Introduction to Quantum Algorithms
Part I: Quantum Gates and Simon's Algorithm

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Overview

Today:

- Introduction to qubits, quantum gates, and circuits.
- Appetizer: Two-bit problem where quantum beats classical!
- The power of quantum computing: Simon's algorithm
- Basic principles used:
  - Computing in superposition
  - Constructive/destructive interference

Tomorrow: Shor's algorithm and Grover's algorithm

Basics: Quantum Information

Quantum–bit (qubit)

A qubit is a normalized state in $\mathbb{C}^2$:

$$\alpha\,|0\rangle + \beta\,|1\rangle\,, \quad \alpha, \beta \in \mathbb{C}, \quad \text{where } |\alpha|^2 + |\beta|^2 = 1.$$

Quantum register

A quantum register of length n is a collection of qubits $q_1, \ldots, q_n$.

Possible operations / dynamics

A **quantum computer** can

- perform unitary operations on quantum registers
- measure single qubits

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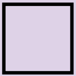
Bits vs Qubits

One classical bit



is in one of the basis states

$$\uparrow \;\widehat{=}\; \textit{High} \;\widehat{=}\; 1$$
$$\downarrow \;\widehat{=}\; \textit{Low} \;\widehat{=}\; 0$$

Boolean state space:

$$\mathbb{F}_2 = \{\uparrow, \downarrow\} = \{0, 1\}$$

consisting of 2 elements

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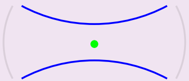

is in a coherent superposition

$$|\Psi\rangle = \alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle$$

of the basis states $|\uparrow\rangle, |\downarrow\rangle$

Quantum state space $\mathcal{H}_2 = \mathbb{C}^2$:

$$\left\{ \alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle : |\alpha|^2 + |\beta|^2 = 1 \right\}$$

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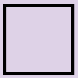

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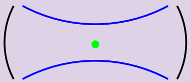

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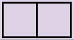
Two Bits vs Two Qubits

Two classical bits

Two bits can hold either of the 4 states in the state space
$\{0,1\} \times \{0,1\} = \{0,1\}^2 = \{\uparrow\uparrow, \downarrow\uparrow, \uparrow\downarrow, \downarrow\downarrow\}$.

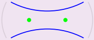


Two coupled qubits

A register of two coupled qubits can hold any of the states

$$|\Psi\rangle = \alpha\,|\uparrow\uparrow\rangle + \beta\,|\downarrow\uparrow\rangle + \gamma\,|\uparrow\downarrow\rangle + \delta\,|\downarrow\downarrow\rangle$$

in the state space $\mathcal{H}_2 \otimes \mathcal{H}_2 = \mathbb{C}^2 \otimes \mathbb{C}^2$.



Two separate qubits

Two separate qubits can hold any of the product states

$$|\Psi_1\rangle \otimes |\Psi_2\rangle = (\alpha_1\,|\uparrow\rangle + \beta_1\,|\downarrow\rangle) \otimes (\alpha_2\,|\uparrow\rangle + \beta_2\,|\downarrow\rangle)$$

in the state space $\mathcal{H}_2 \oplus \mathcal{H}_2 \subset \mathbb{C}^2 \oplus \mathbb{C}^2$.

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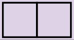

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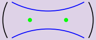

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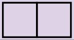

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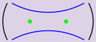

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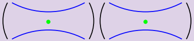
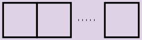


Many Bits vs Many Qubits

Classical register of n bits

 holds one of the 2^n states $\underline{\epsilon} = \epsilon_1 \cdots \epsilon_n$ of the state space $\{0,1\}^n = \{0,1\} \times \cdots \times \{0,1\}$.

Quantum register of n qubits

 can hold any coherent superposition

$$|\Psi\rangle = \sum_{\underline{\epsilon} \in \{0,1\}^n} \alpha_{\epsilon_1 \cdots \epsilon_n} |\epsilon_1\rangle \otimes |\epsilon_2\rangle \otimes \cdots \otimes |\epsilon_n\rangle$$

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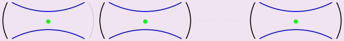
Product states of n qubits

 can only hold a product state

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(thus, only linear scaling of system and dimension)

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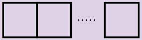 holds one of the 2^n states $\underline{\epsilon} = \epsilon_1 \cdots \epsilon_n$ of the state space $\{0,1\}^n = \{0,1\} \times \cdots \times \{0,1\}$.

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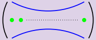 can hold any coherent superposition

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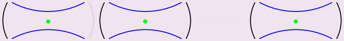 can only hold a product state

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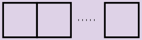 holds one of the 2^n states $\underline{\epsilon} = \epsilon_1 \cdots \epsilon_n$ of the state space $\{0,1\}^n = \{0,1\} \times \cdots \times \{0,1\}$.

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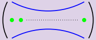 can hold any coherent superposition

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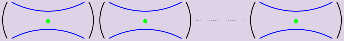 can only hold a product state

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Our Model of Measurements

von Neumann measurements of one qubit

First, specify a basis B for $\mathbb{C}^2$, e. g. $\{|0\rangle, |1\rangle\}$. The outcome of measuring the state $\alpha|0\rangle + \beta|1\rangle$ is described by a random variable X. The probabilities to observe "0" or "1" are given by

$$\Pr(X = 0) = |\alpha|^2, \quad \Pr(X = 1) = |\beta|^2.$$

Measuring a state in $\mathbb{C}^n$ in an orthonormal basis B

- Recall: Orthonormal Basis of $\mathbb{C}^N$

$$B = \{|\psi_i\rangle : i = 1, \ldots, N\}, \quad \text{where } \langle\psi_i|\psi_j\rangle = \delta_{i,j}$$

- Let $|\varphi\rangle = \sum_{i=1}^N \alpha_i |i\rangle$, where $\sum_{i=1}^N |\alpha_i|^2 = 1$. Then measuring $|\varphi\rangle$ in the basis B gives random variable X_B taking values $1, \ldots, N$:

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Two Important Types of Operations

Local operations

$$\mathbf{1}_N \otimes U = \begin{pmatrix} \boxed{U} & & & \\ & \boxed{U} & & \\ & & \ddots & \\ & & & \boxed{U} \end{pmatrix}, \quad \text{where} \quad U \in \mathcal{U}(2).$$

Conditioned operation: the controlled NOT (CNOT)

$$\boxed{\begin{array}{l} |00\rangle \mapsto |00\rangle \\ |01\rangle \mapsto |01\rangle \\ |10\rangle \mapsto |11\rangle \\ |11\rangle \mapsto |10\rangle \end{array}} \;\widehat{=}\; \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \;\widehat{=}$$

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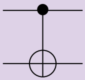
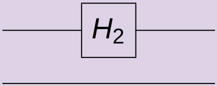
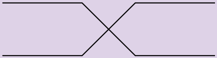
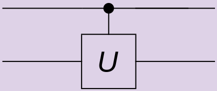


Notation for Quantum Gates

Gate in Feynman notation







Corresponding transformation

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ 1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & u_{11} & u_{12} \\ \cdot & \cdot & u_{21} & u_{22} \end{bmatrix}$$

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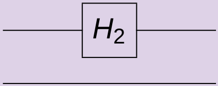

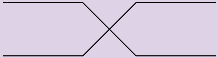

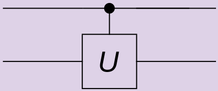

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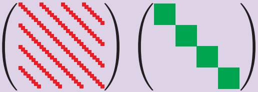
Two Constructions for Matrices

Tensor product

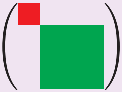
$T_1 \otimes T_2 = (T_1 \otimes I)(I \otimes T_2) =$ 

$I \otimes T_2$ classically parallel

$T_1 \otimes T_2$ quantum parallel $\Longrightarrow$ independent dynamics

classical: expensive, quantum: easy, since local operations

Direct sum

$T_1 \oplus T_2 = (T_1 \oplus I)(I \oplus T_2) =$ 

$I \oplus T_2$ conditional operation

$T_1 \oplus T_2$ CASE operator $\Longrightarrow$ coupled dynamics

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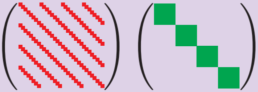
Two Constructions for Matrices

Tensor product

$$\color{red}{T_1} \otimes \color{green}{T_2} = (\color{red}{T_1} \otimes I)(I \otimes \color{green}{T_2}) =$$



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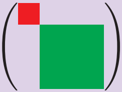

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The Hadamard transform on one qubit

$$H_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\sum_{x,y\in\mathbf{F}_2}(-1)^{xy}\,|x\rangle\langle y|\,.$$

The Hadamard transform on n qubits

The n-fold tensor product of H_2 is given by

$$H_2^{\otimes n} = \frac{1}{2^{n/2}}\sum_{x,y\in\mathbf{F}_2^n}(-1)^{x\cdot y}\,|x\rangle\langle y|\,,$$

where $x \cdot y := \sum_{i=1}^{n} x_i y_i \in \mathbf{F}_2$.

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The Hadamard transform on n qubits

The n-fold tensor product of H_2 is given by

$$H_2^{\otimes n} = \frac{1}{2^{n/2}} \sum_{x,y \in \mathbf{F}_2^n} (-1)^{x \cdot y} |x\rangle \langle y| ,$$

where $x \cdot y := \sum_{i=1}^n x_i y_i \in \mathrm{F}_2$.

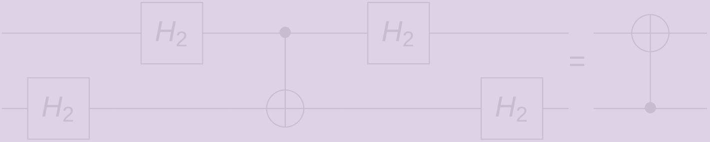
Properties of the Hadamard transform

- By definition H_2 is the following map:

$$|0\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

- Identity involving the Pauli matrices: $H_2 \sigma_x H_2 = \sigma_z$
- Identity involving the CNOT gate:



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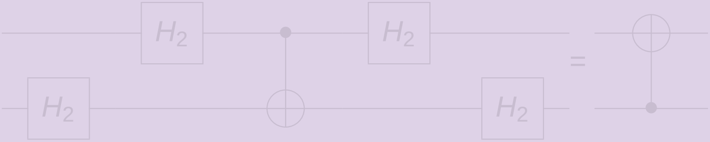

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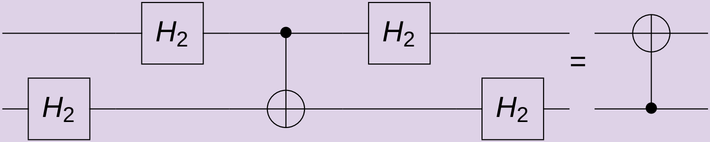
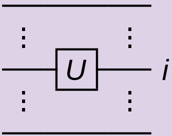
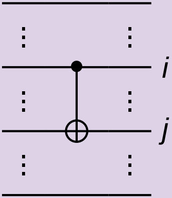


Quantum Gates and Circuits

Elementary quantum gates

$U^{(i)} =$ 

$\mathrm{CNOT}^{(i,j)} =$ 

Universal set of gates

Theorem (Barenco et al., 1995):

$$\mathcal{U}(2^n) = \langle U^{(i)}, \mathrm{CNOT}^{(i,j)} \quad : \quad i,j = 1,\ldots,n, \quad i \neq j \rangle$$

Quantum gates: main problem

Find efficient factorizations for given $U \in \mathcal{U}(2^n)$!

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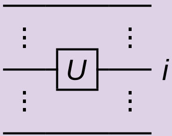

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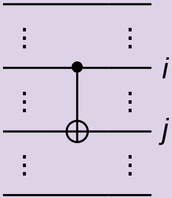

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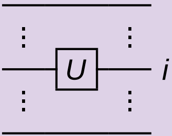

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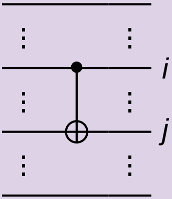

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Different Levels of Abstraction

Unitary matrix

$$U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

Factorized unitary matrix

$$U = \quad (I \otimes H_2) \quad (I \oplus \sigma_x) \quad (H_2 \otimes I)$$

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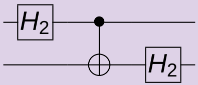

Basic Facts About Boolean Functions

Boolean functions

- Any function $f : \{0,1\}^n \to \{0,1\}^m$, where n is the number of input bits and m the number of output bits is said to be a Boolean function.
- Any Boolean function can be represented by a truth table. If $f = (f_1, \ldots, f_m)$, this is a matrix of size $2^n \times m$ where in column i we have the list of values $f_i(x_1, \ldots, x_n)$, where $x_j \in \{0,1\}$ for $j = 1, \ldots, n$.

The number of Boolean functions

- There are 2^{2^n} Boolean functions $f : \{0,1\}^n \to \{0,1\}$, i. e., functions with n inputs and one output (since we can assign an arbitrary value for each of the 2^n inputs).
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Universal Gates: Classical Computing

Connectives

NOT gate:

x_1	$\overline{x_1}$
0	1
1	0

AND gate:

x_1	x_2	$x_1 \wedge x_2$
0	0	0
0	1	0
1	0	0
1	1	1

OR gate:

x_1	x_2	$x_1 \vee x_2$
0	0	0
0	1	1
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NAND gate:

x_1	x_2	$\overline{x_1 \wedge x_2}$
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Theorem

Any deterministic classical circuit and thereby any deterministic classical computation can be realized by using NAND gates only. Any probabilistic computation can be realized using NAND gates and in addition one gate which realizes a fair coin flip.

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Finite field of two elements

- The set $\{0, 1\}$ can be equipped with a multiplication "$\cdot$" and an addition "$\oplus$" such that the field axioms hold.
- Truth tables for these operations:

x	y	$x \cdot y$
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x	y	$x \oplus y$
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- Then $(\{0, 1\}, \cdot, \oplus)$ becomes a field consisting of two elements only, also denoted by F_2. A finite field with n elements exists if and only if n is a prime power.
- Important identity: $(-1)^{x \oplus y} = (-1)^x \cdot (-1)^y$
- We can also rewrite CNOT: $|x\rangle |y\rangle \mapsto |x\rangle |x \oplus y\rangle$

Loading More Structure Onto Bits

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Reversible Computation of Boolean Functions

Basic issue of reversible computing

Suppose, we want to compute a function $f : \{0,1\}^n \to \{0,1\}^m$ that is not reversible. How can we do this?

One possible solution

Define a new Boolean function which takes $n+m$ inputs and $n+m$ outputs as follows:

$$F(x,y) := (x, y \oplus f(x))$$

Properties of $F(x,y)$

- On the special inputs $(x,0)$, where $x \in \{0,1\}^n$ we obtain that $F(x,0) = (x, f(x))$. Furthermore, F is reversible.
- Theorem (Bennett): If f can be computed using K gates, then F can be computed using $2K+m$ gates.

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The Toffoli gate "TOF"

x	y	z	x'	y'	z'
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0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
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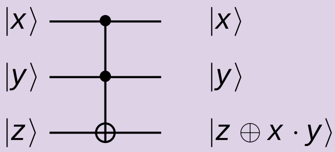

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Any reversible computation can be realized by using TOF gates and ancilla (auxiliary) bits which are initialized to 0.

Universal Gates: Reversible Computing

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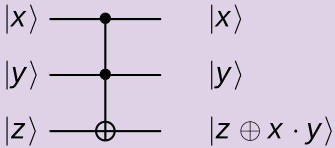

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Reversible Computation of Boolean Functions

More features of the reversible embedding $F(x,y)$

- Possibly, there are reversible functions that compute $G(x,0) = (f(x), \mathrm{junk}(x))$ and use use fewer than $n+m$ bits.
- Actually, only $\lceil \log_2(\max c_y) \rceil$ many extra bits are needed to make f reversible, where $c_y = |\{x : f(x) = y\}|$ are the sizes of the collisions of f.
- The advantage of F is its uniform definition.

Computing a function by a quantum circuit

Any function $f : \{0,1\}^n \to \{0,1\}^m$ can be computed by means of the reversible function $F : \{0,1\}^{n+m} \to \{0,1\}^{n+m}$. Hence we can compute f also by a quantum circuit

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A Two-Bit Problem

The Deutsch Jozsa problem (most simple case)

Given a Boolean function $f : \{0,1\} \to \{0,1\}$. Decide whether the property $f(0) = f(1)$ holds or not.

The four possible functions

x	$f(x)$
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1	0

x	$f(x)$
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Observation

- From querying the function only on one input, we cannot determine whether $f(0) = f(1)$ with certainty. E. g., if the answer is $f(0) = 0$, it could be the first or third function.
- On the other hand, two queries determine the function completely. What is the quantum query complexity?

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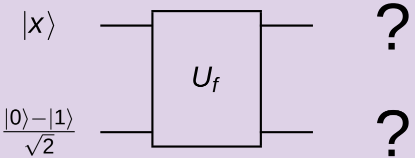
The Phase “Kick-Back” Trick

Computing the function into the phase



Computing the effect for different inputs

$$\begin{aligned}
|x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} \quad &\mapsto \quad |x\rangle \left(\frac{|f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} \right) \\
&= \quad |x\rangle (-1)^{f(x)} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\
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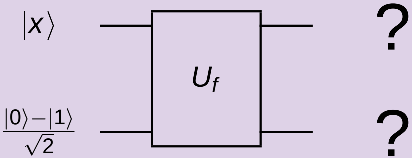

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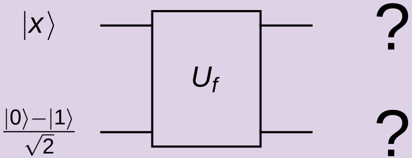

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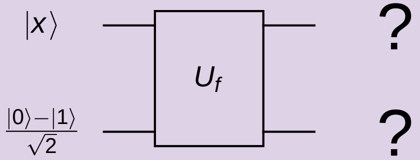

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|x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} &\mapsto |x\rangle \left(\frac{|f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} \right) \\
&= |x\rangle (-1)^{f(x)} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\
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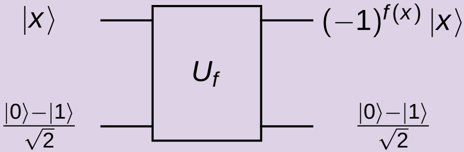
The Phase "Kick-Back" Trick

Computing the function into the phase



Computing the effect for different inputs

$$\begin{aligned}
|x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} \quad &\mapsto \quad |x\rangle \left(\frac{|f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} \right) \\
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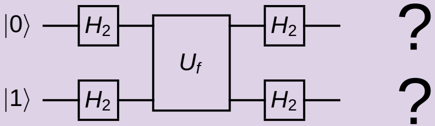
The Phase "Kick-Back" Trick

The quantum algorithm



Phase kick-back in superposition

$$
\begin{aligned}
|0\rangle\,|1\rangle &\xmapsto{H_2^{\otimes 2}} \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\
&\xmapsto{U_f} \frac{1}{\sqrt{2}}(-1)^{f(0)}\,|0\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}(-1)^{f(1)}\,|1\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\
&= \frac{(-1)^{f(0)}}{\sqrt{2}}\left(|0\rangle + (-1)^{f(0)\oplus f(1)}\,|1\rangle\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\
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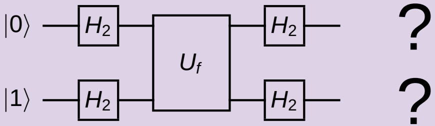

Phase kick-back in superposition

$$|0\rangle\,|1\rangle \overset{H_2^{\otimes 2}}{\longmapsto} \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

$$\overset{U_f}{\longmapsto} \frac{1}{\sqrt{2}}(-1)^{f(0)}\,|0\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}(-1)^{f(1)}\,|1\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

$$= \frac{(-1)^{f(0)}}{\sqrt{2}}\left(|0\rangle+(-1)^{f(0)\oplus f(1)}\,|1\rangle\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

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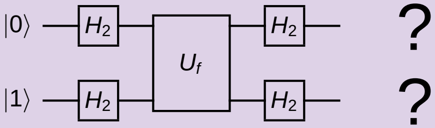

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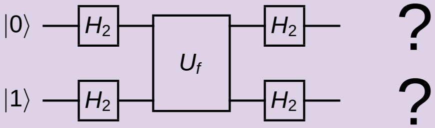

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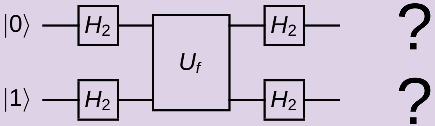

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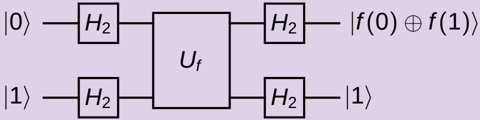

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Hence, we from measuring the first qubit in the computational basis, we obtain the answer $f(0) \oplus f(1)$ which reveals whether $f(0) = f(1)$ or not.

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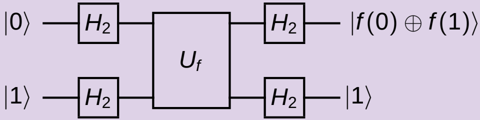

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What can be computed?

The strong Church-Turing thesis

Any reasonable algorithmic process carried out by a physical machine can be efficiently simulated by a probabilistic Turing machine. The slow-down for this is at most polynomial.

Efficient computations

Given a problem of size n, an algorithm is said to have a polynomial running time if the number of steps it needs to find a solution is bounded by $p(n)$, where p is a polynomial.

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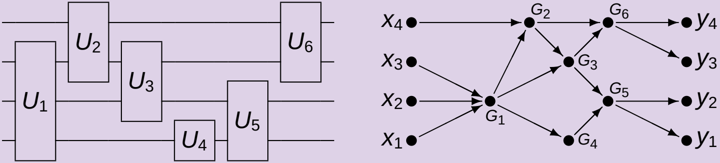
Computational Model: Quantum Circuits

Example: quantum circuit

Quantum circuit and corresponding directed acyclic graph.



Uniform families of quantum circuits

A family $\mathcal{F} := \{\mathcal{C}_n \in \mathcal{U}(2^n) \mid n \in \mathbb{N}\}$ of quantum circuits is called **uniform** if there exists a polynomial-time deterministic Turing machine which computes $n \mapsto \mathcal{C}_n$, where n is the problem size.

Theorem (Yao '93)

Uniform families of quantum circuits and quantum Turing machines (see Bernstein/Vazirani '93) are equivalent.

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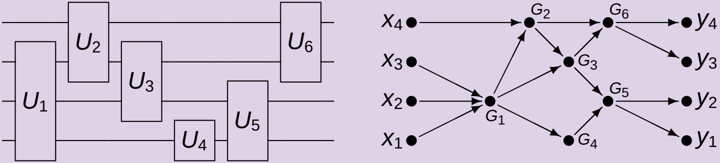

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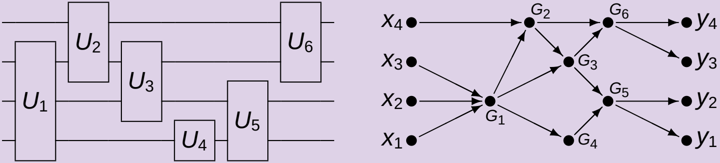

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Asymptotics of Functions

Landau notation

- We use Landau notation to compare the asymptotics of two functions $f, g : \mathbb{N} \to \mathbb{C}$. Typically, f is the running time for an algorithm for input of size n and g is another function.
- $f(n) = O(g(n))$ means that for some m there exists a constant $c > 0$, such that $|f(n)| \leq cg(n)$ for all $n \geq m$.
- $f(n) = \Omega(g(n))$ means that for some m there exists a constant $c > 0$, such that $|f(n)| \geq cg(n)$ for all $n \geq m$.
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Example

Let $f(n)$ be number of operations needed to compute a classical FFT of a vector of length n. Then $f(n) = O(n \log(n))$.

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Computational Complexity

Measuring the problem size

- Usually algorithms can use different kinds of resources. Typical examples for resources which can be measured are **time**, **space**, **depth**, and **queries**.
- The resources needed are always measured as a function of the input size.

Examples

- Consider the problem of multiplying two n-bit numbers. Clearly, this can be done in $O(n^2)$ additions and multiplications using the school method. The best known method uses $O(n \log n \log \log n)$ operations.
- The converse problem of factoring an n-bit number N into its prime factors is much harder. The currently best known algorithm needs $\exp\left(c(\log N)^{1/3}(\log \log N)^{2/3}\right)$ operations.

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Query Problems

Query complexity and black boxes

- Instead of counting the number of operations we can count the number of queries to f in order to solve the problem.
- Often upper and lower bounds can be shown for the query complexity model only.
- Black-box model: we assume that we are given a function f but cannot analyze how f is actually implemented.
- Most real-world problems are actually white-box, for example FACTORING, GRAPH-ISO, etc. Lower bounds in the white-box problems are typically very weak.

Examples

- Black box problems: Simon's algorithm, Grover's algorithm.
- White-box problems: Factoring and dlog, phase estimation.

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Simon's Problem

The XOR bit mask problem

We consider only functions $f : \{0,1\}^n \rightarrow \{0,1\}^n$ for which there exists $s \in \{0,1\}^n$ such that

- $\forall x \in \{0,1\}^n$ we have $f(x) = f(x \oplus s)$,
- $\forall x, y \in \{0,1\}^n$ we have that if $x \neq y \oplus s$, then $f(x) \neq f(y)$.

The task is to find the hidden string s.

Example where $f : \{0,1\}^3 \rightarrow \{0,1\}^3$

0	0	0	1	1	1
0	0	1	1	0	0
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	0	1
1	0	1	0	0	0
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Here $s = (0,1,1)$.

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Some comments about Simon's problem

- There is some similarity to the Deutsch-Jozsa problem, however, here the task is not to distinguish between two cases (constant vs balanced) but between exponentially many cases (one for each unknown string s).
- Problem might seem artificial, but Shor's algorithm has the same underlying idea. Note: This is a promise problem!
- Problem gave the first strong (polynomial vs exponential) separation between quantum and classical computing. Before that only super-polynomial separations were known.

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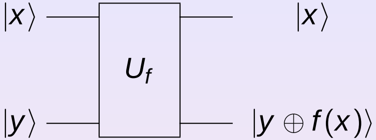
Simon's Problem: Specification

How the function f is given

We are given f as a black box in form of a quantum circuit computing U_f:



How can we query f?

- Classical algorithm: makes queries $x_1, \ldots, x_k$ resulting in the answers $f(x_1), \ldots, f(x_k)$ (counts as k queries).
- Quantum algorithm: can query in superposition, i. e., starting with the state $|\varphi\rangle = \sum_{i=1}^{k} |x_i\rangle$ results in

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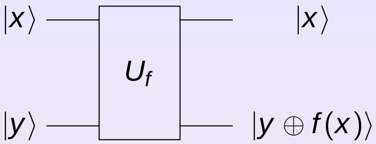

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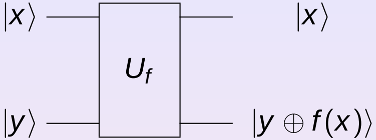

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$$U_f |\varphi\rangle |0\rangle = \sum_{i=1}^{k} U_f |x_i\rangle |0\rangle = \sum_{i=1}^{k} |x_i\rangle |f(x_i)\rangle .$$

This counts as one query!

Partial measurements / “non-demolition” measurements

- Suppose we are given a Boolean function $f : \{0,1\}^n \to \{0,1\}$ and have prepared the state

$$|\psi\rangle = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle |f(x)\rangle = \frac{1}{2^{n/2}} \left(\sum_{x:f(x)=0} \alpha_x |x\rangle |0\rangle + \sum_{x:f(x)=1} \alpha_x |x\rangle |1\rangle \right).$$

- Measuring the last qubit gives a random variable X.

$$\Pr(X=0) = \sum_{x:f(x)=0} |\alpha_x|^2, \; \Pr(X=1) = \sum_{x:f(x)=1} |\alpha_x|^2.$$

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Why partial/"non-demolition" measurements

- We can still work with the collapsed state! For instance if measuring the last qubit of

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Simon's Problem: Preparing Useful States

Creating the uniform superposition

The basic idea is to prepare

$$|0\rangle\,|0\rangle \overset{H_2^{\otimes n}}{\longmapsto} \frac{1}{\sqrt{2^n}} \sum_{x \in \mathbf{F}_2^n} |x\rangle\,|0\rangle$$

$$\overset{U_f}{\longmapsto} \frac{1}{\sqrt{2^n}} \sum_{x \in \mathbf{F}_2^n} |x\rangle\,|f(x)\rangle\,.$$

Collapsing the uniform superposition

Now, measuring the second register will yield a random $s \in \mathbf{F}_2^n$ in the image of f. The state collapses to

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$$\langle s \rangle^{\perp} = \{x \in \mathbb{F}_2^n | x \cdot s = 0\}.$$

What we really want...

... are elements from $\langle s \rangle$ itself (there is only 0 and s itself since $\langle s \rangle$ is one-dimensional). How can we compute $\langle s \rangle$ from $\langle s \rangle^{\perp}$?

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Solution to Simon's Problem

Quantum algorithm

Given: Function $f : \{0,1\}^n \to \{0,1\}^n$ with Simon promise, i. e., preimages of a fixed image have the form x_0 and $x_0 \oplus s$.

Task: Find the unknown bit-string $s \in \mathbf{F}_2^n$.

Repeat the following steps $n-1$ times:

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6. Measure first register: Sample $y \in \mathbb{F}_2^n$ with $y \cdot s = 0$.

Further classical post-processing is necessary!

Solution to Simon's Problem

Quantum algorithm

Given: Function $f : \{0,1\}^n \to \{0,1\}^n$ with Simon promise, i. e., preimages of a fixed image have the form x_0 and $x_0 \oplus s$.

Task: Find the unknown bit-string $s \in \mathbb{F}_2^n$.

Repeat the following steps $n-1$ times:

1. Initialize two quantum registers: $|0\rangle\,|0\rangle$
2. Equal distribution on first register: $\frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle\,|0\rangle$
3. Compute f in superposition: $\frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle\,|f(x)\rangle$
4. Measure second register: $\frac{1}{\sqrt{2}}(|x_0\rangle + |x_0 \oplus s\rangle) = |\varphi_{x_0,s}\rangle$
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Further classical post-processing is necessary!

Classical Postprocessing in Simon's Algorithm

Classical post-processing

- After $n-1$ iterations: $y_1, \ldots, y_{n-1} \in \mathbb{F}_2^n$ with $y_i \cdot s = 0$.
- We have to infer s by a purely classical computation.
- Show high probability of success over the choice of y_i.

Linear algebra over $\mathbb{F}_2$

We are given the linear system of equations

$$A \cdot s = \begin{pmatrix} y_1 \\ \hline \vdots \\ \hline y_{n-1} \end{pmatrix} \cdot s = 0$$

Hence, we have to compute the kernel of $A \in \mathbb{F}_2^{(n-1)\times n}$. If the kernel is one-dimensional, then s is uniquely determined.

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Classical Post-Processing in Simon's Algorithm

The probability of success

Since $\dim(\langle s\rangle) = 1$ we have that $\dim(\langle s\rangle^\perp) = n-1$. Hence we have to bound the probability that $n-1$ random vectors in F_2^n are linear independent:

$$\begin{aligned}
\Pr(rk(A) = n-1) &= \left(\frac{2^{n-1}-1}{2^{n-1}}\right)\left(\frac{2^{n-1}-2}{2^{n-1}}\right)\cdot\ldots\ldots\left(\frac{2^{n-1}-2^{n-2}}{2^{n-1}}\right) \\
&= \left(1-\frac{1}{2^{n-1}}\right)\left(1-\frac{1}{2^{n-2}}\right)\cdot\ldots\ldots\left(1-\frac{1}{4}\right)\cdot\frac{1}{2} \\
&\geq \left(1-\left(\frac{1}{2^{n-1}}+\ldots+\frac{1}{4}\right)\right)\frac{1}{2} \geq \left(1-\frac{1}{2}\right)\cdot\frac{1}{2} \geq \frac{1}{4}
\end{aligned}$$

Complexity of the quantum algorithm

- We have used n iterations and each individual run uses one query, $2n$ Hadamard transforms, and $n+1$ single qubit measurements
- Postprocessing: Computing the kernel of a matrix of size $n \times n$ is linear algebra and can be solved in time $O(n^3)$.
- Overall we have found a polynomial-time quantum algorithm.

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Simon's Problem: Classical Lower Bound

Theorem

Let $\mathcal{A}$ be a classical probabilistic algorithm which determines s using k queries to the function f. Then $k = \Omega(2^{n/2})$.

Computational complexity theory lingo

Hence there exists an oracle $\mathcal{O}$ with respect which we have a separation between classical and quantum computation:

$$\mathrm{BPP}^{\mathcal{O}} \neq \mathrm{BQP}^{\mathcal{O}}$$

- This is a so-called "relativized result", i. e., it holds in a world in which calls to the oracle cost only one query.
- Whether this also holds for our world, i. e., without oracles, is a major open problem in theoretical computer science.
- Looking ahead: Why does the fact that *FACTORING* is in BQP, whereas no classical algorithm is known for it, does not imply that BPP $\neq$ BQP?

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Sketch of proof of the $\Omega(2^{n/2})$ lower bound

- Suppose $\mathcal{A}$ makes k queries $x_1, \ldots, x_k$, where $x_i \neq x_j$. Let $\mathcal{F}_k := \{f(x_i) : i = 1, \ldots, k\}$ and $\mathcal{E}_k := \{x_i \oplus x_j : i \neq j\}$.
- If $|\mathcal{F}_k| < k$ then we have found a collision, i. e. a pair (i_0, j_0) with $f(x_{i_0}) = f(x_{j_0})$. Then $s = x_{i_0} \oplus x_{j_0}$.
- Suppose there was no collision. Then $s \neq \mathcal{E}_k$ and $|\mathcal{E}_k| = \binom{k}{2}$ candidates have been eliminated.
- However, there are $2^n - 1 - \binom{k}{2}$ candidates for s. We show that they are equally likely for a given $\mathcal{F}_k$. Then $k = \Omega(2^n)$.
- Bayes rule:

$$\Pr(s = s_0 \,|\, |\mathcal{F}|_k = k) = \frac{\Pr(|\mathcal{F}_k| = k \,|\, s = s_0) \cdot \Pr(s = s_0)}{\Pr(|\mathcal{F}_k| = k)}.$$

The a priori probabilities $\Pr(s = s_0)$ are equal and by symmetry $\Pr(|\mathcal{F}_k| = k \,|\, s = s_0)$ is independent of s_0. $\square$

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For Further Reading

- G. Alber, Th. Beth, M. Horodecki, P. Horodecki, R. Horodecki, M. Roetteler, H. Weinfurter, and A. Zeilinger.
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  Graduate Studies in Mathematics, vol. 47, AMS, 2002.
- M. Nielsen und I. Chuang.
  Quantum Computation and Information.
  Cambridge University Press, 2000.

Conclusions

- Elementary quantum gates:
  - Controlled NOT gate
  - Local unitary transformations
- A simple quantum algorithm on two qubits which distinguishes constant from balanced functions.
- Separation: 1 query (quantum) vs 2 queries (classical)
- Quantum algorithm for Simon's problem based on:
  - Computing with superpositions
  - Interference of computational paths

Introduction to Quantum Algorithms
Part II: The Algorithms of Shor and Grover

Martin Rötteler

NEC Laboratories America, Inc.
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International Summer School on Quantum Information,
Max-Planck-Institut für Physik komplexer Systeme
Dresden, September 1, 2005

Overview

Today:

- Shor's algorithm
  - Modular exponentiation
  - Period extraction via Quantum Fourier Transform
  - Classical post-processing
- Generalizations of Shor's algorithm
- Grover's algorithm for searching a list
- Universal quantum gates

Outlook:

- On September 19, Markus Grassl will continue with an introduction to quantum error-correcting codes.

The Integer Factorization Problem

Basic problem

Given a natural number N. Find a (prime) factor of N.

Best known classical algorithm

The number field sieve has a complexity of

$$\exp\left((1.923 + o(1))(\log N)^{1/3}(\log\log N)^{2/3}\right)$$

which is (sub)exponential in the number $n = \log N$ of bits of N.

Making money with factoring

The company RSA offers \$ 200.000 for anybody who can factor a certain 617 digit number N. This number is known to be of the form $N = pq$ but finding p and q is infeasible using the best known classical algorithms.

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The RSA Factoring Challenge

RSA-200 is factored!

In May 2005 a small team of people has factored RSA-200. At 663 bits, this is the largest RSA Challenge Number factored.

The classical effort undertaken

Sieving equivalent of 55 years on a single 2.2 GHz Opteron CPU. The matrix step took about 3 months on a cluster of 80 2.2 GHz Opterons. Computed from late 2003 to May 2005.

RSA-200 and its factors

$$N = \begin{array}{l} 27997833911221327870829467638722601621070446786955 \\ 42853756000992932612840010760934567105295536085606 \\ 18223519109513657886371059544820065767750985805576 \\ 13579098734950144178863178946295187237869221823983 \end{array}$$

$$p = \begin{array}{l} 35324619344027701212726049781984643686711974001976 \\ 25023649303468776121253679423200058547956528088349 \end{array}$$

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Shor's Algorithm: Reduction to Order Finding

Reformulating the factoring problem

We can factor N if the following problem can be solved:

- **Input:** A number a with $1 < a < N$.
- **Output:** The order r of a modulo N, i. e., the smallest integer $r > 0$ such that $a^r \equiv 1 \mod N$.

Why is this a reduction?

Suppose we want to find a divisor of N different from $+1$ or -1.

- Pick a random a with $1 < a < N$ and find its order r
- Suppose that r is even (happens with high probability):

$$0 = (a^r - 1) = (a^{r/2} - 1)(a^{r/2} + 1) \mod N.$$

- If $a^{r/2} \neq \pm 1$ then $\gcd(a^{r/2} - 1, N)$ and $\gcd(a^{r/2} + 1, N)$ yield at least one nontrivial divisor of N.

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Shor's Algorithm: Reduction to Order Finding

Remarks about this reduction

- Note that $\gcd(a, b)$ of two n-bit integers can be computed in $poly(\log(n))$ time.
- There were two events in which the reduction fails: (i) we pick a with an odd order r and (ii) $a^{r/2} = \pm 1$. We have to bound the probability for one of these events to occur.

Theorem

Let $N = p_1^{\mu_1} \dots p_m^{\mu_m}$ with $m \geq 2$ and $p_i > 2$. Then

$$\Pr(\text{neither (i) nor (ii) occurs}) \geq 1 - \frac{1}{2^m}$$

The big question

How can we efficiently determine the multiplicative order of a random element a modulo N?

Shor's Algorithm: Reduction to Order Finding

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Defining a Period Function

The modular exponentiation map

Let N be an integer and let $a \in Z_N$.

- Let M be an integer. The modular exponentiation is the map $f : x \mapsto (a^x \bmod N)$ from Z_M to Z_N.
- Result: The map f can be implemented efficiently using standard arithmetic in $O(poly(\log N))$ operations.
- Hence also the map $U_f : |x\rangle |y\rangle \mapsto |x\rangle |a^x \bmod N\rangle$ can be implemented efficiently.
- Recall that the order of a is defined as the smallest integer r such that $a^r = 1 \bmod N$.

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The function $f : x \mapsto (a^x \bmod N)$ is periodic and has period length r, i. e., $f(x) = f(x + r)$ for all inputs x.

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Setting up a Periodic State

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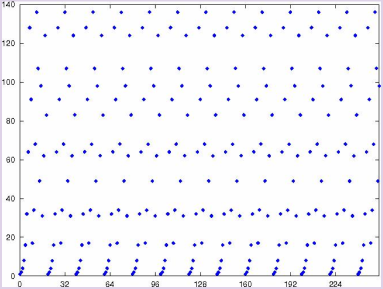
The function $f : x \mapsto a^x \bmod N$ is periodic and has period length r, i. e., $f(x) = f(x+r)$ for all inputs x.

The graph of the function $f(x) = 2^x \bmod 165$

$|y = f(x)\rangle$



$|x\rangle$

Shor's Problem: Preparing Useful States

Creating the graph of f

Let $f(x) = a^x \bmod N$ be the modular exponentiation and let $U_f : |x\rangle |y\rangle \mapsto |x\rangle |y \oplus f(x)\rangle$ be as usual. We compute (letting $M = 2^m$)

$$|0\rangle |0\rangle \overset{H_2^{\otimes m}}{\mapsto} \frac{1}{\sqrt{M}} \sum_{x \in Z_M} |x\rangle |0\rangle \overset{U_f}{\mapsto} \frac{1}{\sqrt{M}} \sum_{x \in Z_M} |x\rangle |f(x)\rangle .$$

Collapsing the graph of f

Now, measuring the second register will yield a random $s \in Z_N$ in the image of f. The state collapses to

$$\frac{1}{\sqrt{N/r}} \sum_{k=0}^{N/r-1} |x_0 + k \cdot r\rangle .$$

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An Application of the DFT: Period Extraction

Motivation

- We would like to apply the trick from Simon's algorithm:

$$|\varphi_{x_0,s}\rangle = \frac{1}{\sqrt{2}}(|x_0\rangle + |x_0 \oplus s\rangle) \mapsto \frac{1}{\sqrt{2^{n+1}}} \sum_{\substack{x \in \mathbb{F}_2^n \\ x \cdot s = 0}} (-1)^{x \cdot x_0} |x\rangle .$$

- The unknown offset x_0 is transfered into the phases.

- The analogue of $|\varphi_{x_0,s}\rangle$ in case of the cyclic group Z_N is

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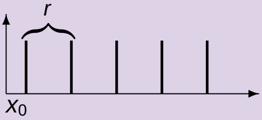

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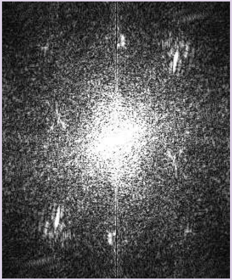
The Discrete Fourier Transformation (DFT)

Definition of the DFT

$$\mathrm{DFT}_N := \frac{1}{\sqrt{N}} \left[\omega_N^{k\cdot\ell} \right]_{k,\ell=0\ldots N-1}, \quad \omega_N = e^{2\pi i/N}$$

Example

$\xrightarrow{\text{DFT}}$



Estimating Character Sums

Useful trick in quantum computing (Character Lemma)

Lemma: For all $i = 0, \ldots, n-1$ the following holds:

$$\sum_{j=0}^{n-1} \omega_n^{ij} = n \cdot \delta_{i,0}$$

Proof: Let $S := \sum_{j=0}^{n-1} \omega_n^{ij}$. Then

$$\omega_n^i S = \sum_{j=0}^{n-1} \omega_n^i \omega_n^{ij} = \sum_{j=0}^{n-1} \omega_n^{i(j+1)} = \sum_{j=0}^{n-1} \omega_n^{ij} = S$$

- If $i \neq 0$ then $\omega_n^i \neq 1$, i. e., $(1 - \omega_n^i) \neq 0$. Hence $S = 0$.
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Extracting the Period of a Function

Theorem (Fourier duality)

Let $N \in \mathbb{N}$ and let $r \in Z_N$ be a divisor of N, and let $x_0 \in Z_N$. Then

$$\mathrm{DFT}_N \, |\psi_{x_0,r}\rangle = \mathrm{DFT}_N \left(\frac{1}{\sqrt{N/r}} \sum_{k=0}^{N/r-1} |x_0 + k \cdot r\rangle \right) = \frac{1}{\sqrt{r}} \sum_{\ell=0}^{r-1} \omega_N^{\ell x_0 \frac{N}{r}} \left| \ell \frac{N}{r} \right\rangle$$

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Constructive / Destructive Interference

Computing the coefficients α_i

For each $i = 0, \ldots, N-1$ we have to compute $\alpha_i = \sum_{k=0}^{N/r-1} \omega_N^{ikr}$.

- Case 1: $i = \frac{N}{r}\ell$ for some $\ell = 0, \ldots, r-1$. Then

$$\alpha_i = \sum_{k=0}^{N/r-1} \omega_N^{\frac{N}{r}\ell kr} = \sum_{k=0}^{N/r-1} 1 = \frac{N}{r}.$$

- Case 2: $i \neq \frac{N}{r}\ell$ for all $\ell = 0, \ldots, r-1$. Then

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by the Character Lemma.

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$$\mathrm{DFT}_N \left|\psi_{x_0,r}\right\rangle = \frac{\sqrt{r}}{N}\sum_{i=0}^{N-1}\omega_N^{ix_0}\left(\sum_{k=0}^{N/r-1}\omega_N^{ikr}\right)|i\rangle = \frac{\sqrt{r}}{N}\sum_{i=0}^{N-1}\omega_N^{ix_0}\alpha_i\,|i\rangle$$

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What happens if r does not divide N?

In this case the state can be approximated very accurately by

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Given: Modular exponentiation function $U_a : |x\rangle\,|0\rangle \mapsto |x\rangle\,|a^x \bmod N\rangle$.

Task: Find the order r of a modulo N.

Repeat the following steps one time: (w/o normalizations, $M = 2^m >> N$)

1. Initialize two quantum registers: $|0\rangle\,|0\rangle$
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How can we classically reconstruct r from $\frac{p}{q}$?

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Solution to the Period Extraction Problem

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Classical Post-Processing

How to obtain r from $\frac{p}{q}$?

Since we know N and ℓ_0, we could easily compute r from $\ell_0 \frac{N}{r}$. However, we are just given $\frac{p}{q}$ for which we only know that the following holds

$$\left| \frac{p}{q} - \frac{\ell_0}{r} \right| < \frac{1}{2r^2}$$

Diophantine approximation

We apply the continued fractions algorithm to $\frac{p}{q}$. This will lead to several principal fractions and actually $\frac{\ell_0}{r}$ will be one of them. Note that we can check whether a candidate r is indeed the order.

Theorem (Shor '94)

$$\text{FACTORING} \in \text{BQP}.$$

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The continued fractions algorithm

Input: $x \in \mathbb{R}$. Output: Sequence of $b_i \in \mathbb{Z}$ (possibly infinite) which represents x. Let $b_0 := \lfloor x \rfloor$, $x_1 := \frac{1}{x - b_0}$, $b_1 := \lfloor x_1 \rfloor$, $x_2 := \frac{1}{x_1 - b_1}$, ... Then

$$x = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \frac{1}{\dots}}}$$

Example with rational input

Suppose that $x = \frac{5021264471}{8589934592}$. The algorithm results in (vector of b_i's):

$$[1, 1, 2, 2, 5, 3, 1, 1, 3, 1, 11, 1, 1, 21, 1, 2, 5, 1, 1, 1, 1, 1, 2, 1, 2, 1, 3]$$

Consider the *convergents* C_n which are obtained by truncating after n steps:

n	1	2	3	4	5	6	7	8	9	10
C_n	1	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{7}{12}$	$\frac{38}{65}$	$\frac{121}{207}$	$\frac{159}{272}$	$\frac{280}{479}$	$\frac{999}{1709}$	$\frac{1279}{2188}$

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Let $x \in \mathbb{Q}$ and assume that we are given $\frac{p}{q} \in \mathbb{Q}$ such that

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Then x is a convergent C_n of $\frac{p}{q}$, namely that for which $|C_n - \frac{p}{q}| < \frac{1}{2q^2}$.

Example (cont'd)

Let $y = \frac{31415}{53742}$ be the inverse period. Since denominator of y is $\leq 2^{16}$ we can work with precision $\leq 2^{33}$. Suppose we measure $\frac{p}{q} = \frac{5021264471}{8589934592}$:

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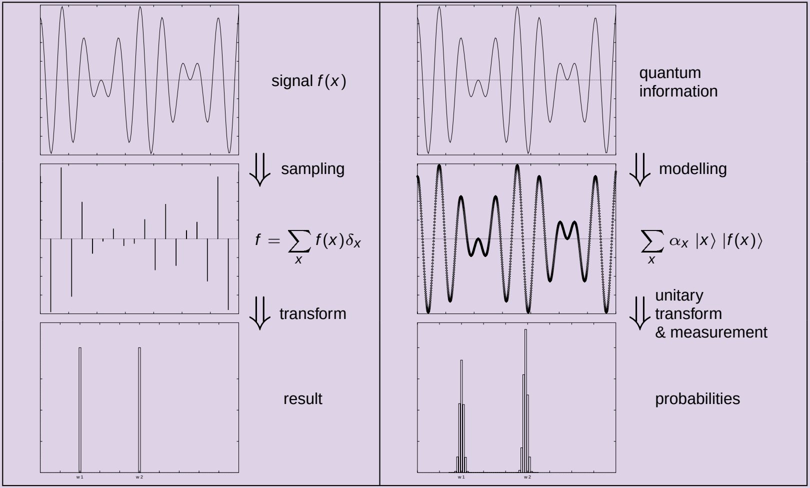
Parallels between DSP and QC

Digital Signal Processing | Quantum Computing



Example: Fast Fourier Transform (FFT)

Cooley-Tukey FFT

The matrix $\mathrm{DFT}_N = \frac{1}{\sqrt{N}} \left[\omega_N^{k \cdot \ell} \right]_{k,\ell=0\ldots N-1}$, where $\omega_N = e^{2\pi i/N}$ can be written as a short product of sparse matrices.

$$\mathrm{DFT}_4 = \Pi_{rev} \cdot (\mathbf{1}_2 \otimes \mathrm{DFT}_2) \cdot \mathrm{diag} \cdot (\mathrm{DFT}_2 \otimes \mathbf{1}_2)$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & i \end{bmatrix} \cdot \begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & 1 & & -1 \end{bmatrix}$$

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Multiplication with DFT_N can be performed classically in $O(N \log N)$ elementary operations.

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Cooley-Tukey Formula

$$\Pi_n \, \mathrm{DFT}_{2^n} = \left(\begin{array}{c|c} \mathrm{DFT}_{2^{n-1}} & \mathrm{DFT}_{2^{n-1}} \\ \hline \mathrm{DFT}_{2^{n-1}} \, D_n & -\mathrm{DFT}_{2^{n-1}} \, D_n \end{array}\right)$$

$$= (\mathbf{1}_2 \otimes \mathrm{DFT}_{2^{n-1}}) \cdot (\mathbf{1}_{2^{n-1}} \oplus D_n) \cdot (\mathrm{DFT}_2 \otimes \mathbf{1}_{2^{n-1}})$$

Factorization of the **twiddle factors**

$$D_n := \begin{pmatrix} 1 & & & \\ & \omega_{2^n} & & \\ & & \omega_{2^n}^2 & & \\ & & & \ddots & \\ & & & & \omega_{2^n}^{2^{n-1}-1} \end{pmatrix} = \begin{pmatrix} 1 & \\ & \omega_{2^n}^{2^{n-2}} \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 1 & \\ & \omega_{2^n} \end{pmatrix}$$

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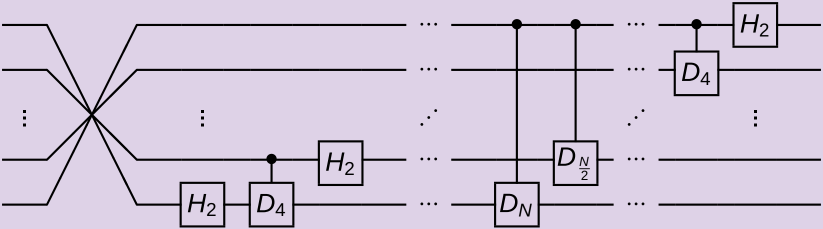
Cooley-Tukey Realization of DFT

Quantum circuit for DFT_N



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Classical Computer	Quantum Computer
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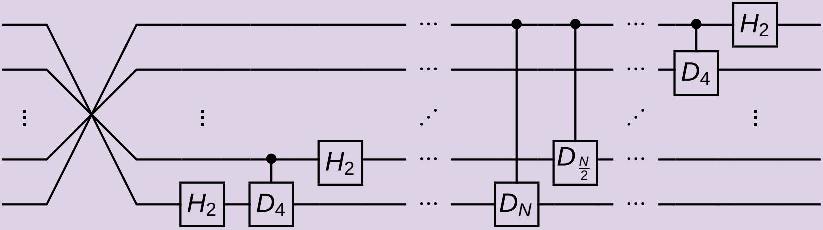

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Shor's Algorithm: Gate Counts

Modular exponentiation

$U: |x\rangle\,|0\rangle \mapsto |x\rangle\,|a^x \bmod N\rangle$, where N is the k-bit number to be factored, and x is a $2k$-bit number. Implementation using $396k(k^2 + O(k))$ elementary operations [Beckman et al.].

Quantum Fourier Transform:

DFT_{2^n} needs $\frac{1}{2}n(n-1)$ two-qubit gates and n one-qubit gates.

An upper bound on the resources for k-bit number N

About $400k^3$ operations are needed (can be improved to $O(k^2 \log k \log\log k))$). The space needed is $5k + 1$ qubits.

Example: Factoring 128-bit and 1024-bit numbers

For 128-bit we need $840 \cdot 10^6$ operations and 641 qubits. For 1024-bit number $429 \cdot 10^9$ operations and 5121 qubits.

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HSP: History of the Problem

D. Simon, 1994

Hidden Subgroups in $(\mathbb{Z}_2)^n$.

P. Shor, 1994

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Definition of the problem

Given: Finite group G, finite set S, map $f : G \to S$

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Problem: Find generators for H.

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- This is a natural generalization of Simon's problem.
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- In fact, the HSP for *any* subgroup $H \leq \mathbb{F}_2^n$ can be solved efficiently.

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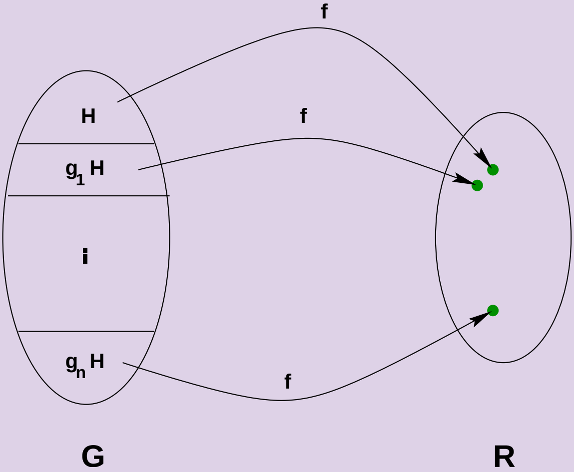
HSP: Separation of Pre-images

Visualization of the cosets of G



Examples for Hidden Subgroup Problems

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Find hidden subgroup in $G = Z_2^n$ of a *black-box* function $f : G \to \{0,1\}^m$, where $m \le n$ and $x \in y + H \Leftrightarrow f(x) = f(y)$.

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Find hidden subgroup in $G = Z_M$ with respect to the function

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Can be reduced to the problem of finding certain hidden subgroups of order 2 in the non-abelian group $G = S_n \wr S_2$.

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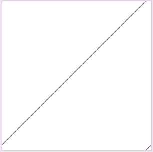
Duality of the Fourier Transform

Basic identity

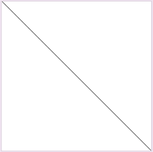
$$\mathrm{DFT}_A\left(\frac{1}{\sqrt{|U|}}\sum_{\mathbf{x}\in U+c}|\mathbf{x}\rangle\right)=\frac{1}{\sqrt{|U^\perp|}}\sum_{\mathbf{y}\in U^\perp}\varphi_{c,y}\cdot|\mathbf{y}\rangle$$

Geometric interpretation

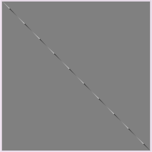
Shifted Line



DFT $\longrightarrow$



Amplitude



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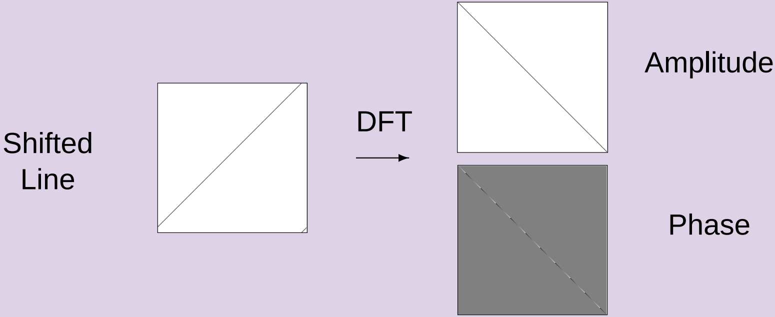

Generalized Fourier Transforms

Definition of DFT_G

Any isomorphism $\Phi : \mathbb{C}[G] \longrightarrow \bigoplus_{k=1}^{m} \mathbb{C}^{d_k \times d_k}$ of the group algebra and a direct sum of irreducible matrix algebras.

Some properties of DFT_G

- Defined for arbitrary finite groups.
- The isomorphism Φ is realized by a unitary matrix

$$\mathrm{DFT}_G = \frac{1}{\sqrt{|G||H|}} \sum_{\rho,i,j} \sqrt{d_\rho} \sum_{h \in H} \rho_{ij}(gh) \left|\rho, i, j\right\rangle \left\langle g\right| .$$

- DFT_G decomposes the regular representation ϕ of G:

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Hidden Subgroup Problems: Standard Algorithm

Quantum algorithm

Given: finite group G with hidden subgroup $H \leq G$.
Task: Find a set of generators for H.

Repeat the following steps $poly(n)$ many times:

1. Initialize two quantum registers: $|0\rangle\,|0\rangle$
2. Equal distribution on first register: $\frac{1}{\sqrt{|G|}} \sum_{x \in G} |x\rangle\,|0\rangle$
3. Compute f in superposition: $\frac{1}{\sqrt{|G|}} \sum_{x \in G} |x\rangle\,|f(x)\rangle$
4. Measure second register: $\frac{1}{\sqrt{|H|}} \sum_{x \in cH} |x\rangle\,|f(c)\rangle$
5. Compute DFT_G on first register: $$\frac{1}{\sqrt{|G||H|}} \sum_{\rho,i,j} \sqrt{d_\rho} \sum_{h \in H} \rho_{ij}(ch)\,|\rho, i, j\rangle$$
6. Measure first register: Sample (ρ, j) with probability $\sum_i \frac{d_\rho}{|G|} \left| \sum_{h \in H} \rho_{ij}(ch) \right|^2$.

Further classical post-processing necessary!

Searching for a satisfying assignment

Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$, find an $x \in \{0,1\}^n$ such that $f(x) = 1$. Such an x is also called "satisfying assignment" and f itself is also called "predicate". Note that this search problem includes NP-complete problems such as 3-SAT.

How the search problem is specified

Given: List X of $N = 2^n$ items and a predicate f on X which is given by the operator

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Question: What is the relation between U_f and V_f?

Realizing V_f from U_f

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Grover's Algorithm

The diffusion operator

$$D_n := \begin{pmatrix} -1 + \frac{2}{2^n} & \frac{2}{2^n} & \cdots & \frac{2}{2^n} \\ \frac{2}{2^n} & -1 + \frac{2}{2^n} & \cdots & \frac{2}{2^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{2^n} & \frac{2}{2^n} & \cdots & -1 + \frac{2}{2^n} \end{pmatrix}$$

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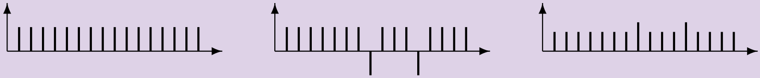

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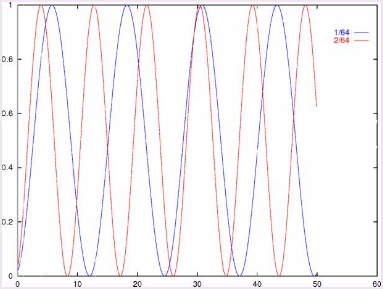

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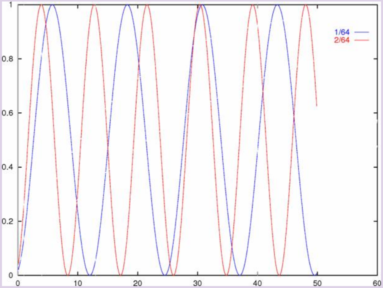

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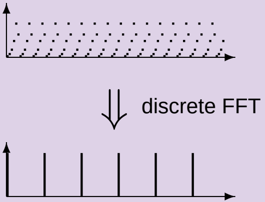
quantum: $O(\mathrm{poly}(\log n))$



- feature extraction using signal transforms
- leads to the idea of "hidden subgroup problems"
- highly regular, in general huge speed-ups can be expected

Searching

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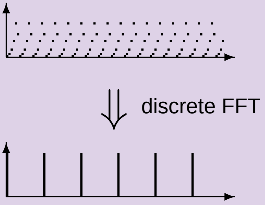

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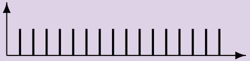

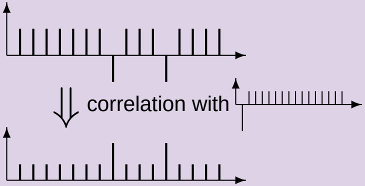

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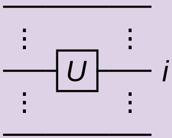
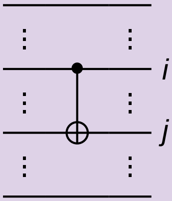
Quantum Gates and Circuits

Elementary quantum gates

$U^{(i)} =$ 

$\mathrm{CNOT}^{(i,j)} =$ 

Universal set of gates

Theorem (Barenco et al., 1995):

$$\mathcal{U}(2^n) = \langle U^{(i)}, \mathrm{CNOT}^{(i,j)} \quad : \quad i,j = 1,\ldots,n, \quad i \neq j \rangle$$

How to prove this result?

Next: breaking down the proof into several small steps

Examples for quantum circuits

Realizing a cyclic shift

How to realize $P_n : x \mapsto x + 1 \mod 2^n$, which cyclically shifts the basis states of an n qubit register?

Solution 1

Purely classical realization

Solution 2

Genuinely quantum realization

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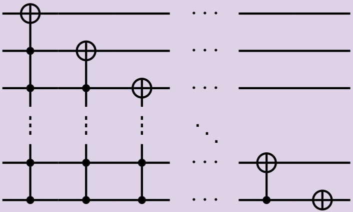

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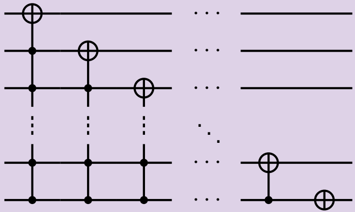

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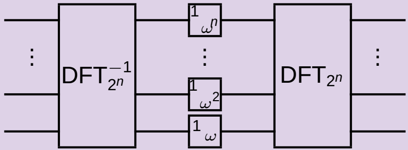

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Universality of CNOT and Local Gates

Proof outline

- Given a unitary matrix $U \in \mathcal{U}(2^n)$.
- Write $U = U_1 \cdot \ldots \cdot U_M$, where U_i acts on pairs of states
- Factorize each U_i using multiply-controlled $\Lambda_n(V)$ gates, where $V \in \mathcal{U}(2)$.
- Write each V in the form $(A^\dagger \sigma_x A)(B^\dagger \sigma_x B)$ with $A, B \in \mathcal{U}(2)$.
- Use this to write each $\Lambda_n(V)$ in terms of local gates and $\Lambda_n(\sigma_x)$ (i. e., generalized CNOTs).
- Implement $\Lambda_n(\sigma_x)$ recursively using $\Lambda_k(\sigma_x)$, where $k < n$.

Literature

A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo et al.
Elementary gates for quantum computation.
Physical Review A, 52(5):3457–3467, 1995.

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- Use this to write each $\Lambda_n(V)$ in terms of local gates and $\Lambda_n(\sigma_x)$ (i. e., generalized CNOTs).
- Implement $\Lambda_n(\sigma_x)$ recursively using $\Lambda_k(\sigma_x)$, where $k < n$.

Literature

A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo et al.
Elementary gates for quantum computation.
Physical Review A, 52(5):3457–3467, 1995.

Rotation on a subspace spanned by the states s_1 and s_2

$$T(s_1,s_2) = \begin{array}{l} \begin{array}{ccccccccccc} & & & s_1 & & & & s_2 & & & \end{array} \\ \left(\begin{array}{ccccccccccc} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & * & & & & * & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & * & & & & * & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{array}\right) \begin{array}{c} \\ \\ \\ s_1 \\ \\ \\ \\ s_2 \\ \\ \\ \\ \end{array} \end{array}$$

Theorem

Every $U \in \mathcal{U}(2^n)$ can be written in the form

$$U = \prod_{s_1,s_2 \in \{0,1\}^n} T(s_1,s_2).$$

Operations on Subspaces

Rotation on a subspace spanned by the states s_1 and s_2

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Conditional gates with multiple controls

Let $U \in \mathcal{U}(2)$. Then $\Lambda_k(U) \in \mathcal{U}(2^{k+1})$ is defined by

$$\Lambda_k := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \boxed{U} \end{pmatrix} = \mathbf{1}_{2^{k+1}-2} \oplus U.$$

Alternative description of $\Lambda_k(U)$

$$\Lambda_k(U)\,|x_1,\ldots,x_n\rangle\,|y\rangle = \begin{cases} |x_1,\ldots,x_n\rangle\,|y\rangle & \text{if } \exists i : x_i \neq 1 \\ |x_1,\ldots,x_n\rangle\,U\,|y\rangle & \text{if } \forall i : x_i = 1 \end{cases}$$

Elementary Quantum Gates

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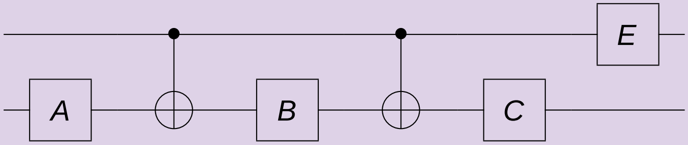
Realizing a singly controlled $\Lambda_1(U)$ gate

$\Lambda_1(U) =$



Description of the local gates used in this circuit

- $U = \exp(i\phi)W$,
- $W \in \mathrm{SU}(2)$,
- $E = \begin{pmatrix} 1 & 0 \\ 0 & \exp(i\phi) \end{pmatrix}$,
- $ABC = I$, $A\sigma_x B\sigma_x C = W$.
- The matrices A,B, and C are obtained from the decomposition $U = (A^\dagger\sigma_x A)(B^\dagger\sigma_x B)$.

Elementary Quantum Gates

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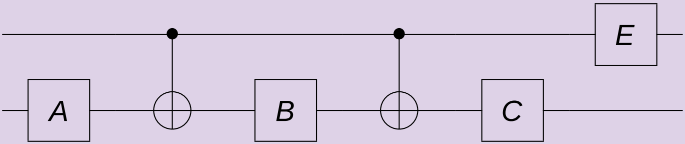
$\Lambda_1(U) =$ 

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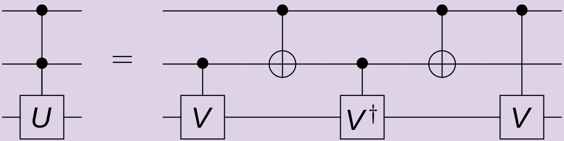
Elementary Quantum Gates

Realizing a two-fold controlled gate $\Lambda_2(U)$



Comments

- Here we have to find a unitary V with $V^2 = U$.
- This idea can be generalized to arbitrary $\Lambda_k(U)$, however, the complexity obtained from this factorization scales exponentially with k.
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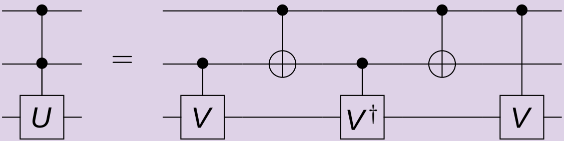

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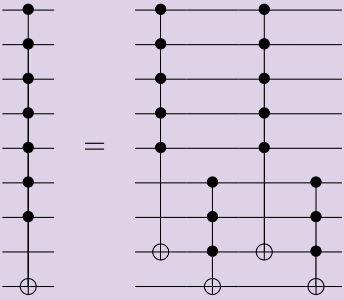
Efficient Break-Down Strategies

$\Lambda_{n-2}(\sigma_x)$



$\Longrightarrow$ linear in n

$\Lambda_{n-1}(W)$ with $W \in \mathrm{SU}(2)$

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Important result about multiply-controlled gates

Let $U \in \mathcal{U}(2)$. Then any $\Lambda_{n-1}(U)$ gate operating on n qubits can be implemented using at most $O(n)$ elementary gates.

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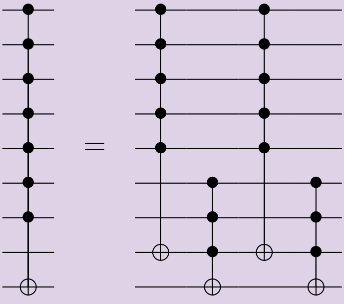

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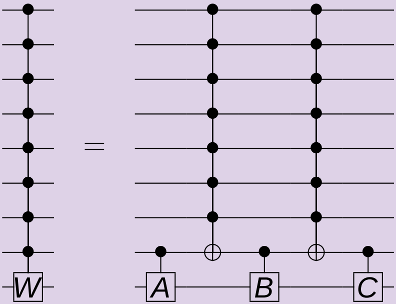

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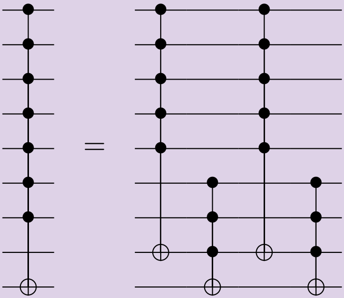

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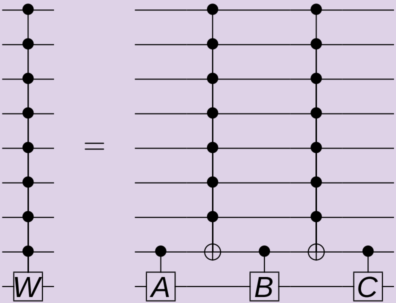

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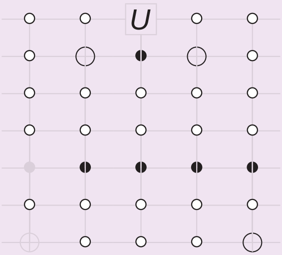
Operations on a two-dimensional subspace

Consider a subspace with basis $|i\rangle$, $|j\rangle$:

- $\Lambda_{n-1}(U)$, if i and j differ in only one bit
- Use Gray code sequence to connect i and j.

Example: $n = 7$, $i = 5$, $j = 100$

0000101
0000100
0100100
1100100



$\Longrightarrow$ Complexity is $O(n^3)$ elementary gates.

Elementary Quantum Gates

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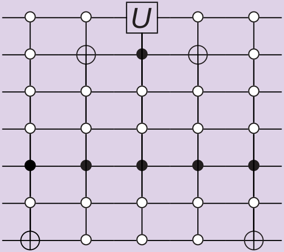

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Saving Even More Gates and the Final Result

The Cybenko trick (2001)

We can save almost all of the permutation gates necessary for the Gray code by using CNOT gates at the different positions:

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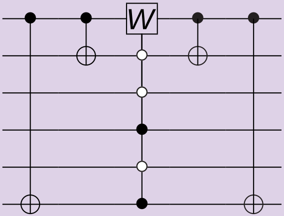
1100101
1100100



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Theorem

Any unitary transformation on n qubits can be implemented using at most 4^n elementary gates. This bound is tight and for almost all elements of $\mathcal{U}(2^n)$ we need $\Theta(4^n)$ gates.

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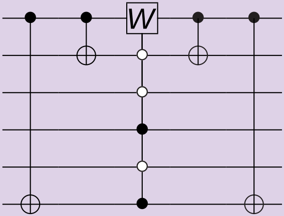

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Conclusions

- Shor's algorithm for factoring integers
  - Reducing factoring to order finding
  - Setting up a periodic state
  - Efficient extraction of the period by computing a QFT
- Grover's algorithm: searching N items in time $O(\sqrt{N})$.
- Elementary quantum gates:
  - Controlled NOT gate
  - Local unitary transformations

Dedicated to Thomas Beth, *16.11.1949, † 17.08.2005