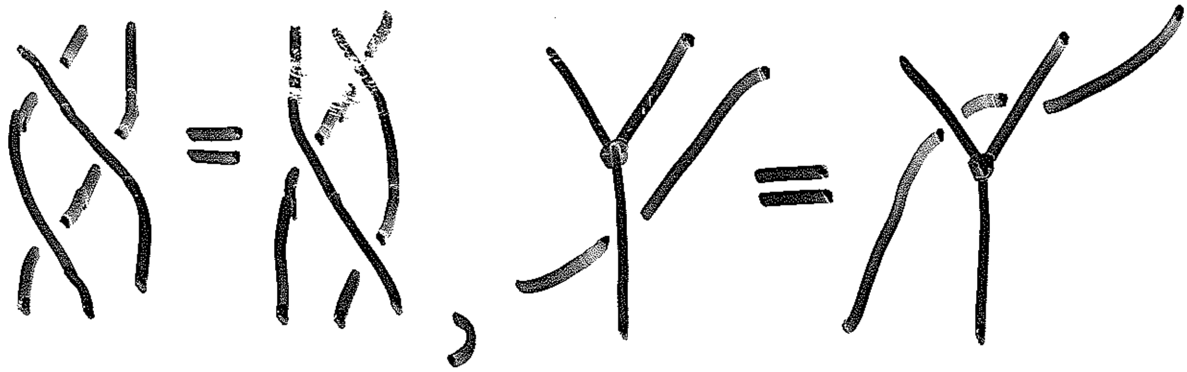
braiding operators





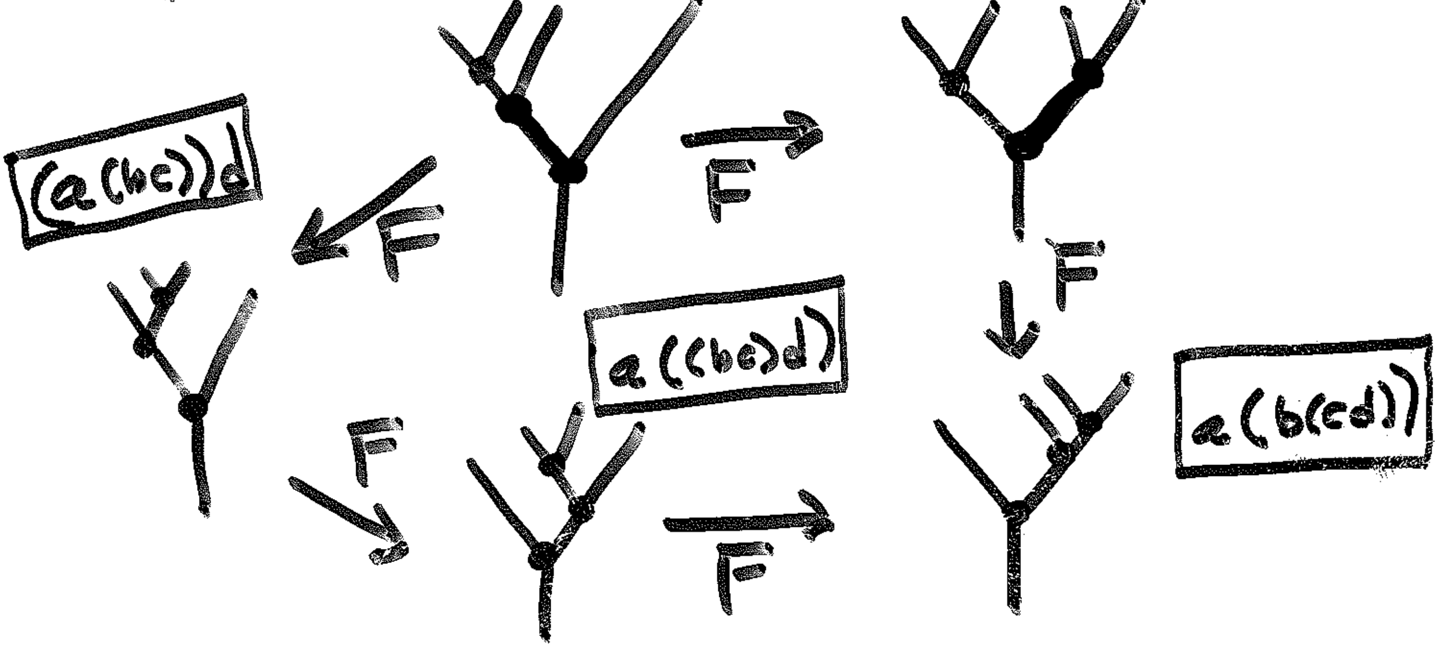
recoupling formulas

Identities : $\left\{ \begin{array}{l} \text{braiding} \\ \text{pentagon} \\ \text{naturality} \\ \text{hexagon} \end{array} \right\}$

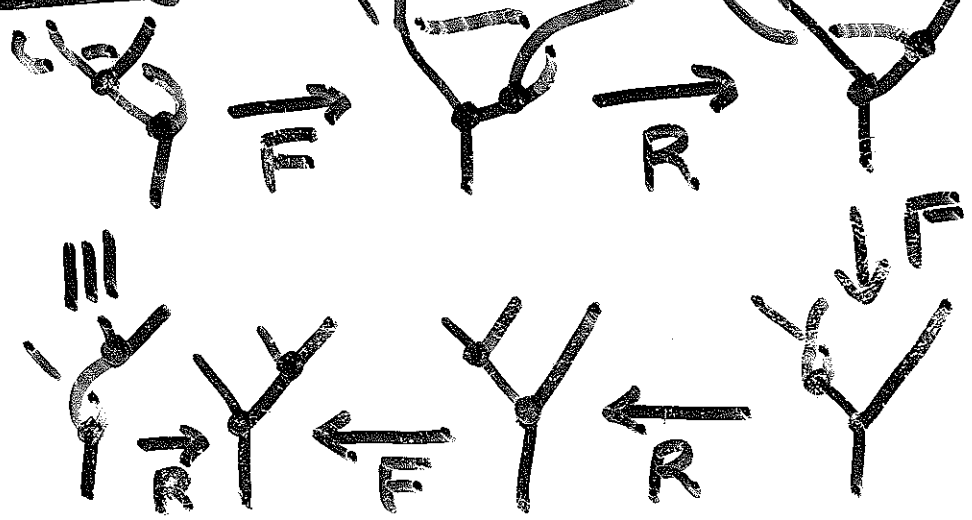


- braiding and naturality

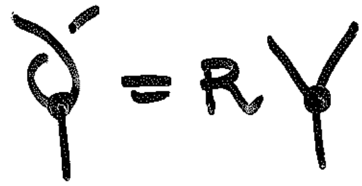
- pentagon $\boxed{((ab)c)d}$ $\boxed{(ab)(cd)}$



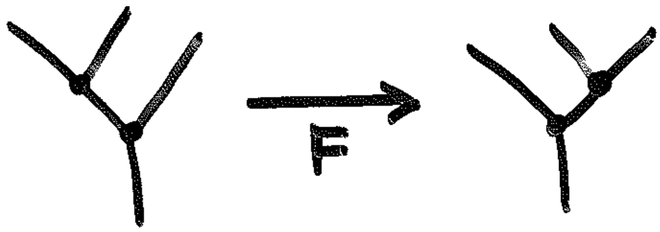
- hexagon



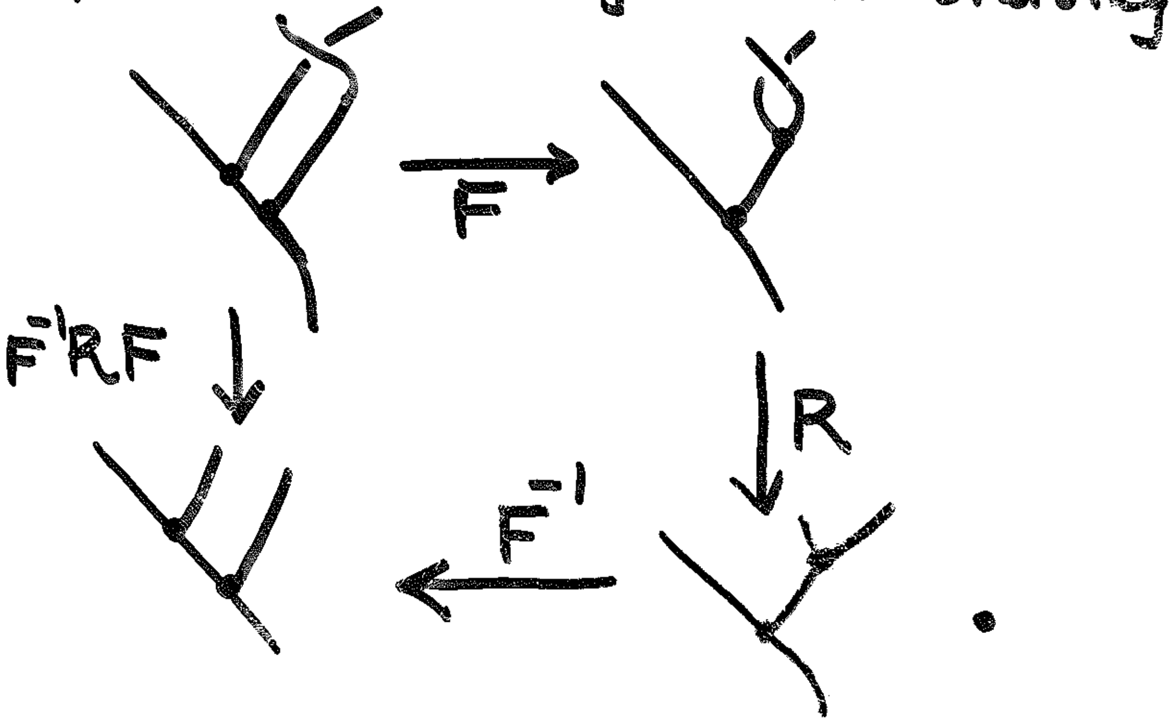
The purpose for TQFT in the present discussion is to produce a flexible source of representations of the braid groups.

Local braiding 

plus recoupling



produces more general braiding



Temperley – Lieb Recoupling Theory

(See Book by Kauff + Lins, PUP 1994)

- version of q-deformed spin-network theory based on the bracket polynomial model for the Jones polynomial.

$$K O = K\delta, \quad \delta = -A^{2} - A^{-2}$$

- <u>all the architecture of this theory is based on knot polynomial evaluations.</u>
- pentagon, hexagon, naturality are all automatic consequences of the topological structure of the models.
- leads to many unitary reps of braid group
- Fibonacci model is simplest case

Penrose Spin Networks

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$SL(2,\mathbb{C}) = \left\{ A \mid A \text{ 2x2 matrix } \& \ A\epsilon A^T = \epsilon \right\}$$

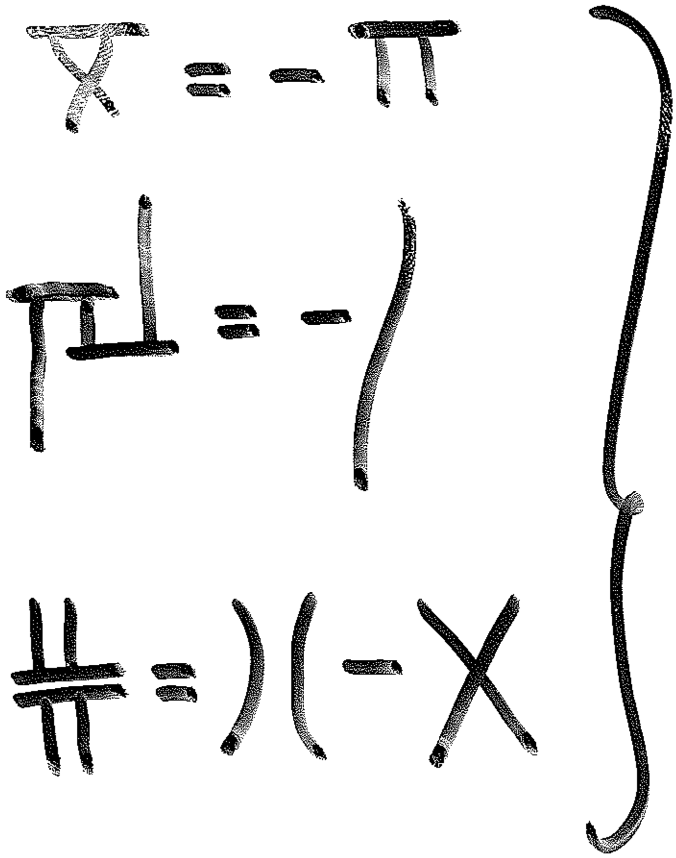
Let  =  $= \epsilon_{ab}$

 = vector v^a

$\langle v, w \rangle$ =  $= \epsilon_{ab} v^a w^b$

$: \ \epsilon^{ab}\epsilon_{cd} = \delta^a_c \delta^b_d - \delta^a_d \delta^b_c$

<u>Wanted</u> diagrammatic representation that is convenient and topologically invariant in the plane.



$\sqcap_{ab} = \varepsilon_{ab}$

$)^{a}_{b} = \delta^{a}_{b}$

$\sqcup\!\sqcap = \sum_{a,b} \varepsilon_{ab}\varepsilon^{ab} = 2$

Penrose adjusts the tensors: $\cup = \sqrt{-1}\ \sqcup$

$\cap = \sqrt{-1}\ \sqcap$

$\times \longmapsto -\times$

(i.e. $\times^{ab}_{cd} = -\delta^{a}_{d}\delta^{b}_{c}$

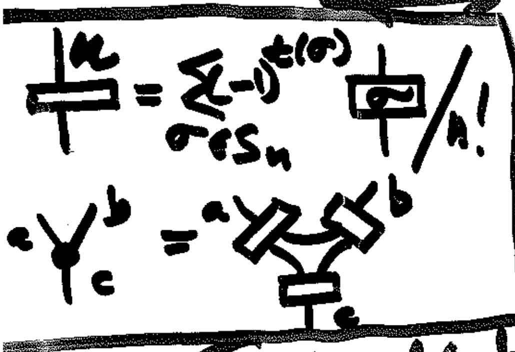
Then: $\left\{ \begin{array}{c} \asymp +)(+ \times = \emptyset \\ \bigcirc = -2 \end{array} \right\}$

$\boxed{\times = -\asymp -)(}$

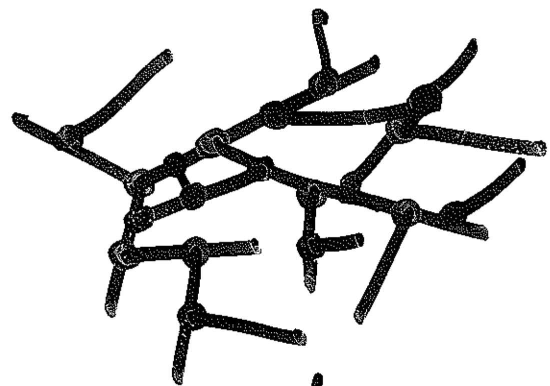
16.3

Recoupling calculus for Spin Nets

$\longleftrightarrow$ Version of SU(2) angular momentum recoupling theory.



Spin Nets :



$= \frac{1}{2!}\left[\,)(- \times\,\right]$

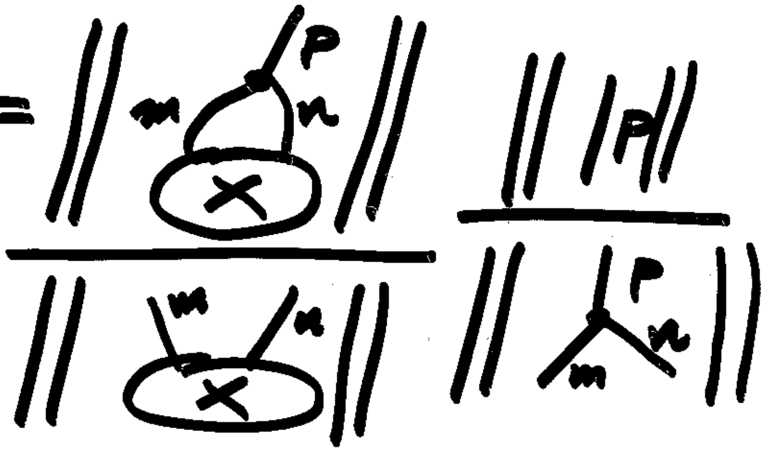
Arbitrary trivalent graphs... (labeled with SU(2) reps...)

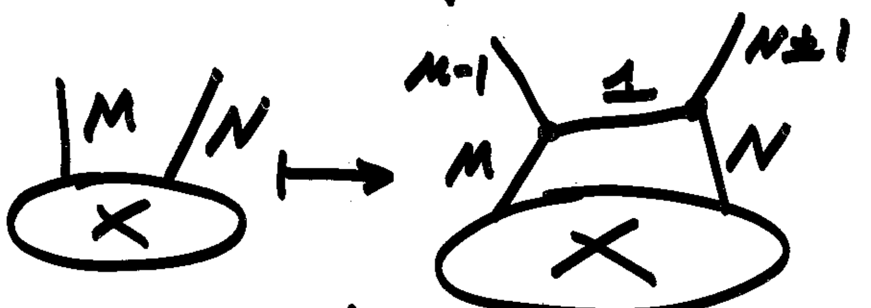
Penrose Spin Geometry Theorem

$\boxed{\asymp +)(+ \times = 0}$

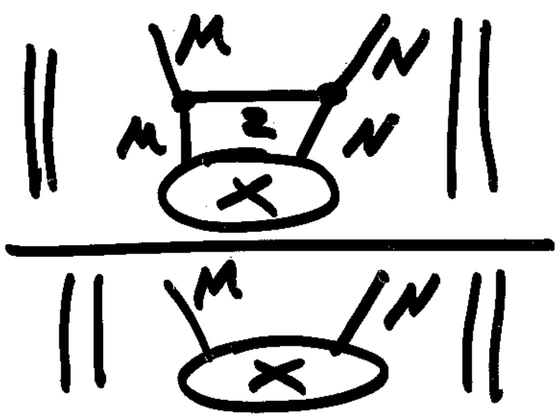
Spin net evaluations can be used to define angles between free-ends. For appropriate (experimentally repeatable) spin nets the angles satisfy restrictions of a set of directions in 3-space.

Spin Geometry

Prob("p") = 

Process: 

Assume probability unaffected by iteration.

Then $\frac{1}{2}\cos(\theta) \cong$ 

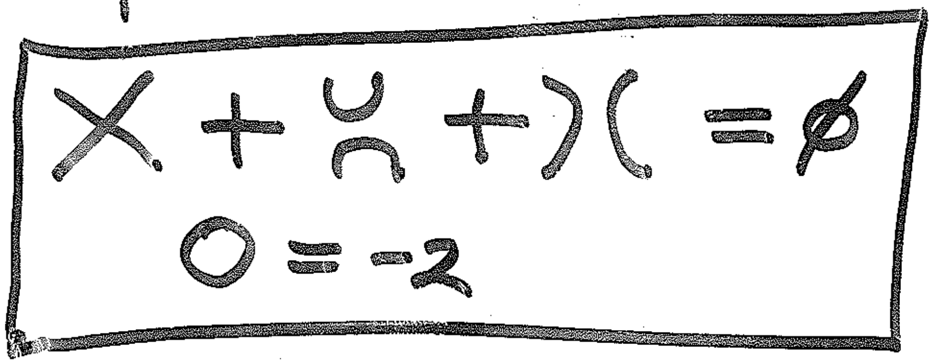
satisfies $\sphericalangle$s in $\mathbb{R}^3$

for $\left\{\begin{array}{l} \bullet \text{ large } M, N \\ \bullet \text{ iteration independence} \end{array}\right\}$.

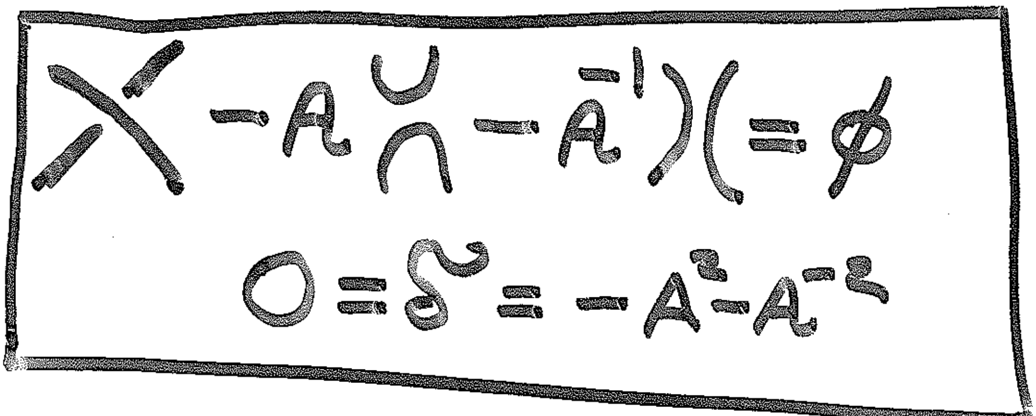
A) A - Deformed Spin Nets

"q"

Replace

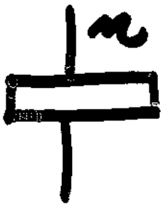


by

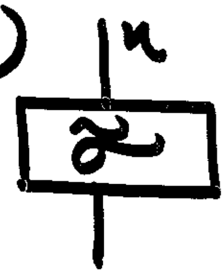


Underly calculations replaced by bracket polynomial evaluations.

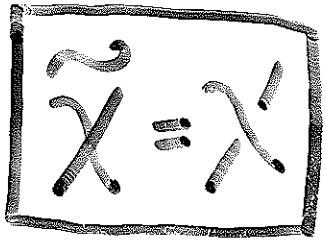
B) Projectors



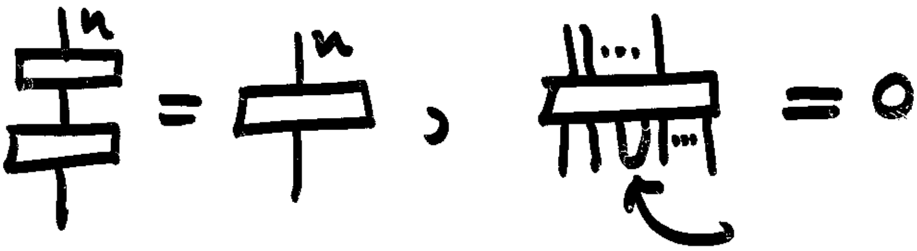
$$= \frac{1}{\{n\}!} \sum_{\sigma \in S_n} (A^{-3})^{t(\sigma)}$$



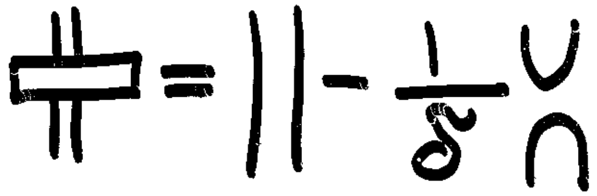
$$\{n\}! = \sum_{\sigma \in S_n} (A^{-4})^{t(\sigma)}$$



$\Rightarrow$

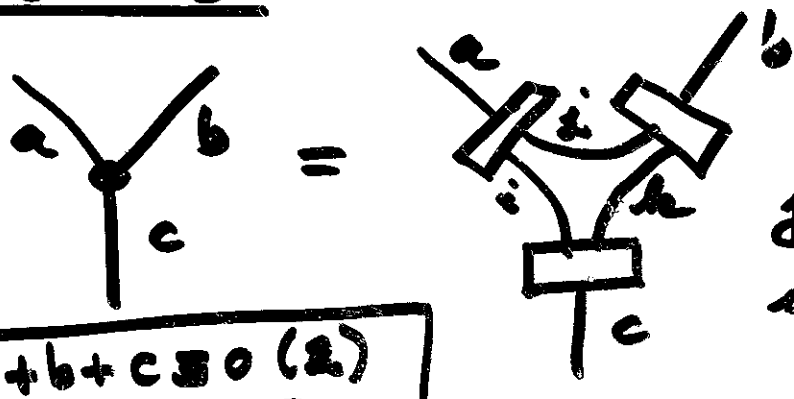


example:



after simplification

C) Vertices

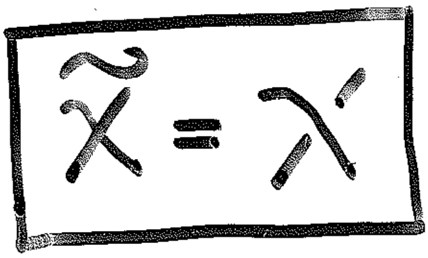


$i+j=a$
$j+k=b$
$i+k=c$

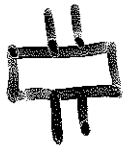
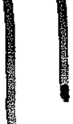
$$\boxed{\begin{array}{l} a+b+c \equiv 0 \ (2) \\ a+b \geq c \ \forall \ ③ \end{array}}$$

 $= \frac{1}{\{n\}!} \sum_{\sigma \in S_n} (A^{-3})^{t(\sigma)}$ 

$$\{n\}! = \sum_{\sigma \in S_n} (A^{-4})^{t(\sigma)}$$



$$\{2\}! = 1 + A^{-4}$$

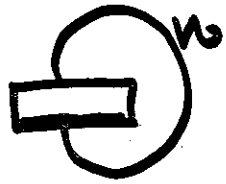
 $= \frac{1}{1+A^{-4}} \Big[$  $+ A^{-3}$  $\Big]$

$= \frac{1}{1+A^{-4}} \Big[$  $+ A^{-3} \big[A$  $+ A^{-1}$  $\big] \Big]$

$= \frac{1}{1+A^{-4}} \Big[(1+A^{-4})$  $+ A^{-2}$  $\Big]$

$=$  $+ \frac{1}{A^{2}+A^{-2}}$ 

 $=$  $- \frac{1}{\delta}$ 

$\Delta_n =$  n-strands

$$\Delta_{-1} = 0 \ , \ \Delta_0 = 1$$

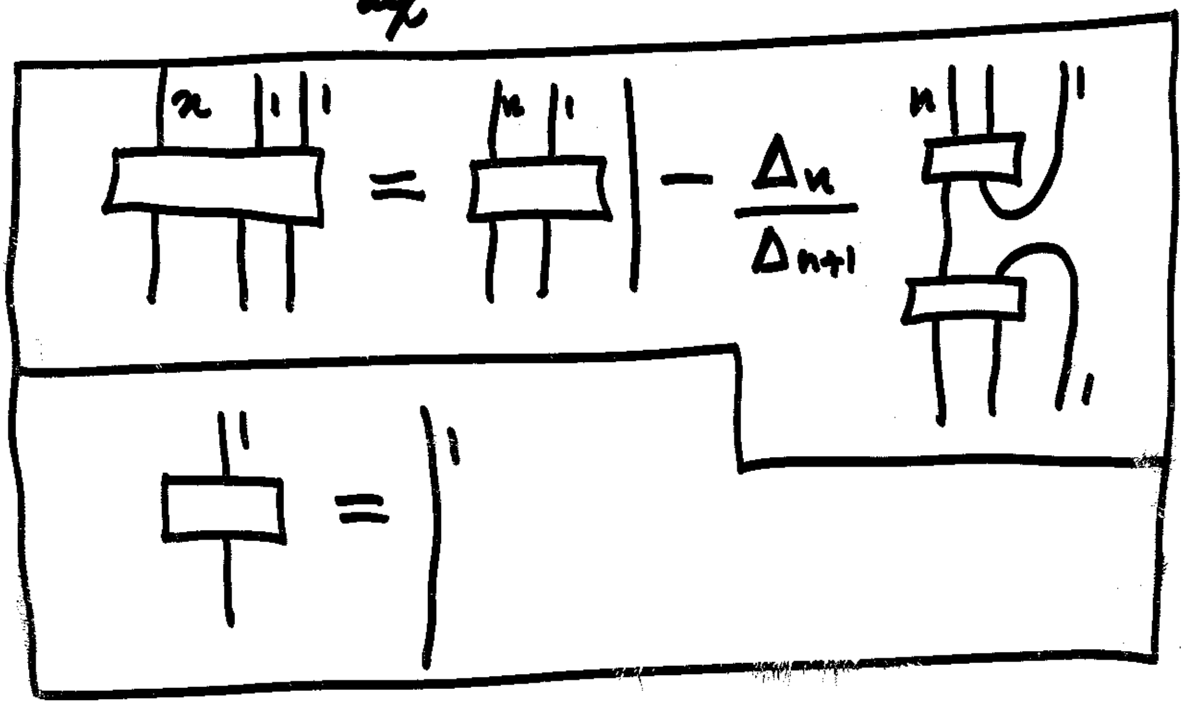
$$\Delta_{n+1} = \delta \Delta_n - \Delta_{n-1} \ , \ \delta = -A^2 - A^{-2}$$

$$\Delta_n = (-1)^n \frac{A^{2n+2} - A^{-2n-2}}{A^2 - A^{-2}}$$

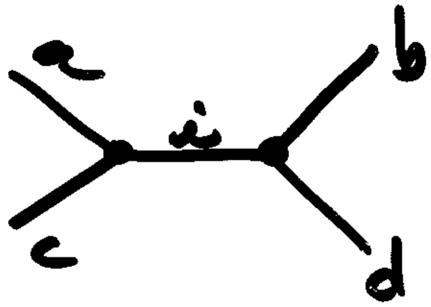
$$A = e^{i\pi/2r} \Rightarrow \Delta_n = (-1)^n \frac{\sin((n+1)\pi/r)}{\sin(\pi/r)}$$

$$\Rightarrow \begin{cases} \Delta_n \neq 0 & 0 \leq n \leq r-2 \\ \Delta_{r-1} = 0 \end{cases}$$

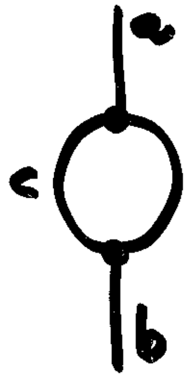
$$[n] \underset{def}{=} (-1)^{n-1} \Delta_{n-1}$$



D) Recoupling


$$= \sum_{j} \begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix}$$


E) "Schur's Lemma"


$$= \frac{\Theta(a,c,d)}{\Delta_a} \Bigg| \; \delta^a_b$$

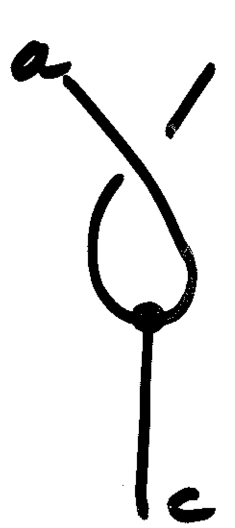
$\Big\langle$  $\Big\rangle = \Theta(a,c,d)$

F) 6j evaluations

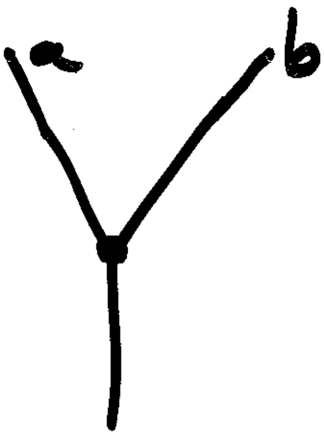
$$\begin{Bmatrix} a & b & i \\ c & d & k \end{Bmatrix} = \frac{\mathrm{Tet}\begin{bmatrix} a & b & i \\ c & d & k \end{bmatrix} \Delta_k}{\Theta(a,b,k)\,\Theta(c,d,k)}$$

$\Big\langle$  $\Big\rangle = \mathrm{Tet}\begin{bmatrix} a & b & i \\ c & d & k \end{bmatrix}$

6) Braiding



$$= \lambda^{ab}_{c}$$



$$\lambda^{ab}_{c} = (-1)^{(a+b-c)/2} \, A^{(a'+b'-c')/2}$$

$$x' = x(x+2)$$

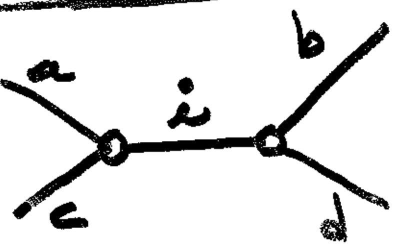
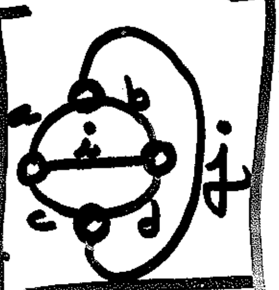
All the TQFT identities follow directly from the underlying topological invariance of the theory.

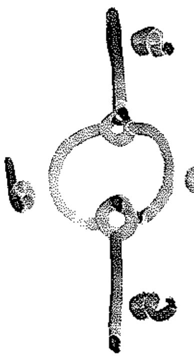
Symmetry and Unitarity

Note: Let  $= \dfrac{\sqrt{\sqrt{\Delta_a \Delta_b \Delta_c}}}{\sqrt{\Theta(a,b,c)}}$ 

(Re-weight the vertices.)

Theorem:

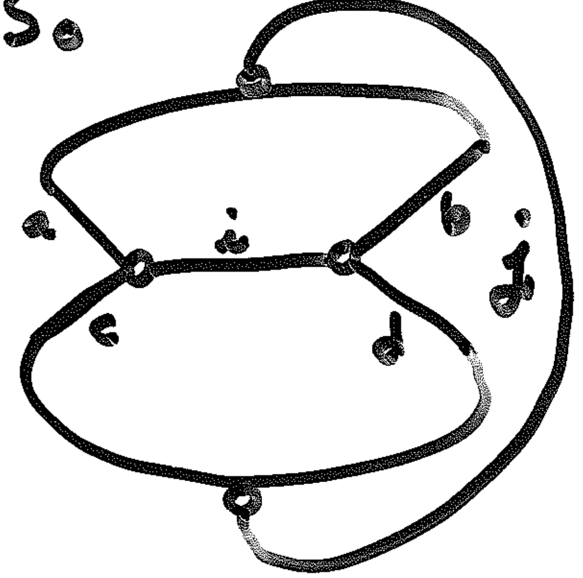
 $= \displaystyle\sum_j \left[\frac{}{\sqrt{\Delta_a \Delta_b \Delta_c}} \right]$  

Proof.  $= \dfrac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\Theta(a,b,c)}$ 

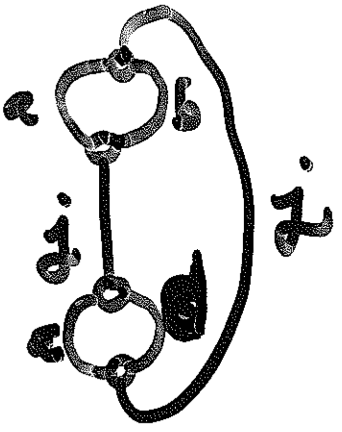
$= \dfrac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\Theta(a,b,c)} \; \dfrac{\Theta(a,b,c)}{\Delta_a}$ 

$= \sqrt{\dfrac{\Delta_b \Delta_c}{\Delta_a}}$ 

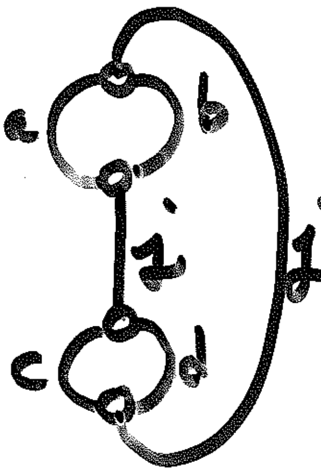
So



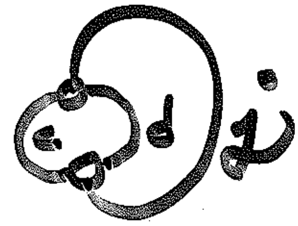
$$= \boxed{\begin{matrix} a & b \\ c & d \end{matrix}}_{ij}$$



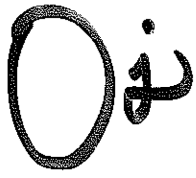
and



$$= \sqrt{\frac{\Delta_a \Delta_b}{\Delta_j}}$$

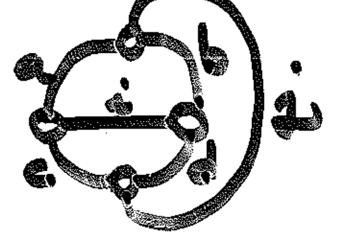


$$= \sqrt{\frac{\Delta_a \Delta_b}{\Delta_j}} \sqrt{\frac{\Delta_c \Delta_d}{\Delta_j}}$$



$$= \sqrt{\Delta_a \Delta_b \Delta_c \Delta_d}$$

$$\Rightarrow \boxed{\begin{matrix} a & b \\ c & d \end{matrix}}_{ij} =$$



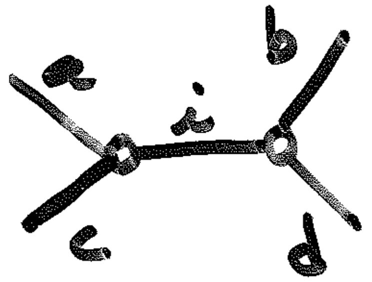
$$\sqrt{\Delta_a \Delta_b \Delta_c \Delta_d}$$

//

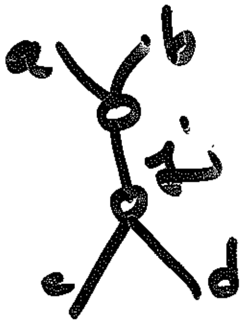
With this adjustment,

$$\boxed{\begin{matrix} a\,b \\ c\,d \end{matrix}}^{-1} = \boxed{\begin{matrix} a\,b \\ c\,d \end{matrix}}^{T}$$

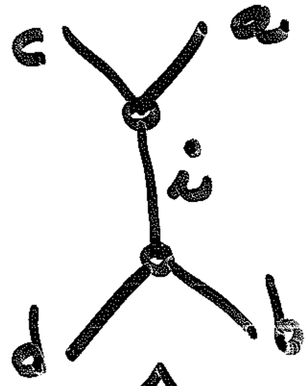
So when the recoupling coeffs are real, the recoupling transformations are unitary.



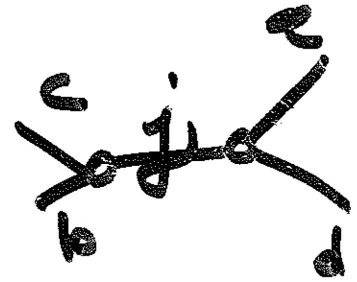
$$= \sum_j \boxed{\begin{matrix} a\,b \\ c\,d \end{matrix}}_{ij}$$



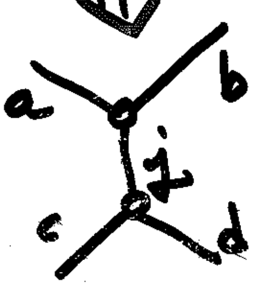
$$\Downarrow$$



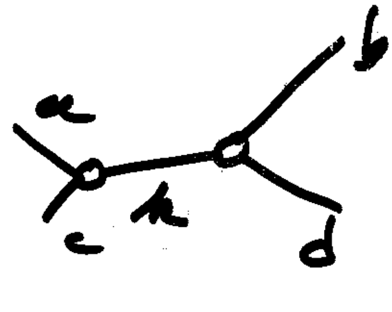
$$= \sum_j \boxed{\begin{matrix} a\,b \\ c\,d \end{matrix}}_{ij}$$

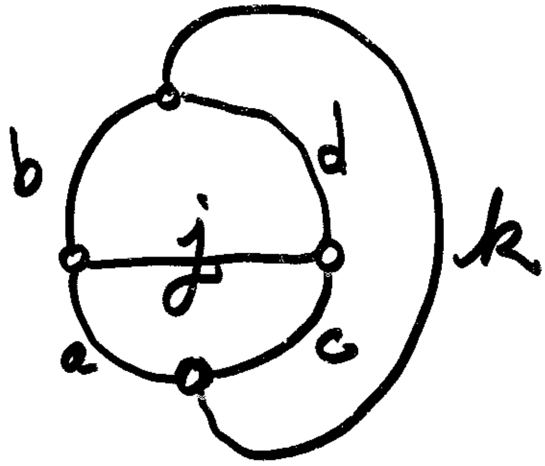


$$\Updownarrow$$

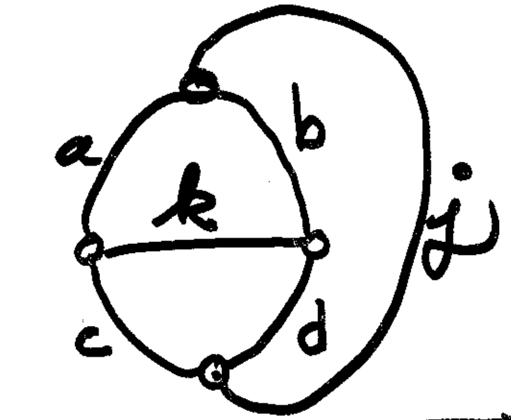


$$= \sum_k \boxed{\begin{matrix} b\,d \\ a\,c \end{matrix}}_{jk}$$





$$\frac{}{\sqrt{\Delta_a \Delta_b \Delta_c \Delta_d}} = \frac{}{\sqrt{\Delta_a \Delta_b \Delta_c \Delta_d}}$$

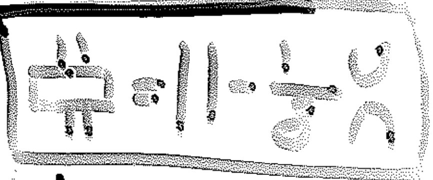


$$\Longrightarrow \quad \boxed{\begin{matrix} a & b \\ c & d \end{matrix}}^{-1} = \boxed{\begin{matrix} a & b \\ c & d \end{matrix}}^{T}$$

<u>Corollary</u>. Lots of unitary braid group representations at roots of unity via the Temperley–Lieb Recoupling Theory.

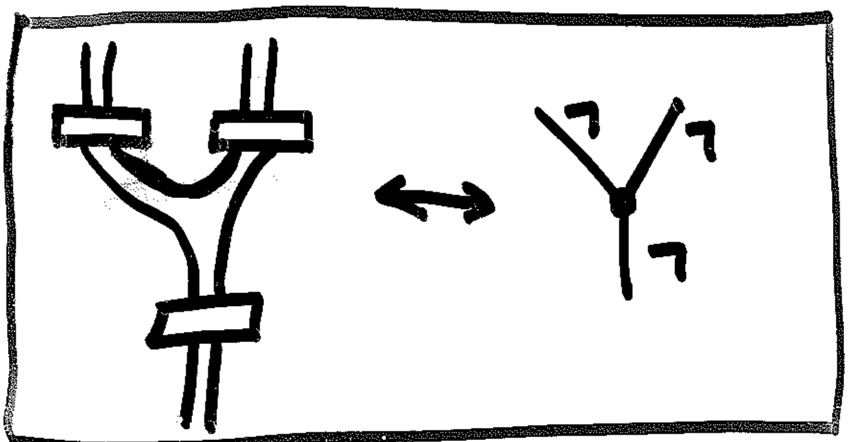
So Where are the Fibonacci Anyons?

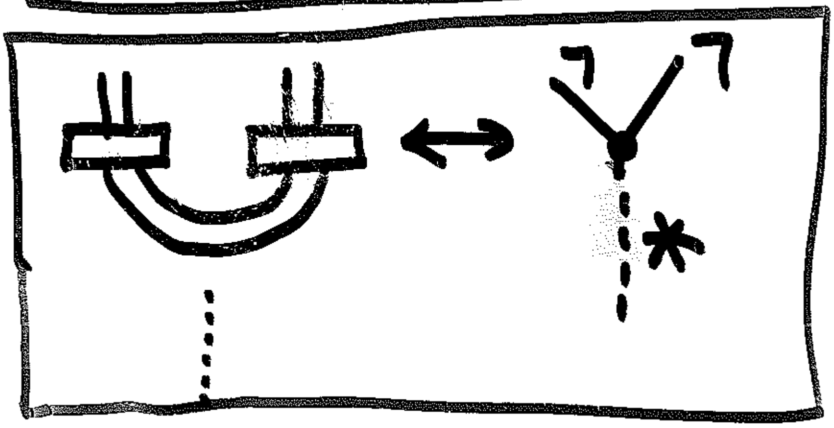
Answer:  = "7".



for $A = e^{i3\pi/5}$

Note: $7 \leftrightarrow \delta = \frac{1+\sqrt{5}}{2}$.

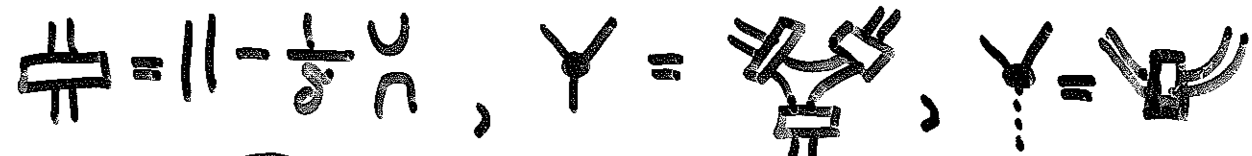




(The root of unity makes  $= \phi$

so  is forbidden.)

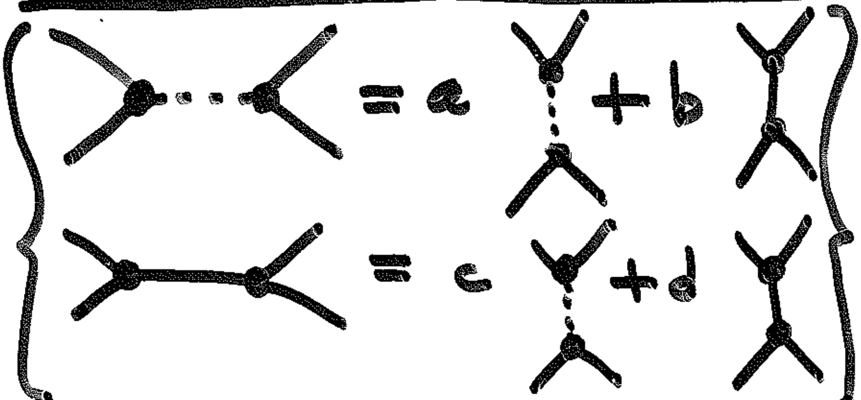
A sketch of the derivation



$\Delta =$  $= \delta^2 - 1$

$\Theta =$  $= \left(\delta - \frac{1}{\delta}\right)^2 \delta - \Delta/\delta$

$T =$  $= \left(\delta - \frac{1}{\delta}\right)^2 (\delta^2 - 2) - 2\,\Theta/\delta$



$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Rightarrow F = \begin{pmatrix} 1/\Delta & \Delta/\Theta \\ \Theta/\Delta^2 & \Delta T/\Theta^2 \end{pmatrix}$

$\Delta^2 = \Delta + 1$
So $\Delta^2 = \delta^2$.
Take $\Delta = \delta$.

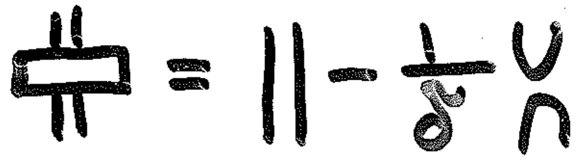
$F^2 = I \Rightarrow \frac{1}{\Delta} + \frac{1}{\Delta^2} = 1$

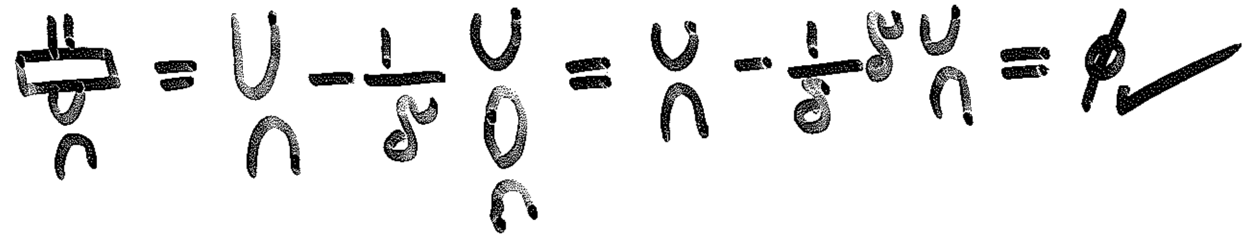
With $\delta^2 = \delta + 1$ (So $\delta = \frac{1+\sqrt{5}}{2}$)
$\Delta = \delta^2 - 1 = \delta$ and above ok.
Then $F = \begin{pmatrix} 1/\Delta & \Theta/\Delta^2 \\ \Delta/\Theta & -1/\Delta \end{pmatrix}$.

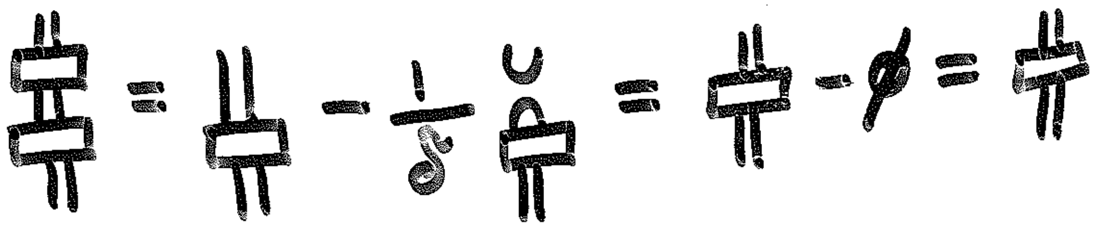
Replace each vertex v by $\alpha \cdot v$ where $\alpha^4 = \Delta^3/\Theta^2$. Then

$F = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}, \quad \tau = \frac{1}{\Delta}$.

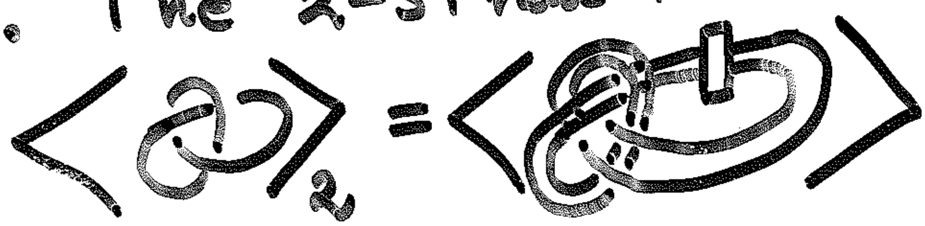
17.2

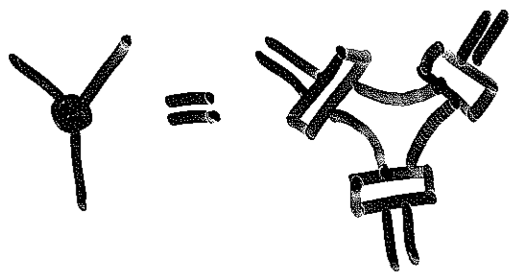


1. 

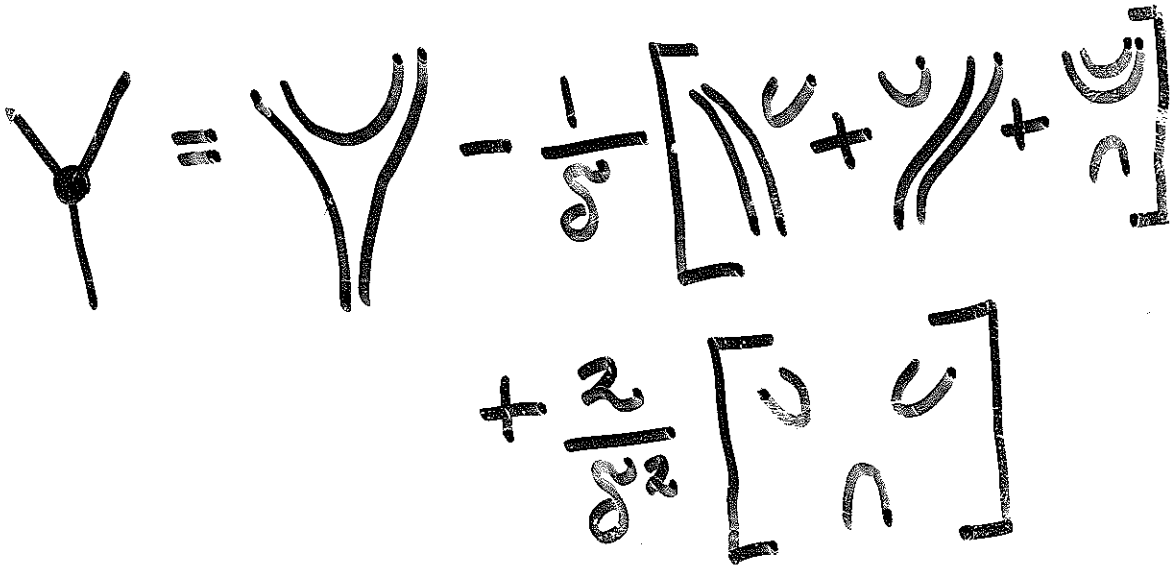
2. 

3. The 2-strand invariant

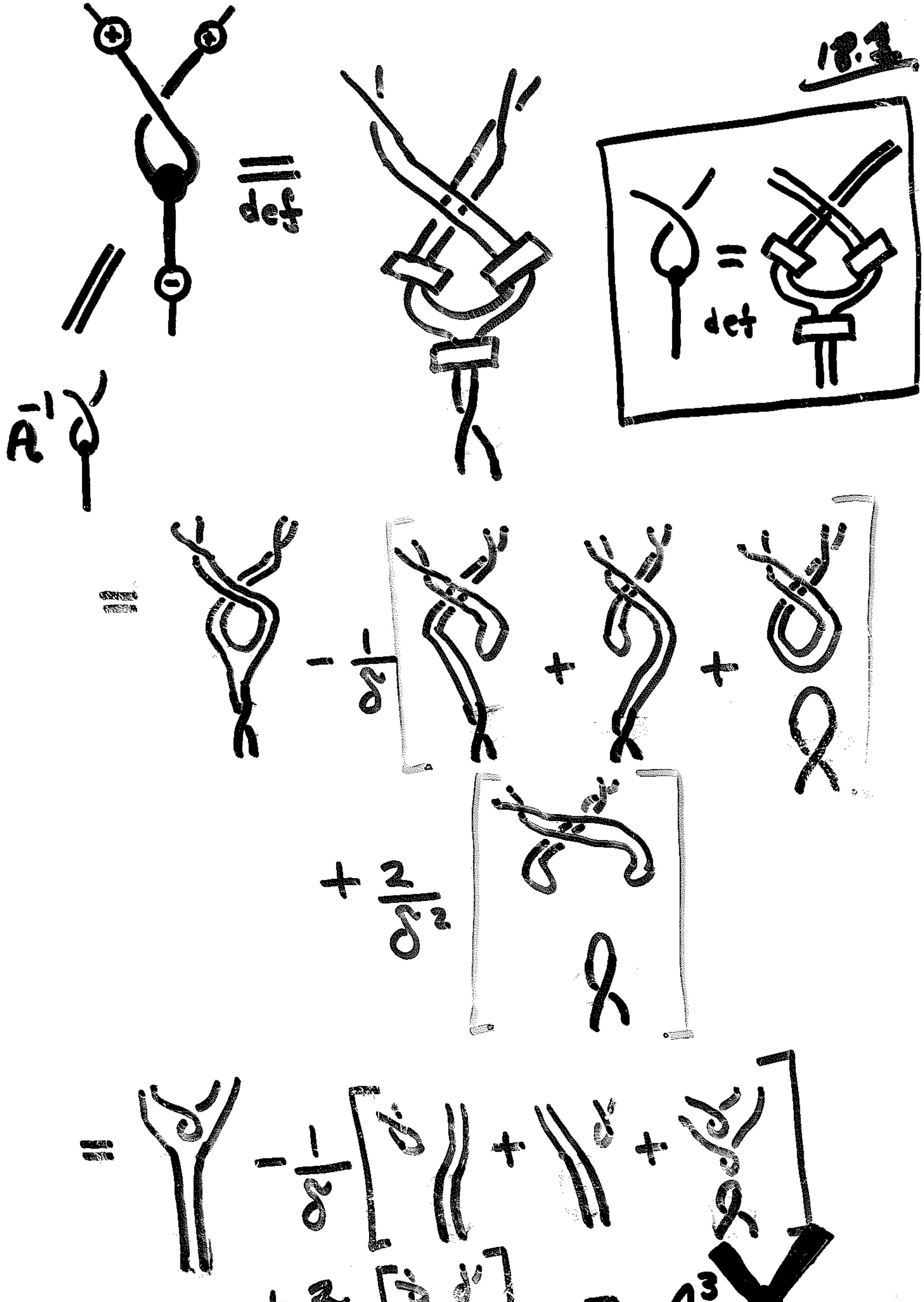


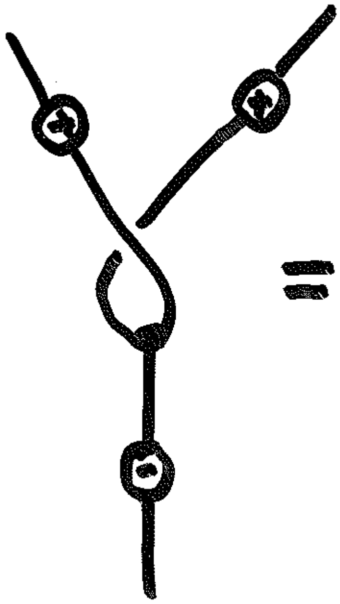
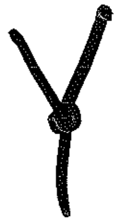
4. 

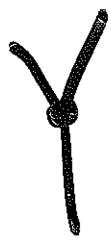
n=2



18.1



 $= -A^3$ 

A^{-1}  $= -A^3$ 

 $= -A^4$ 

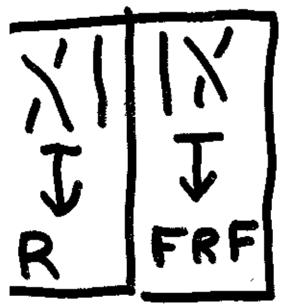
Questions

1. Do there exist physical realizations for these anyonic braiding representations?

2. How efficiently can standard gates be made or approximated from R, F and the recoupling apparatus? (Solovay Kitaev Thm)

3. How do questions about topological entanglement versus quantum entanglement resurrect in this context?

4. Spin Nets and q-deformed spin nets have been used a "substitutes" for spacetime and measurement operators in quantum gravity. Reexamine quantum gravity from this point of view (that unitary transfs are generated by spin network braiding ...).

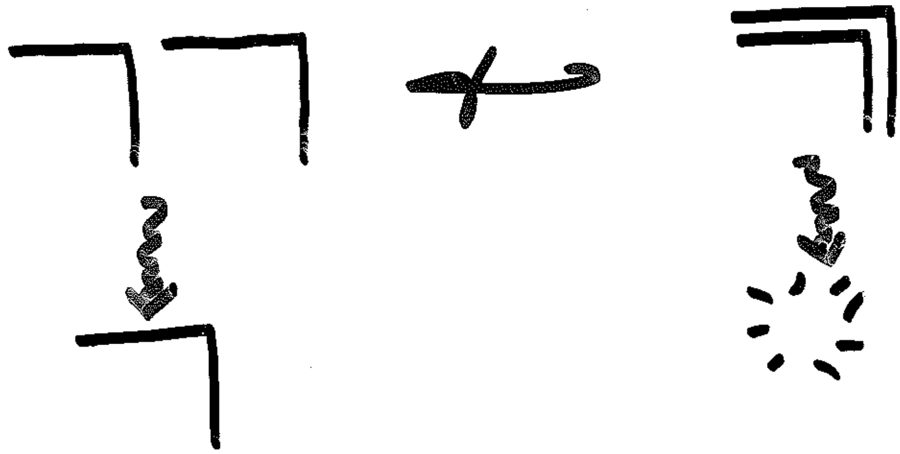


5. Express quantum algorithms in recoupling language.

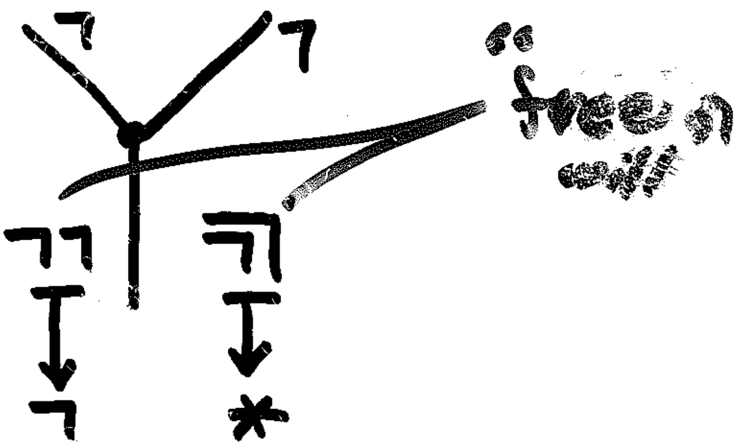
Digression on Laws of Form

One "logical particle" $\rceil$

Two ~~forms~~ of self-interaction



We will "quantize" in sense that



Boolean algebra arises in the calculus of the mark: $\overline{\overline{a}\rceil}\rceil = a$, $a\rceil = \rceil$, $\overline{\overline{a}\rceil\,\overline{b}\rceil}\rceil\, c = \overline{\overline{ac}\rceil\,\overline{bc}\rceil}\rceil$

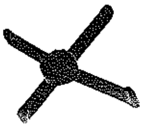
etc. where $\overline{a}\rceil$ fixes $\rceil$ as an operator.

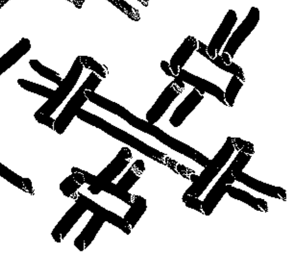
Remark on the knot invariant

$\mathcal{L}(K) = \langle K^{(2)} \rangle$

$\mathcal{L}($  $) = \langle$  $\rangle$.

$\mathcal{L}($  $) = A^{4}\mathcal{L}($  $) + A^{-4}\mathcal{L}($  $) + \delta\,\mathcal{L}($  $)$

where  = 

$\Big(\mathcal{L}($  $) = \langle$  $\rangle\Big)$

$\Rightarrow$

$\mathcal{L}($  $) - \mathcal{L}($  $) = (A^{4} - A^{-4})\big[\mathcal{L}($  $) - \mathcal{L}($  $)\big]$

$\mathcal{L}($  $) = A^{8}\mathcal{L}(\frown)$

$\mathcal{L}($  $) = A^{-8}\mathcal{L}(\frown)$

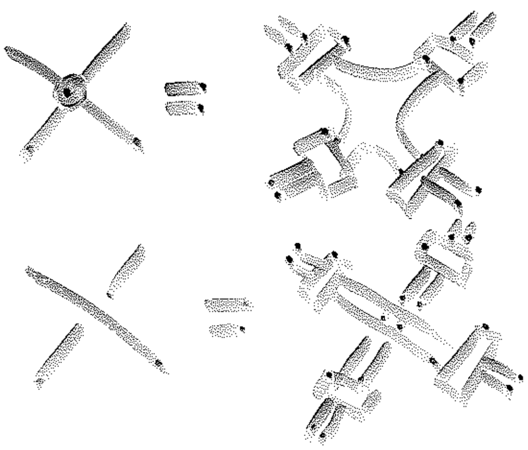
Special case ~~of the~~ "Dubrovnik" 2-variable polynomial.

For $A = e^{i3\pi/5}$,

$\mathcal{L}_K(A) = d(K)$ is an (unnormalized) invariant ~~of~~ 3-manifolds.

2.7.1

$$\mathcal{L}(\times) = A^{+}\mathcal{L}(=) + A^{-}\mathcal{L}(\supset\subset) + \delta\,\mathcal{L}(\otimes)$$



Is a state sum for this invariant and extends naturally to virtual knots and links.

Hence this specialization of Dubrovnik polynomial extends to virtuals.

(In general there is a problem having invariance under

for extensions of quantum invariants)

$$\mathcal{L}_{\diagup\!\!\!\diagdown} - \mathcal{L}_{\diagdown\!\!\!\diagup} = z\left(\mathcal{L}_{\asymp} - \mathcal{L}_{)(}\right)$$

$$\mathcal{L}_{\frown\!\!\circ\!\!\frown} = \alpha\,\mathcal{L}_{\sim}$$

$$\mathcal{L}_{\smile\!\!\circ\!\!\smile} = \alpha^{-1}\mathcal{L}_{\sim}$$

$$\mathcal{L}_{OK} = \delta\,\mathcal{L}_{K}\ , \quad \delta = \frac{\alpha - \alpha^{-1}}{z} + 1$$

$$\Phi = \frac{1+\sqrt{5}}{2}\ , \quad A = e^{3\pi i/5}$$

$$\Rightarrow \quad \alpha = A^{8} = -e^{\pi i/5}$$

$$z = -\Phi$$

$$\delta = \Phi$$

$$)\cdots(\; = \;)(\; = \frac{1}{\Delta}\,{\smile \atop \frown} + \frac{\Theta}{\Delta^{2}}\,\succ\!\!\prec$$

$$\frac{1}{\Delta} = \delta^{-1} \qquad \frac{\Theta}{\Delta^{2}} = \frac{\delta - 1}{\delta^{2}} = -\delta^{-3}.$$

So $\boxed{\mathcal{L}_{)(} = R\,{\smile \atop \frown} + S\,\succ\!\!\prec}$

This makes $\mathcal{L}$ into a 3-mfld invariant.