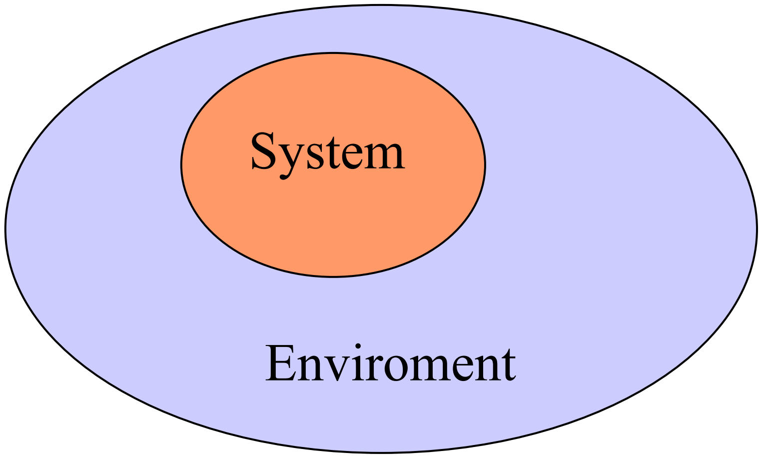
Quantum Evolution



Unitary evolution of initially uncoupled system and environment:

$$\rho_{\text{sys}} \otimes \rho_{\text{env}} \rightarrow U\rho_{\text{sys}} \otimes \rho_{\text{env}} U^{\dagger}$$

Effectively for the system alone, this amounts to a trace preserving completely positive map:

$$\rho_{\text{sys}} \rightarrow \text{Tr}_{\text{env}}\left(U\rho_{\text{sys}} \otimes \rho_{\text{env}} U^{\dagger}\right) = \sum_i K_i \rho_{\text{sys}} K_i^{\dagger} \qquad \sum_i K_i^{\dagger} K_i = \mathbf{I}$$

Kraus operators

Does Classical Communication Help in Cloning of Entangled States?

$$i\hbar \frac{\partial \psi}{\partial t} = \Bigg(-\frac{\hbar^2}{2m}\Delta + V +$$

$$\Bigg)\psi$$

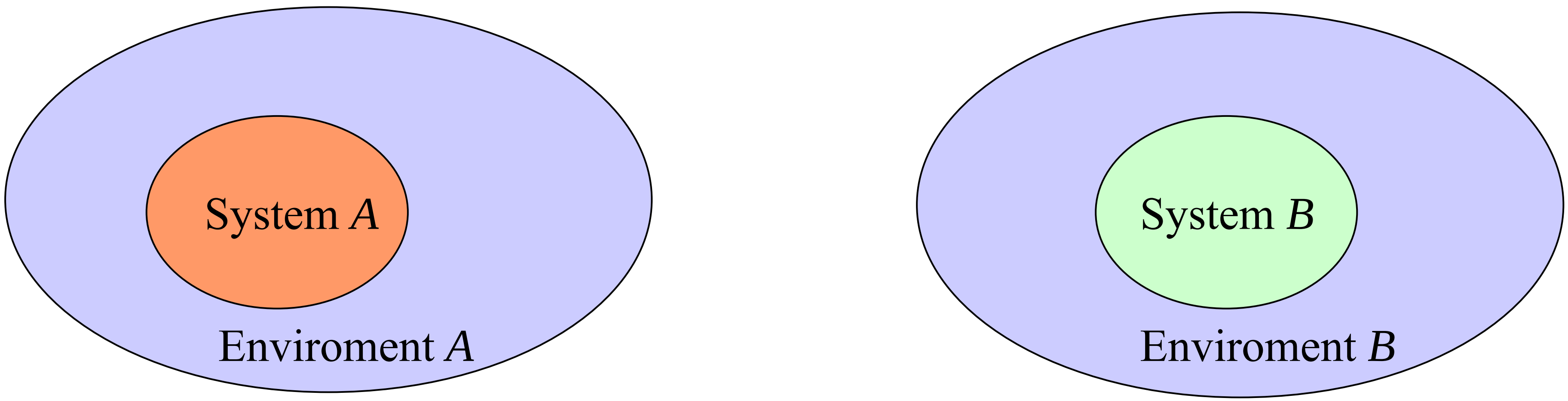
Rafał Demkowicz-Dobrzański (CFT PAN)

Distant subsystems – local operations

Independent, local, unitary operations:

$$\rho_{AB} \rightarrow U_A \otimes U_B \rho_{AB} U_A^{\dagger} \otimes U_B^{\dagger}$$

Distant subsystems – local operations

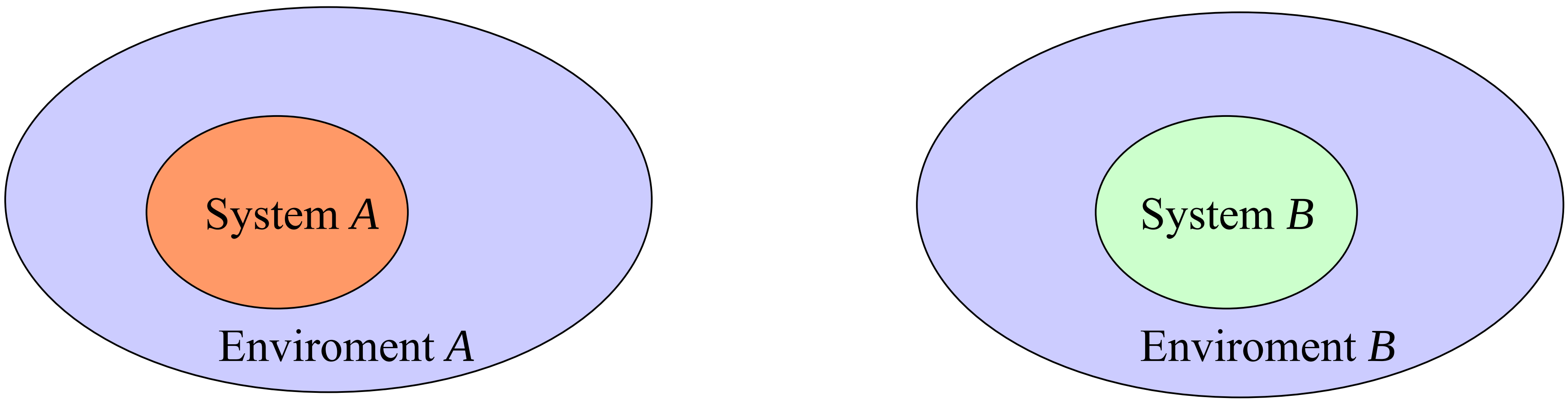


Independent, local, completely positive maps:

$$\rho_{AB} \to \sum_j \mathbf{I}_A \otimes K_{B,j} \left(\sum_i K_{A,i} \otimes \mathbf{I}_B \rho_{AB} K_{A,i}^{\dagger} \otimes \mathbf{I}_B \right) \mathbf{I}_A \otimes K_{B,j}^{\dagger}$$

$$\sum_i K_{A,i}^{\dagger} K_{A,i} = \mathbf{I}_A \qquad \sum_j K_{B,j}^{\dagger} K_{B,j} = \mathbf{I}_B$$

Distant subsystems – local operations

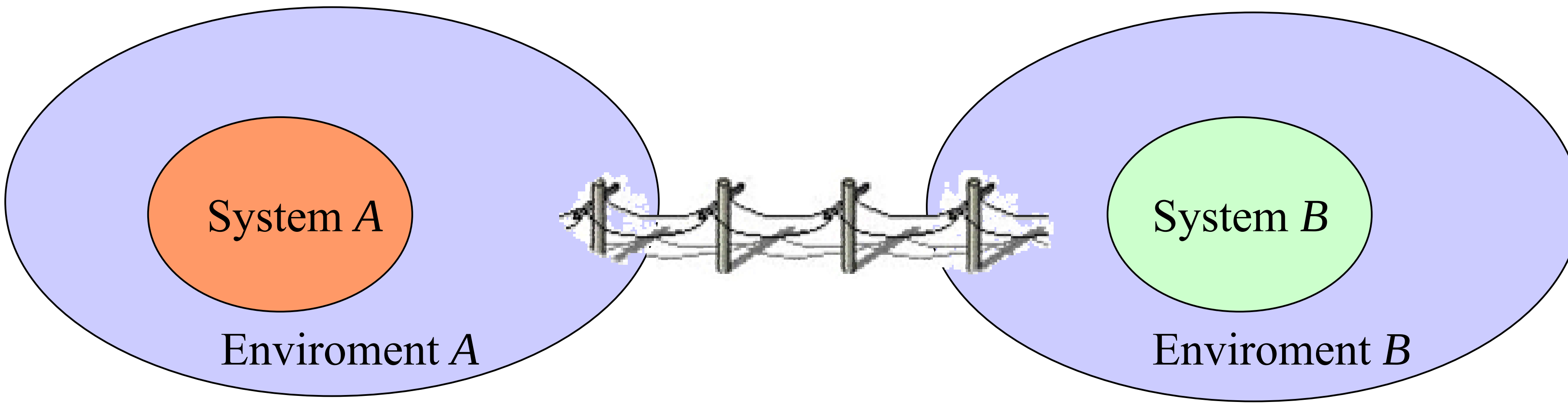


Independent, local, completely positive maps:

$$\rho_{AB} \rightarrow \sum_{ij} K_{A,i} \otimes K_{B,j} \rho_{AB} K_{A,i}^{\dagger} \otimes K_{B,j}^{\dagger}$$

$$\sum_{i} K_{A,i}^{\dagger} K_{A,i} = \mathbf{I}_{A} \qquad \sum_{j} K_{B,j}^{\dagger} K_{B,j} = \mathbf{I}_{B}$$

+ Classical Communication



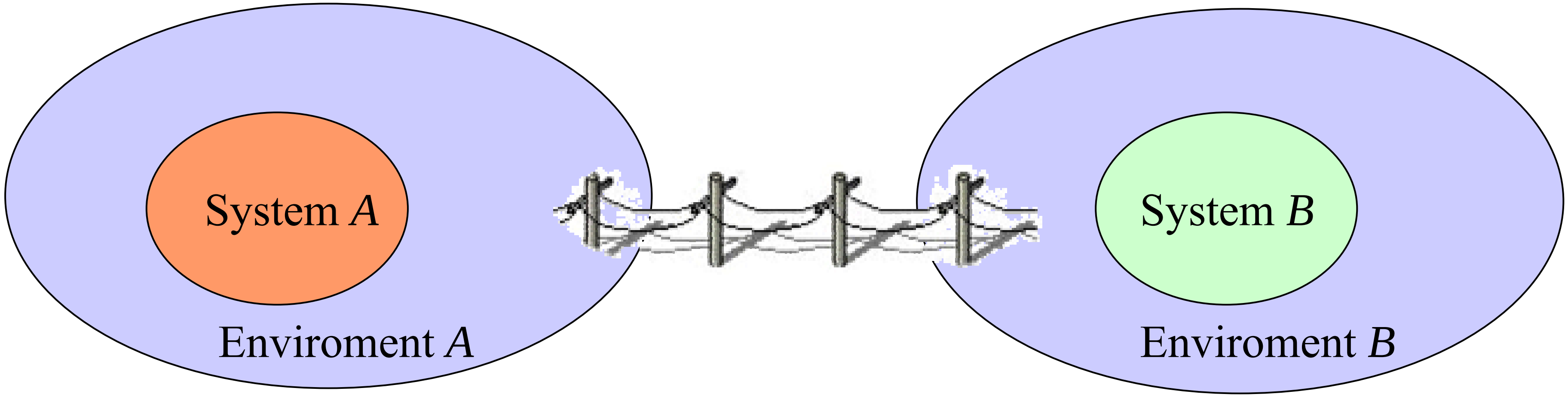
One way communication from *A* to *B*:

$$\rho_{AB} \rightarrow \sum_{ij} \mathbf{I}_A \otimes K_{B,ij} \left(K_{A,i} \otimes \mathbf{I}_B \rho_{AB} K_{A,i}^{\dagger} \otimes \mathbf{I}_B \right) \mathbf{I}_A \otimes K_{B,ij}^{\dagger}$$

$$\sum_i K_{A,i}^{\dagger} K_{A,i} = \mathbf{I}_A$$

For every *i*: $$\sum_j K_{B,ij}^{\dagger} K_{B,ij} = \mathbf{I}_B$$

+ Classical Communication



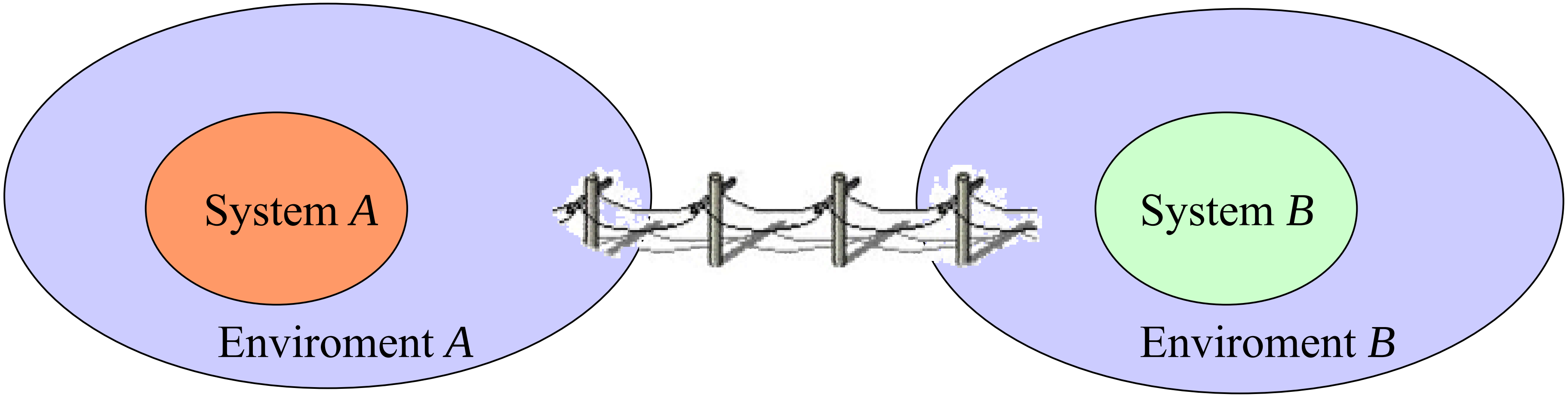
A communicates to B and subsequently B communicates to A:

$$\sum_{ijk} K_{A,ijk} \otimes \mathbf{I}_B \left(\mathbf{I}_A \otimes K_{B,ij} \left(K_{A,i} \otimes \mathbf{I}_B \rho_{AB} K_{A,i}^{\dagger} \otimes \mathbf{I}_B \right) \mathbf{I}_A \otimes K_{B,ij}^{\dagger} \right) K_{A,ijk}^{\dagger} \otimes \mathbf{I}_B$$

For every i: For every ij:

$$\sum_i K_{A,i}^{\dagger} K_{A,i} = \mathbf{I}_A \qquad \sum_j K_{B,ij}^{\dagger} K_{B,ij} = \mathbf{I}_B \qquad \sum_j K_{A,ijk}^{\dagger} K_{A,ijk} = \mathbf{I}_A$$

+ Classical Communication



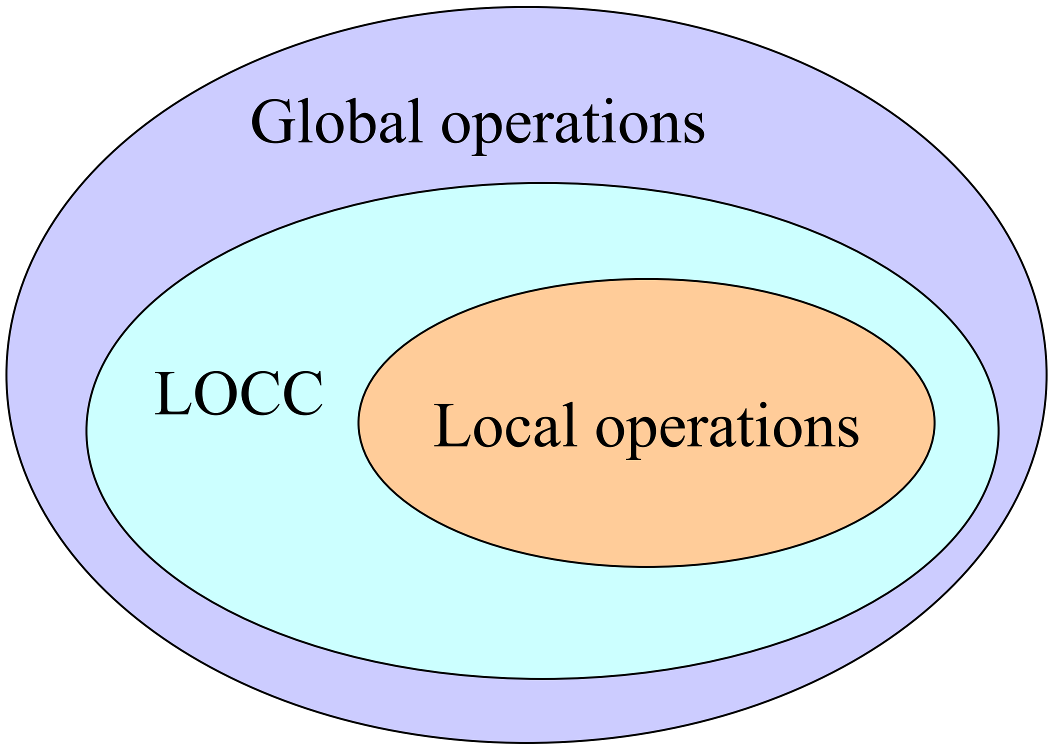
Arbitrary amount of communication:

$$\rho_{AB} \to \sum_{ij\ldots} \ldots \left(\mathbf{I}_A \otimes K_{B,ij}\left(K_{A,i} \otimes \mathbf{I}_B \rho_{AB} K^{\dagger}_{A,i} \otimes \mathbf{I}_B\right)\mathbf{I}_A \otimes K^{\dagger}_{B,ij}\right)\ldots$$

LOCC – **L**ocal **O**perations and **C**lassical **C**ommunication

There is no known applicable criterion to check whether a given operation is LOCC (this problem is even harder than checking separability of a mixed state)

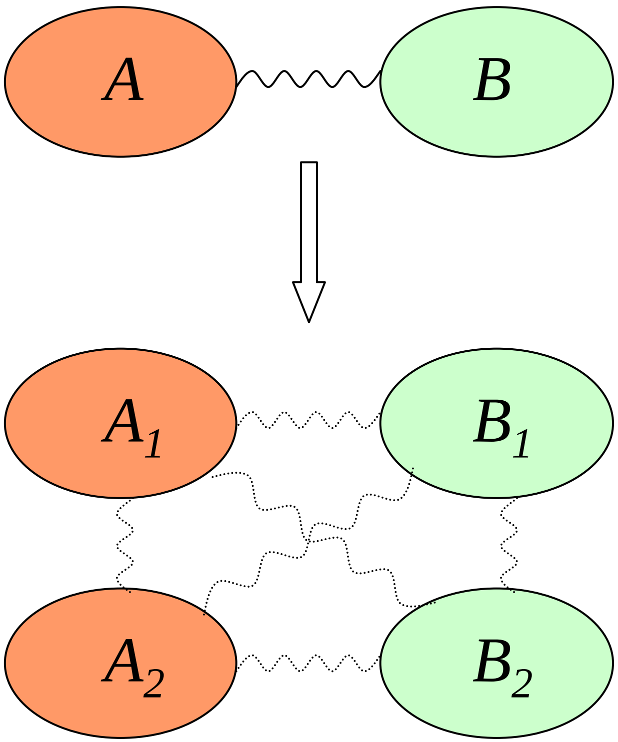
Eggs in Eggs



Any separable states can be created by an LOCC operation starting with a product state

LOCC operations, cannot increase entanglement

Cloning of Entangled Qubits

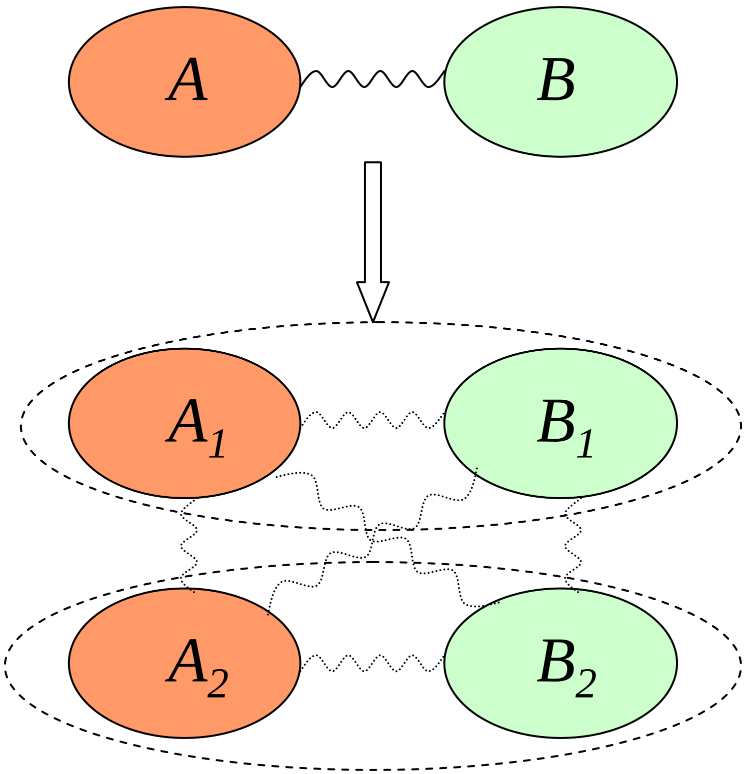


$$|\psi\rangle_{AB} \in H_A \otimes H_B$$

$$\Downarrow$$

$$\rho_{A_1 B_1 A_2 B_2}$$

Cloning of Entangled Qubits



$$|\psi\rangle_{AB} \in H_A \otimes H_B$$

First clone:

$$\rho_{A_1B_1} = \mathrm{Tr}_{A_2B_2}\left(\rho_{A_1B_1A_2B_2}\right)$$

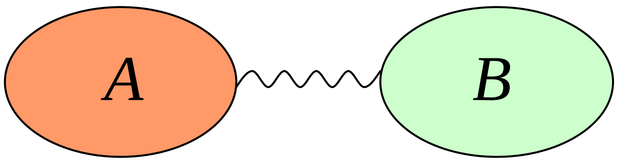
Second clone:

$$\rho_{A_2B_2} = \mathrm{Tr}_{A_1B_1}\left(\rho_{A_1B_1A_2B_2}\right)$$

Symmetry: $\rho_{A_1B_1} = \rho_{A_2B_2}$

Fidelity: $F = \langle \psi_{AB} | \rho_{A_1B_1} | \psi_{AB} \rangle$

Locally equivalent states



$$|\psi\rangle_{AB} \in H_A \otimes H_B$$

Locally unitarily equivalent states (local orbit):

$$|\psi\rangle_{AB} = U_A \otimes U_B \left(\alpha|00\rangle + \sqrt{1-\alpha^2}|11\rangle\right)$$

$$0 \leq \alpha \leq \tfrac{1}{\sqrt{2}}$$

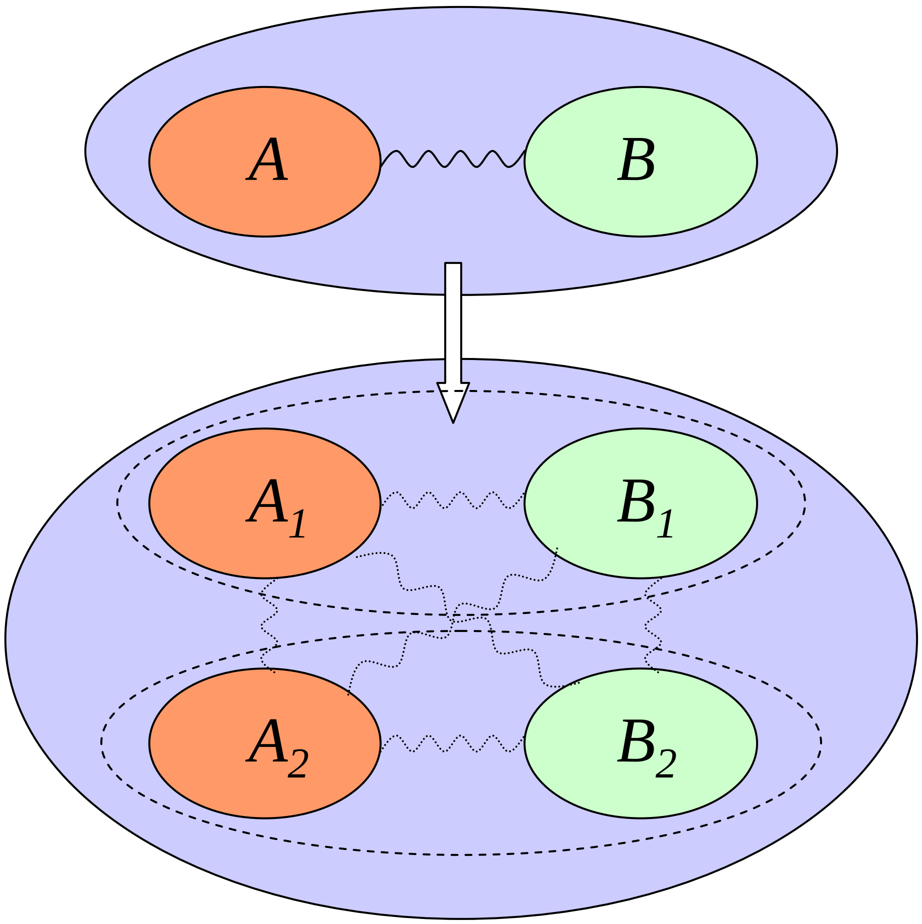
product states (at $\alpha = 0$)

max. entangled states (at $\alpha = \tfrac{1}{\sqrt{2}}$)

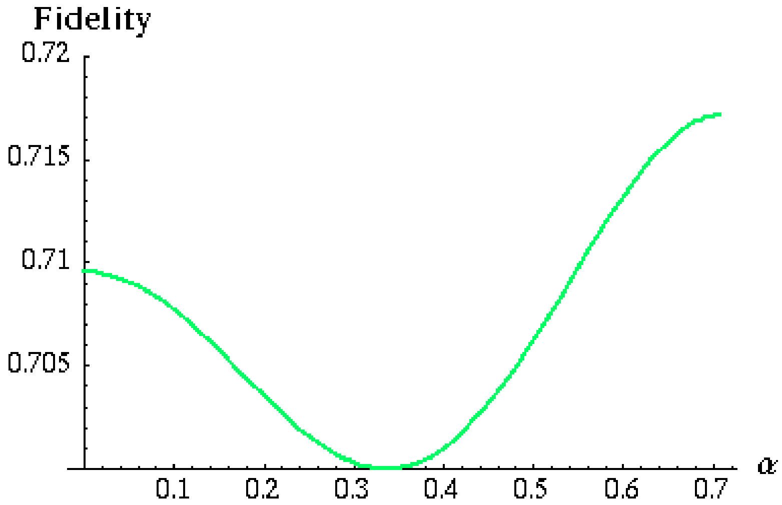
We want to clone an unknown state from a given orbit

Perfect cloning is impossible, as there are nonorthogonal states in an orbit

Optimal Cloning with Global Operations



Fidelity of the optimal cloning with no restrictions on allowed operations:



Maximally entangled states are easiest to clone.

Minimum: $\alpha = \sqrt{1/2 - \sqrt{15}/10} \approx 0.335$

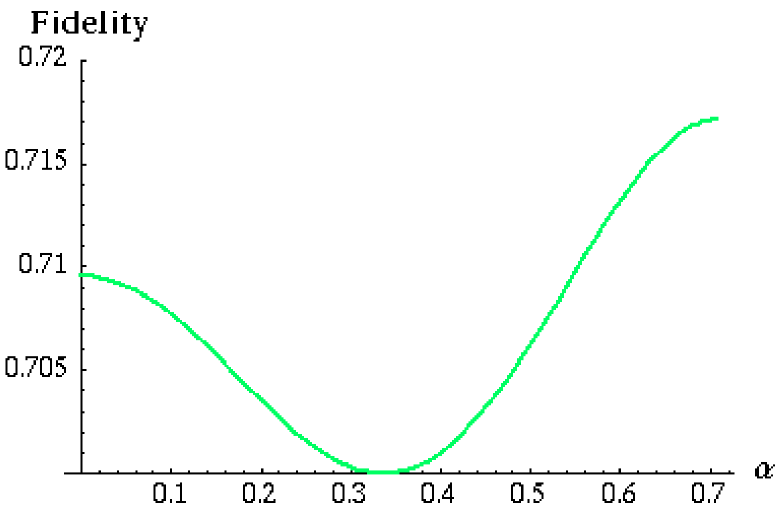
If the cloned state was totally unknown, the optimal fidelity would be F=0.7

Fidelity „~” 1/Volume

Fidelity of the optimal cloning with no restrictions on allowed operations :

Novotny, Alber,Jex, quant-ph/0411105 (for global fidelity)
RDD, Lewenstein, Sen, Sen(De), Bruss, quant-ph/0502103

$$F(\alpha) = \text{a lengthy expression}$$



Minimum: $\alpha = \sqrt{1/2 - \sqrt{15}/10} \approx 0.335$

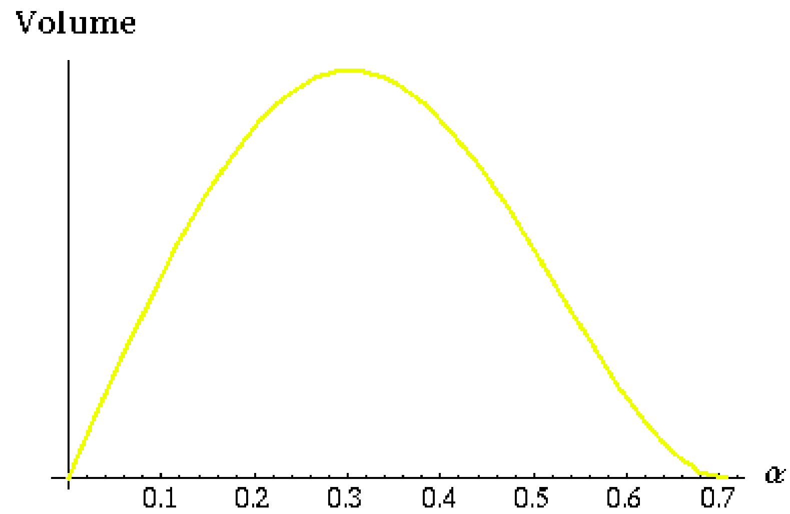
The Fubini-Study metric:

$$\mathrm{d}l^2 = 4\left(1 - \left|\langle \psi + \delta\psi | \psi \rangle\right|^2\right)$$

Volume of local orbits:

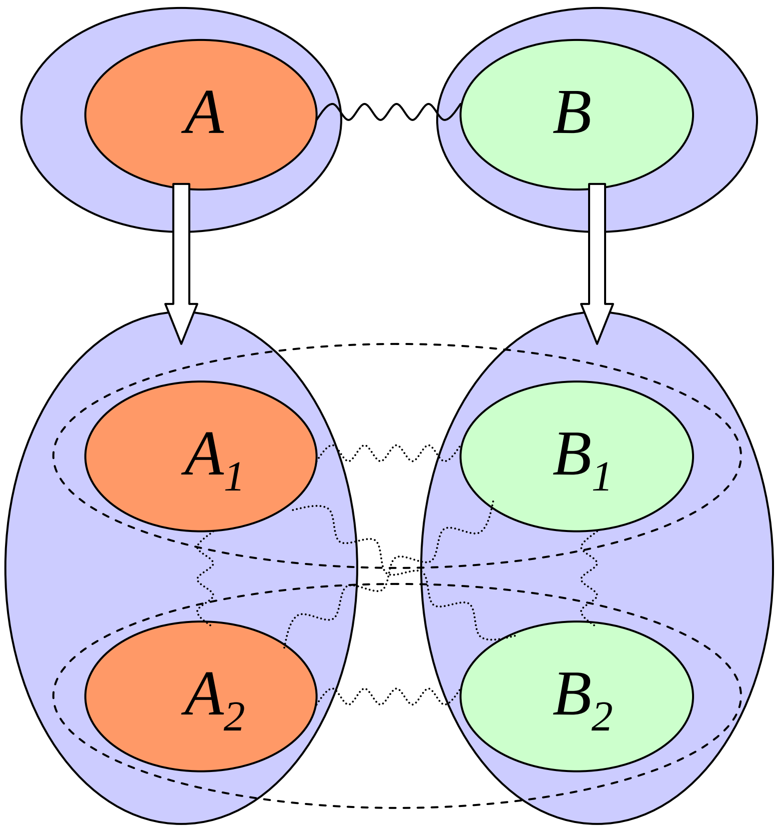
Bengtsson,Brannlund,Życzkowski, Int. J. Mod. Phys. A, 17, 4675 (2002)

$$\mathrm{vol}(\alpha) \propto 2\alpha\left(1 - 2\alpha^2\right)^2 \sqrt{1 - \alpha^2}$$

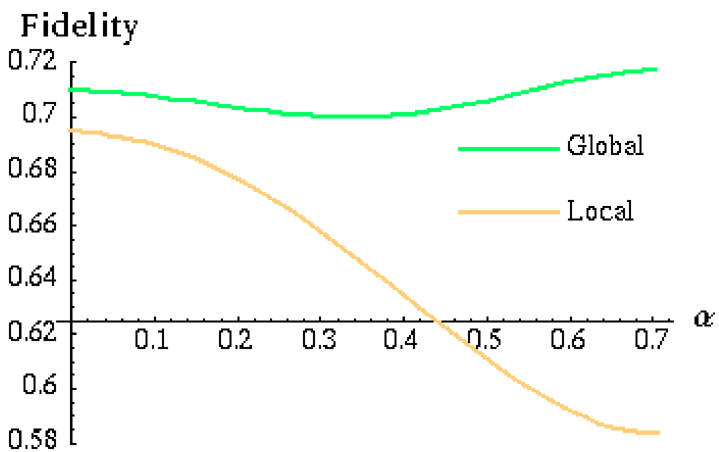


Maximum: $\alpha = \sqrt{1/2 - \sqrt{6}/6} \approx 0.303$

Optimal Cloning with Local Operations



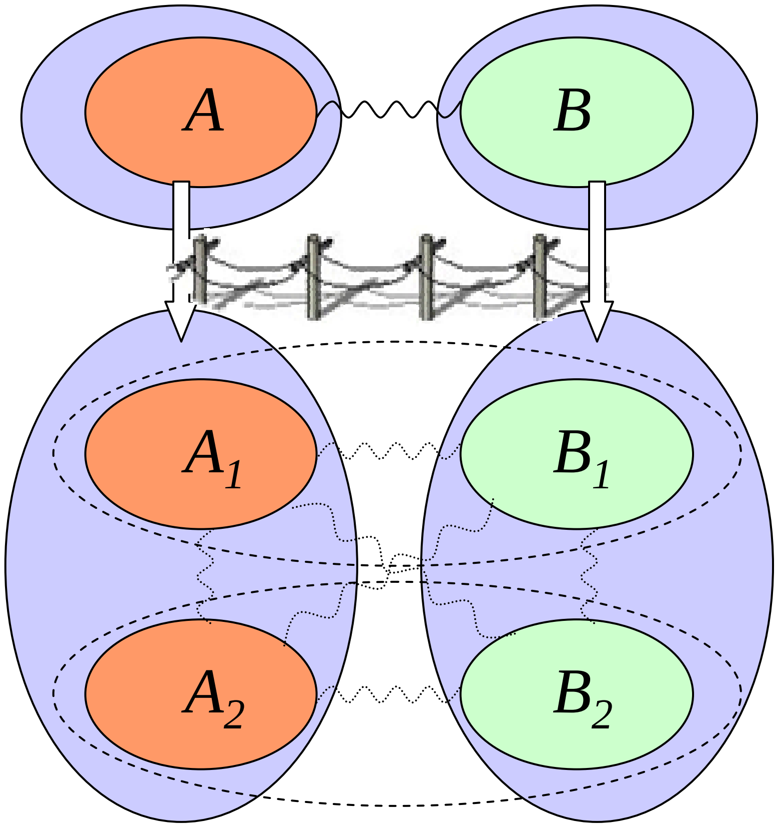
Optimal strategy: application of two independent, optimal universal qubit cloners:



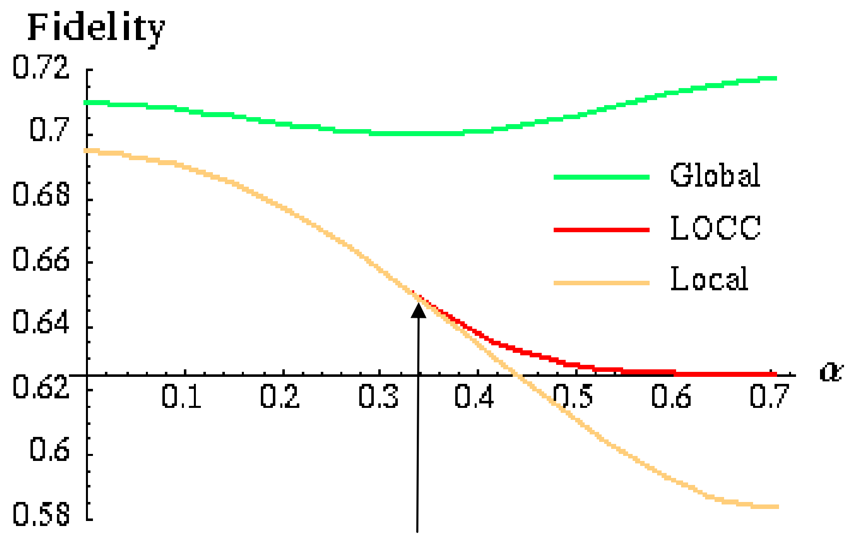
Lower fidelity for more entangled states

Can classical communication help?

Optimal Cloning with LOCC



Threshold for usefulness of classical communication:



Threshold exactly at the point where „Global" has minimum.

The optimal LOCC protocol requires one bit of classical communication

Conclusions

• Classical communication is useful in cloning of states from local orbits, but only for states with entanglement above threshold

• Threshold corresponds to minimum of the optimal cloning fidelity when no restrictions on operations are imposed. Maybe there is something deeper in states with $\alpha = \sqrt{1/2 - \sqrt{15}/10}$

• Our optimisation procedure also revealed that any use of bound entangled states as an additional resource will not increase cloning fidelity.

• Possible applications: local attack on secret sharing protocols

More details:

RDD, M. Lewenstein, A. Sen(De), U. Sen, D. Bruss, „*Usefulness of classical communication for local cloning of entangled states*", quant-ph/0502103

Technical Appendix

Covariant operations

When looking for optimal cloning machines cloning states from an orbit of a group, one can restrict oneself to covariant operations. In our case, covariant means:

$$\Lambda\left(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger\right) = U_A \otimes U_B \otimes U_A \otimes U_B \rho_{A_1 B_1 A_2 B_2} U_A^\dagger \otimes U_B^\dagger \otimes U_A^\dagger \otimes U_B^\dagger$$

Every completely positive map can be associated with a positive operator (Jamiołkowski isomorphism):

$$\Lambda : L(H_{\mathrm{in}}) \rightarrow L(H_{\mathrm{out}}) \qquad P_\Lambda = \Lambda \otimes \mathbf{I}\left(\left|\Psi\right\rangle\left\langle\Psi\right|\right) \qquad P_\Lambda \in L\left(H_{\mathrm{out}} \otimes H_{\mathrm{in}}\right)$$

$$\Lambda(\rho) = \mathrm{Tr}_{H_{\mathrm{in}}}\left(\mathbf{I} \otimes \rho^T \, P_\Lambda\right)$$

$\left|\Psi\right\rangle$ - maximally entangled state on $H_{\mathrm{in}} \otimes H_{\mathrm{in}}$

Covariance condition on $P_\Lambda \in L\left(H_{1A} \otimes H_{1B} \otimes H_{2A} \otimes H_{2B} \otimes H_A \otimes H_B\right)$

$$[P_\Lambda, U_A \otimes U_B \otimes U_A \otimes U_B \otimes U_A^* \otimes U_B^*] = 0$$

After reordering subspaces: $\tilde{P}_\Lambda \in L\left(H_{1A} \otimes H_{2A} \otimes H_A \otimes H_{1B} \otimes H_{2B} \otimes H_B\right)$

Covariant operations

$$[\widetilde{P}_{\Lambda}, U_A \otimes U_A \otimes U_A^* \otimes U_B \otimes U_B \otimes U_B^*] = 0$$

Simpler situation:

$$[A, U_A \otimes U_A \otimes U_A] = 0$$

Decomposition into irreducible subspaces:

$$H_{A1} \otimes H_{A2} \otimes H_A = M_1 \oplus M_2 \oplus M_3$$

$$8 = 2 + 2 + 4$$

T_i - projection operator on the subspace M_i

$T_{12}\ (T_{21})$ - isomorphism from M_1 to M_2 $(M_2$ to $M_1)$

$T_4 = T_{12} + T_{21}$ $\qquad$ $T_5 = iT_{12} - iT_{21}$

General form of A: $\quad A = \sum_i a_i T_i \quad$ 5 real parameters

General form of $\widetilde{P}_{\Lambda}$: $\quad \widetilde{P}_{\Lambda} = \sum a_{ij} T_i \otimes T_j \quad$ 25 real parameters (instead of 4096)

Constraints

$$\widetilde{P}_\Lambda = \sum a_{ij} T_i \otimes T_j$$

- Positivity: $\widetilde{P}_\Lambda \geq 0$ (semidefinite problem linear in a_{ij})
- Trace preservation constraint: $\mathrm{Tr}_{H_{\mathrm{out}}}\left(P_\Lambda\right) = \mathbf{I}$
- Symmetry – clones should be identical $\rho_{A_1 B_1} = \rho_{A_2 B_2}$

} equations linear in a_{ij}

- LOCC condition !!!

There is no known method how to effectively impose LOCC condition

Necessary conditions for LOCC:

- Separability of operation. There is a Kraus decomposition in which all Kraus operators or products. $\widetilde{P}_\Lambda$ is a separable state. (still very demanding)
- $\widetilde{P}_\Lambda$ is positive after partial transposition (PPT) (semidefinite problem linear in a_{ij})

Constraints

$$\widetilde{P}_\Lambda = \sum a_{ij} T_i \otimes T_j$$

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Semidefinite Programming

$$\widetilde{P}_\Lambda = \sum a_{ij} T_i \otimes T_j$$

Varying over 25 parameters a_{ij}

Optimise fidelity: $F = \langle \psi_{AB} | \rho_{A_1 B_1} | \psi_{AB} \rangle$ (linear in a_{ij})

Under constraints

- Positivity: $\widetilde{P}_\Lambda \geq 0$
- Trace preservation constraint: $\mathrm{Tr}_{H_{\mathrm{out}}}(P_\Lambda) = \mathbf{I}$
- Symmetry – clones should be identical $\rho_{A_1 B_1} = \rho_{A_2 B_2}$
- $\widetilde{P}_\Lambda$ is positive after partial transposition (PPT)

SeDuMi package for Matlab

Nature is Not Malicious

After obtaining numerical results, we could see which parameters ca be put to zero. We were left five parameters with which we could deal symbolically.

As a result we obtain a PPT operation. But it does not have to be necessary LOCC.

Having found P_Λ, we have to check whether it is separable.... It is!!!

(we have found explicit decomposition)

After that one have to check whether it is LOCC.... It is!!!

(we have found the explicit LOCC protocol involving just one bit of classical communication)

PPT constraint was enough to find the optimal LOCC operation, that means that using bound entangled states as an additional resource would not improve cloning fidelity.