Introduction to Quantum Algorithms
Part I: Quantum Gates and Simon’s Algorithm

Martin Rötteler

NEC Laboratories America, Inc.
4 Independence Way, Suite 200
Princeton, NJ 08540, U.S.A.

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Overview

Today:

- Introduction to qubits, quantum gates, and circuits.
- Appetizer: Two-bit problem where quantum beats classical!
- The power of quantum computing: Simon’s algorithm

Basic principles used:

- Computing in superposition
- Constructive/destructive interference

Tomorrow: Shor’s algorithm and Grover’s algorithm
Quantum–bit (qubit)

A qubit is a normalized state in $\mathbb{C}^2$:

$$\alpha \left| 0 \right\rangle + \beta \left| 1 \right\rangle, \quad \alpha, \beta \in \mathbb{C}, \quad \text{where} \quad |\alpha|^2 + |\beta|^2 = 1.$$  

Quantum register

A quantum register of length $n$ is a collection of qubits $q_1, \ldots, q_n$.

Possible operations / dynamics

A quantum computer can
- perform unitary operations on quantum registers
- measure single qubits
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Bits vs Qubits

One classical bit

is in one of the basis states

\[ \uparrow \equiv \text{High} \equiv 1 \]
\[ \downarrow \equiv \text{Low} \equiv 0 \]

Boolean state space:

\[ \mathbb{F}_2 = \{\uparrow, \downarrow\} = \{0, 1\} \]

Consisting of 2 elements

One quantum bit (qubit)

is in a coherent superposition

\[ |\Psi\rangle = \alpha |\uparrow\rangle + \beta |\downarrow\rangle \]

of the basis states \(|\uparrow\rangle, |\downarrow\rangle\)

Quantum state space \(\mathcal{H}_2 = \mathbb{C}^2:\)

\[ \left\{ \alpha |\uparrow\rangle + \beta |\downarrow\rangle : |\alpha|^2 + |\beta|^2 = 1 \right\} \]

of 2 dimensions
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of 2 dimensions
Two Bits vs Two Qubits

Two classical bits

Two bits can hold either of the 4 states in the state space \( \{0, 1\} \times \{0, 1\} = \{0, 1\}^2 = \{\uparrow\uparrow, \downarrow\uparrow, \uparrow\downarrow, \downarrow\downarrow\} \).

Two coupled qubits

A register of two coupled qubits can hold any of the states

\[ |\Psi\rangle = \alpha |\uparrow\uparrow\rangle + \beta |\downarrow\uparrow\rangle + \gamma |\uparrow\downarrow\rangle + \delta |\downarrow\downarrow\rangle \]

in the state space \( \mathcal{H}_2 \otimes \mathcal{H}_2 = \mathbb{C}^2 \otimes \mathbb{C}^2 \).

Two separate qubits

Two separate qubits can hold any of the product states

\[ |\Psi_1\rangle \otimes |\Psi_2\rangle = (\alpha_1 |\uparrow\rangle + \beta_1 |\downarrow\rangle) \otimes (\alpha_2 |\uparrow\rangle + \beta_2 |\downarrow\rangle) \]

in the state space \( \mathcal{H}_2 \oplus \mathcal{H}_2 \subset \mathbb{C}^2 \oplus \mathbb{C}^2 \).
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Many Bits vs Many Qubits

Classical register of $n$ bits

\[\begin{array}{c}
\text{holds one of the } 2^n \text{ states } \epsilon = \epsilon_1 \cdots \epsilon_n \text{ of the state space } \\
\{0, 1\}^n = \{0, 1\} \times \cdots \times \{0, 1\}.
\end{array}\]

Quantum register of $n$ qubits

\[\begin{array}{c}
\text{can hold any coherent superposition} \\
|\Psi\rangle = \sum_{\epsilon \in \{0, 1\}^n} \alpha_{\epsilon_1 \cdots \epsilon_n} |\epsilon_1\rangle \otimes |\epsilon_2\rangle \otimes \cdots \otimes |\epsilon_n\rangle
\end{array}\]

in the $2^n$ dimensional space $\mathcal{H}_{2^n} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = \mathbb{C}^{2^n}$.

Product states of $n$ qubits

\[\begin{array}{c}
\text{can only hold a product state} \\
|\Psi_1\rangle \otimes |\Psi_2\rangle \otimes \cdots \otimes |\Psi_n\rangle = (\alpha_1 |\uparrow\rangle + \beta_1 |\downarrow\rangle) \otimes \cdots \otimes (\alpha_n |\uparrow\rangle + \beta_n |\downarrow\rangle)
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(thus, only linear scaling of system and dimension)
Many Bits vs Many Qubits

Classical register of \( n \) bits

- Holds one of the \( 2^n \) states \( \epsilon = \epsilon_1 \cdots \epsilon_n \) of the state space \( \{0, 1\}^n = \{0, 1\} \times \cdots \times \{0, 1\} \).

Quantum register of \( n \) qubits

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Our Model of Measurements

von Neumann measurements of one qubit

First, specify a basis $B$ for $\mathbb{C}^2$, e.g. $\{|0\rangle, |1\rangle\}$. The outcome of measuring the state $\alpha |0\rangle + \beta |1\rangle$ is described by a random variable $X$. The probabilities to observe “0” or “1” are given by

$$\Pr(X = 0) = |\alpha|^2, \quad \Pr(X = 1) = |\beta|^2.$$ 

Measuring a state in $\mathbb{C}^n$ in an orthonormal basis $B$

- Recall: Orthonormal Basis of $\mathbb{C}^N$

  $$B = \{|\psi_i\rangle : i = 1, \ldots, N\}, \quad \text{where } \langle \psi_i | \psi_j \rangle = \delta_{i,j}$$

- Let $|\varphi\rangle = \sum_{i=1}^{N} \alpha_i |i\rangle$, where $\sum_{i=1}^{N} |\alpha_i|^2 = 1$. Then measuring $|\varphi\rangle$ in the basis $B$ gives random variable $X_B$ taking values $1, \ldots, N$:

  $$\Pr(X_B = 1) = |\langle \psi_1 | \varphi \rangle|^2, \ldots, \Pr(X_B = N) = |\langle \psi_N | \varphi \rangle|^2.$$
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Two Important Types of Operations

Local operations

\[ 1_N \otimes U = \begin{pmatrix} U & \cdots & \cdots & \cdots \\ \cdots & U & \cdots & \cdots \\ \cdots & \cdots & \cdots & U \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad \text{where} \quad U \in \mathcal{U}(2). \]

Conditioned operation: the controlled NOT (CNOT)

\[ |00\rangle \rightarrow |00\rangle, \quad |01\rangle \rightarrow |01\rangle, \quad |10\rangle \rightarrow |11\rangle, \quad |11\rangle \rightarrow |10\rangle \]

\[ \approx \begin{pmatrix} 1 & \cdots & \cdots & \cdots \\ \cdots & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \approx \]

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\]

\[ \mathbb{1} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}, \quad \text{where} \quad U \in \mathcal{U}(2). \]
Notation for Quantum Gates

Gate in Feynman notation

\[ H_2 \]

Corresponding transformation

\[ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \]

\[ \begin{bmatrix} 1 \\ \ldots \\ u_{11} \ u_{12} \\ \ldots \\ u_{21} \ u_{22} \end{bmatrix} \]
Gate in Feynman notation

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\[ U \]

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\[
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\cdots & \cdots & \cdots \\
\cdots & 1 & \cdots \\
\cdots & \cdots & 1
\end{bmatrix}
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Two Constructions for Matrices

Tensor product

\[ T_1 \otimes T_2 = (T_1 \otimes I)(I \otimes T_2) = \left(\begin{array}{c}
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\[ I \otimes T_2 \quad \text{classically parallel} \]

\[ T_1 \otimes T_2 \quad \text{quantum parallel} \rightleftharpoons \text{independent dynamics} \]

classical: expensive, quantum: easy, since local operations

Direct sum

\[ T_1 \oplus T_2 = (T_1 \oplus I)(I \oplus T_2) = \left(\begin{array}{c}
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Two Constructions for Matrices

**Tensor product**

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The Hadamard Transform

The Hadamard transform on one qubit

\[
H_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \sum_{x, y \in \mathbb{F}_2} (-1)^{xy} |x\rangle \langle y|.
\]

The Hadamard transform on \(n\) qubits

The \(n\)-fold tensor product of \(H_2\) is given by

\[
H_2^\otimes n = \frac{1}{2^{n/2}} \sum_{x, y \in \mathbb{F}_2^n} (-1)^{x \cdot y} |x\rangle \langle y|,
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where \(x \cdot y := \sum_{i=1}^n x_i y_i \in \mathbb{F}_2\).
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The Hadamard Transform

Properties of the Hadamard transform

- By definition $H_2$ is the following map:

\[
|0\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)
\]

\[
|1\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
\]

- Identity involving the Pauli matrices: $H_2\sigma_x H_2 = \sigma_z$

- Identity involving the CNOT gate:
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Quantum Gates and Circuits

Elementary quantum gates

$$U^{(i)} = \begin{array}{c}
\vdots \\
U \\
\vdots \\
\end{array} \quad i$$

$$\text{CNOT}^{(i,j)} = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\bullet \\
\vdots \\
\vdots \\
\end{array} \quad i \quad j$$

Universal set of gates

Theorem (Barenco et al., 1995):

$$\mathcal{U}(2^n) = \langle U^{(i)}, \text{CNOT}^{(i,j)} : i, j = 1, \ldots, n, \ i \neq j \rangle$$

Quantum gates: main problem

Find efficient factorizations for given $$U \in \mathcal{U}(2^n)!$$
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\hline \\
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\hline \\
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Quantum gates: main problem

Find efficient factorizations for given \( U \in \mathcal{U}(2^n) \)!
### Different Levels of Abstraction

#### Unitary matrix

\[ U = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \]

#### Factorized unitary matrix

\[ U = (I \otimes H_2)(I \oplus \sigma_x)(H_2 \otimes I) \]

\[ = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \sigma_x \\ \sigma_x & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]

#### Quantum circuit

![Quantum Circuit](image-url)
Different Levels of Abstraction

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Quantum circuit
Different Levels of Abstraction

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Quantum circuit

[Diagram of a quantum circuit with two Hadamard gates (H)]
Basic Facts About Boolean Functions

Boolean functions

- Any function \( f : \{0, 1\}^n \rightarrow \{0, 1\}^m \), where \( n \) is the number of input bits and \( m \) the number of output bits is said to be a Boolean function.

- Any Boolean function can be represented by a truth table. If \( f = (f_1, \ldots, f_m) \), this is a matrix of size \( 2^n \times m \) where in column \( i \) we have the list of values \( f_i(x_1, \ldots, x_n) \), where \( x_j \in \{0, 1\} \) for \( j = 1, \ldots, n \).

The number of Boolean functions

- There are \( 2^{2^n} \) Boolean functions \( f : \{0, 1\}^n \rightarrow \{0, 1\} \), i.e., functions with \( n \) inputs and one output (since we can assign an arbitrary value for each of the \( 2^n \) inputs).

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### Boolean functions

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- Any Boolean function can be represented by a truth table. If $f = (f_1, \ldots, f_m)$, this is a matrix of size $2^n \times m$ where in column $i$ we have the list of values $f_i(x_1, \ldots, x_n)$, where $x_j \in \{0, 1\}$ for $j = 1, \ldots, n$.

### The number of Boolean functions

- There are $2^{2^n}$ Boolean functions $f : \{0, 1\}^n \rightarrow \{0, 1\}$, i.e., functions with $n$ inputs and one output (since we can assign an arbitrary value for each of the $2^n$ inputs).

- There are $2^{m2^n}$ Boolean functions $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$ with $n$ inputs and $m$ outputs.
### Universal Gates: Classical Computing

#### Connectives

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<th>$x_1 \land \overline{x_2}$</th>
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#### Theorem

Any deterministic classical circuit and thereby any deterministic classical computation can be realized by using NAND gates only. Any probabilistic computation can be realized using NAND gates and in addition one gate which realizes a fair coin flip.

---

**Martin Rötteler, NEC Laboratories America**

**Introduction to Quantum Algorithms**
Universal Gates: Classical Computing

Connectives

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<th>$x_1$</th>
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NOT gate:

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OR gate:

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<tr>
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AND gate:

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NAND gate:

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Finite field of two elements

The set \( \{0, 1\} \) can be equipped with a multiplication “\( \cdot \)” and an addition “\( \oplus \)” such that the field axioms hold.

Truth tables for these operations:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>x ( \cdot ) y</th>
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Then \( (\{0, 1\}, \cdot, \oplus) \) becomes a field consisting of two elements only, also denoted by \( \mathbb{F}_2 \). A finite field with \( n \) elements exists if and only if \( n \) is a prime power.

Important identity: \((-1)^{x \oplus y} = (-1)^x \cdot (-1)^y\)

We can also rewrite CNOT: \(|x\rangle |y\rangle \mapsto |x\rangle |x \oplus y\rangle\)
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\begin{array}{c|c|c}
  x & y & x \cdot y \\
  \hline
  0 & 0 & 0 \\
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
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  \hline
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Basic issue of reversible computing

Suppose, we want to compute a function \( f : \{0, 1\}^n \rightarrow \{0, 1\}^m \) that is not reversible. How can we do this?

One possible solution

Define a new Boolean function which takes \( n + m \) inputs and \( n + m \) outputs as follows:

\[
F(x, y) := (x, y \oplus f(x))
\]

Properties of \( F(x, y) \)

- On the special inputs \((x, 0)\), where \( x \in \{0, 1\}^n \) we obtain that \( F(x, 0) = (x, f(x)) \). Furthermore, \( F \) is reversible.
- Theorem (Bennett): If \( f \) can be computed using \( K \) gates, then \( F \) can be computed using \( 2K + m \) gates.
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The Toffoli gate “TOF”

<table>
<thead>
<tr>
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\[ |x\rangle \quad |y\rangle \quad |z\rangle \quad |z \oplus x \cdot y\rangle \]

Theorem (Toffoli, 1981)

Any reversible computation can be realized by using TOF gates and ancilla (auxiliary) bits which are initialized to 0.
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More features of the reversible embedding $F(x, y)$

- Possibly, there are reversible functions that compute $G(x, 0) = (f(x), \text{junk}(x))$ and use fewer than $n + m$ bits.
- Actually, only $\lceil \log_2(\max c_y) \rceil$ many extra bits are needed to make $f$ reversible, where $c_y = |\{x : f(x) = y\}|$ are the sizes of the collisions of $f$.
- The advantage of $F$ is its uniform definition.

Computing a function by a quantum circuit

Any function $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$ can be computed by means of the reversible function $F : \{0, 1\}^{n+m} \rightarrow \{0, 1\}^{n+m}$. Hence we can compute $f$ also by a quantum circuit

$$U_f |x\rangle |y\rangle = |x\rangle |y \oplus f(x)\rangle.$$
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The Deutsch Jozsa problem (most simple case)

Given a Boolean function $f : \{0, 1\} \rightarrow \{0, 1\}$. Decide whether the property $f(0) = f(1)$ holds or not.

### The four possible functions

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
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### Observation

- From querying the function only on one input, we cannot determine whether $f(0) = f(1)$ with certainty. E.g., if the answer is $f(0) = 0$, it could be the first or third function.
- On the other hand, two queries determine the function completely. What is the quantum query complexity?
A Two-Bit Problem

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The Phase “Kick-Back” Trick

Computing the function into the phase

\[ |x\rangle, |0\rangle - |1\rangle \quad \frac{\sqrt{2}}{U_f} \]

Computing the effect for different inputs

\[ |x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} \quad \rightarrow \quad |x\rangle \left( \frac{|f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} \right) \]

\[ = |x\rangle (-1)^{f(x)} \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \]

\[ = (-1)^{f(x)} |x\rangle \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \]
The Phase “Kick-Back” Trick

Computing the function into the phase

\[ |x\rangle \quad \rightarrow \quad U_f \]

\[ \left| 0 \right\rangle - \left| 1 \right\rangle \quad \rightarrow \quad \sqrt{2} \]

Computing the effect for different inputs

\[ |x\rangle \left( \left| 0 \right\rangle - \left| 1 \right\rangle \right) \quad \mapsto \quad |x\rangle \left( \frac{|f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} \right) \]

\[ = \quad |x\rangle (-1)^{f(x)} \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \]

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The Phase “Kick-Back” Trick

Computing the function into the phase

\[ |x\rangle \quad U_f \quad ? \]
\[ \frac{|0\rangle - |1\rangle}{\sqrt{2}} \quad ? \]

Computing the effect for different inputs

\[ |x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} \rightarrow |x\rangle \left( \frac{|f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} \right) \]
\[ = |x\rangle (-1)^{f(x)} \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \]
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Computing the function into the phase

\[ |x\rangle \quad \rightarrow \quad U_f \quad \rightarrow \quad |x\rangle \]

\[ \frac{|0\rangle - |1\rangle}{\sqrt{2}} \quad \rightarrow \quad \frac{|0\rangle - |1\rangle}{\sqrt{2}} \]

Computing the effect for different inputs

\[ |x\rangle \quad \frac{|0\rangle - |1\rangle}{\sqrt{2}} \quad \rightarrow \quad |x\rangle \left( \frac{|f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} \right) \]

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Martin Rötteler, NEC Laboratories America
Introduction to Quantum Algorithms
The Phase “Kick-Back” Trick

Computing the function into the phase

\[ |x\rangle \quad \xrightarrow{\text{U}_f} \quad (-1)^{f(x)} |x\rangle \]

\[ \frac{|0\rangle - |1\rangle}{\sqrt{2}} \quad \xrightarrow{\text{U}_f} \quad \frac{|0\rangle - |1\rangle}{\sqrt{2}} \]

Computing the effect for different inputs

\[ |x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} \quad \rightarrow \quad |x\rangle \left( \frac{|f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} \right) \]

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The Phase “Kick-Back” Trick

The quantum algorithm

|0⟩ → $H_2$ $U_f$ $H_2$ ?

|1⟩ → $H_2$ $U_f$ $H_2$ ?

Phase kick-back in superposition

$|0⟩ |1⟩ \xrightarrow{H_2 \otimes 2} \left( \frac{|0⟩ + |1⟩}{\sqrt{2}} \right) \left( \frac{|0⟩ - |1⟩}{\sqrt{2}} \right)$

$U_f \xrightarrow{} \frac{1}{\sqrt{2}} (-1)^{f(0)} |0⟩ \left( \frac{|0⟩ - |1⟩}{\sqrt{2}} \right) + \frac{1}{\sqrt{2}} (-1)^{f(1)} |1⟩ \left( \frac{|0⟩ - |1⟩}{\sqrt{2}} \right)$

$= \left( -1 \right)^{f(0)} \frac{1}{\sqrt{2}} \left( |0⟩ + (-1)^{f(0) \oplus f(1)} |1⟩ \right) \left( \frac{|0⟩ - |1⟩}{\sqrt{2}} \right)$

$\xrightarrow{H_2 \otimes 2} \left( -1 \right)^{f(0)} |f(0) \oplus f(1)⟩ |1⟩$
The Phase “Kick-Back” Trick

The quantum algorithm

|0⟩ → $H_2$ → $U_f$ → $H_2$ → ?

|1⟩ → $H_2$ → $H_2$ → ?

Phase kick-back in superposition

\[
|0⟩ |1⟩ \xrightarrow{H_2^\otimes 2} \left( \frac{|0⟩ + |1⟩}{\sqrt{2}} \right) \left( \frac{|0⟩ - |1⟩}{\sqrt{2}} \right)
\]

\[
U_f \mapsto \frac{1}{\sqrt{2}} (-1)^{f(0)} |0⟩ \left( \frac{|0⟩ - |1⟩}{\sqrt{2}} \right) + \frac{1}{\sqrt{2}} (-1)^{f(1)} |1⟩ \left( \frac{|0⟩ - |1⟩}{\sqrt{2}} \right)
\]

\[
= \frac{(-1)^{f(0)}}{\sqrt{2}} \left( |0⟩ + (-1)^{f(0)\oplus f(1)} |1⟩ \right) \left( \frac{|0⟩ - |1⟩}{\sqrt{2}} \right)
\]

\[
H_2^\otimes 2 \mapsto (-1)^{f(0)} |f(0) \oplus f(1)⟩ |1⟩
\]
### The Phase “Kick-Back” Trick

#### The quantum algorithm

\[
\begin{align*}
|0\rangle & \xrightarrow{H_2} H_2 |0\rangle & \xrightarrow{U_f} & H_2 |0\rangle & \xrightarrow{H_2} & \text{?} \\
|1\rangle & \xrightarrow{H_2} H_2 |1\rangle & \xrightarrow{U_f} & H_2 |1\rangle & \xrightarrow{H_2} & \text{?}
\end{align*}
\]

#### Phase kick-back in superposition

\[
\begin{align*}
|0\rangle |1\rangle & \xrightarrow{H_2^\otimes2} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\
U_f & \xrightarrow{1} \frac{1}{\sqrt{2}} (-1)^{f(0)} |0\rangle \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) + \frac{1}{\sqrt{2}} (-1)^{f(1)} |1\rangle \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\
& = \frac{(-1)^{f(0)}}{\sqrt{2}} \left( |0\rangle + (-1)^{f(0) \oplus f(1)} |1\rangle \right) \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\
H_2^\otimes2 & \xrightarrow{1} (-1)^{f(0)} |f(0) \oplus f(1)\rangle |1\rangle
\end{align*}
\]
The Phase "Kick-Back" Trick

The quantum algorithm

\[ |0\rangle \xrightarrow{H_2} H_2 \xrightarrow{U_f} H_2 \xrightarrow{?} ? \]
\[ |1\rangle \xrightarrow{H_2} H_2 \xrightarrow{?} ? \]

Phase kick-back in superposition

\[ |0\rangle |1\rangle \xrightarrow{H_2^\otimes 2} \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \]
\[ U_f \xleftarrow{\frac{1}{\sqrt{2}} (-1)^{f(0)} |0\rangle} \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) + \frac{1}{\sqrt{2}} (-1)^{f(1)} |1\rangle \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \]
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\[ |0\rangle \xrightarrow{H_2} |0\rangle \quad \xrightarrow{U_f} |0\rangle \quad \xrightarrow{H_2} |0\rangle \quad ? \]

\[ |1\rangle \xrightarrow{H_2} |1\rangle \quad \xrightarrow{U_f} |1\rangle \quad \xrightarrow{H_2} |1\rangle \quad ? \]

Phase kick-back in superposition

\[ |0\rangle |1\rangle \xrightarrow{H_2 \otimes 2} \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \]

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\[ \xrightarrow{H_2 \otimes 2} (-1)^{f(0)} |f(0) \oplus f(1)\rangle |1\rangle \]
The Phase “Kick-Back” Trick

The quantum algorithm

\[ |0\rangle \xrightarrow{H_2} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \xrightarrow{U_f} \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \xrightarrow{H_2} |f(0) \oplus f(1)\rangle \]

\[ |1\rangle \xrightarrow{H_2} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \xrightarrow{H_2} |1\rangle \]

Phase kick-back in superposition

\[ |0\rangle |1\rangle \xrightarrow{H_2 \otimes 2} \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \]

\[ U_f \xrightarrow{\frac{1}{\sqrt{2}}(-1)^{f(0)} |0\rangle \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) + \frac{1}{\sqrt{2}}(-1)^{f(1)} |1\rangle \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) } \]

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Hence, we from measuring the first qubit in the computational basis, we obtain the answer \( f(0) \oplus f(1) \) which reveals whether \( f(0) = f(1) \) or not.
The Phase “Kick-Back” Trick

The quantum algorithm

\[ |0\rangle \xrightarrow{H_2} \quad U_f \quad \xrightarrow{H_2} \quad |f(0) \oplus f(1)\rangle \]

\[ |1\rangle \xrightarrow{H_2} \quad \xrightarrow{H_2} \quad |1\rangle \]

Phase kick-back in superposition

\[
|0\rangle |1\rangle \xrightarrow{H_2^\otimes 2} \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\
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H_2^\otimes 2 \xrightarrow{(-1)^{f(0)} |f(0) \oplus f(1)\rangle |1\rangle}
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Hence, we from measuring the first qubit in the computational basis, we obtain the answer \( f(0) \oplus f(1) \) which reveals whether \( f(0) = f(1) \) or not.
What can be computed?

The strong Church-Turing thesis
Any reasonable algorithmic process carried out by a physical machine can be efficiently simulated by a probabilistic Turing machine. The slow-down for this is at most polynomial.

Efficient computations
Given a problem of size $n$, an algorithm is said to have a polynomial running time if the number of steps it needs to find a solution is bounded by $p(n)$, where $p$ is a polynomial.

How do quantum algorithms fit in?
They can solve problems which are believed to be intractable for classical computers. There are physically reasonable algorithmic processes carried which seem to be hard to simulate for any classical probabilistic Turing machine.
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Computational Model: Quantum Circuits

Example: quantum circuit

Quantum circuit and corresponding directed acyclic graph.

Uniform families of quantum circuits

A family \( \mathcal{F} := \{ C_n \in U(2^n) \mid n \in \mathbb{N} \} \) of quantum circuits is called uniform if there exists a polynomial-time deterministic Turing machine which computes \( n \mapsto C_n \), where \( n \) is the problem size.

Theorem (Yao ’93)

Uniform families of quantum circuits and quantum Turing machines (see Bernstein/Vazirani ’93) are equivalent.
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Asymptotics of Functions

Landau notation

- We use Landau notation to compare the asymptotics of two functions $f, g : \mathbb{N} \rightarrow \mathbb{C}$. Typically, $f$ is the running time for an algorithm for input of size $n$ and $g$ is another function.
- $f(n) = O(g(n))$ means that for some $m$ there exists a constant $c > 0$, such that $|f(n)| \leq cg(n)$ for all $n \geq m$.
- $f(n) = \Omega(g(n))$ means that for some $m$ there exists a constant $c > 0$, such that $|f(n)| \geq cg(n)$ for all $n \geq m$.
- $f(n) = \Theta(g(n))$ means that both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ hold.

Example

Let $f(n)$ be number of operations needed to compute a classical FFT of a vector of length $n$. Then $f(n) = O(n \log(n))$. 
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**Example**

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Computational Complexity

Measuring the problem size

- Usually algorithms can use different kinds of resources. Typical examples for resources which can be measured are time, space, depth, and queries.
- The resources needed are always measured as a function of the input size.

Examples

- Consider the problem of multiplying two $n$-bit numbers. Clearly, this can be done in $O(n^2)$ additions and multiplications using the school method. The best known method uses $O(n \log n \log \log n)$ operations.
- The converse problem of factoring an $n$-bit number $N$ into its prime factors is much harder. The currently best known algorithm needs $\exp \left( c \left( \log N \right)^{1/3} \left( \log \log N \right)^{2/3} \right)$ operations.
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Query Problems

Query complexity and black boxes

- Instead of counting the number of operations we can count the number of queries to $f$ in order to solve the problem.
- Often upper and lower bounds can be shown for the query complexity model only.
- Black-box model: we assume that we are given a function $f$ but cannot analyze how $f$ is actually implemented.
- Most real-world problems are actually white-box, for example FACTORING, GRAPH-ISO, etc. Lower bounds in the white-box problems are typically very weak.

Examples

- Black box problems: Simon’s algorithm, Grover’s algorithm.
- White-box problems: Factoring and dlog, phase estimation.
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Examples

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- White-box problems: Factoring and dlog, phase estimation.
Simon’s Problem

The XOR bit mask problem

We consider only functions $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$ for which there exists $s \in \{0, 1\}^n$ such that

- $\forall x \in \{0, 1\}^n$ we have $f(x) = f(x \oplus s)$,
- $\forall x, y \in \{0, 1\}^n$ we have that if $x \neq y \oplus s$, then $f(x) \neq f(y)$.

The task is to find the hidden string $s$.

Example where $f : \{0, 1\}^3 \rightarrow \{0, 1\}^3$

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Here $s = (0, 1, 1)$.
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Martin Rötteler, NEC Laboratories America

Introduction to Quantum Algorithms
Some comments about Simon’s problem

- There is some similarity to the Deutsch-Jozsa problem, however, here the task is not to distinguish between two cases (constant vs balanced) but between \textit{exponentially many} cases (one for each unknown string $s$).

- Problem might seem artificial, but Shor's algorithm has the same underlying idea. \textit{Note:} This is a promise problem!

- Problem gave the first strong (polynomial vs exponential) separation between quantum and classical computing. Before that only \textit{super}-polynomial separations were known.

Literature

- D. R. Simon.
  On the Power of Quantum Computation.
Some comments about Simon’s problem

- There is some similarity to the Deutsch-Jozsa problem, however, here the task is not to distinguish between two cases (constant vs balanced) but between exponentially many cases (one for each unknown string \( s \)).

- Problem might seem artificial, but Shor’s algorithm has the same underlying idea. **Note:** This is a promise problem!

- Problem gave the first strong (polynomial vs exponential) separation between quantum and classical computing. Before that only super-polynomial separations were known.

**Literature**

Simon’s Problem

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D. R. Simon.
On the Power of Quantum Computation.
How the function $f$ is given

We are given $f$ as a black box in form of a quantum circuit computing $U_f$:

$$|x⟩ \rightarrow U_f \rightarrow |x⟩$$

$$|y⟩ \rightarrow U_f \rightarrow |y ⊕ f(x)⟩$$

How can we query $f$?

- Classical algorithm: makes queries $x_1, \ldots, x_k$ resulting in the answers $f(x_1), \ldots, f(x_k)$ (counts as $k$ queries).

- Quantum algorithm: can query in superposition, i.e., starting with the state $|ϕ⟩ = \sum_{i=1}^{k} |x_i⟩$ results in

$$U_f |ϕ⟩ |0⟩ = \sum_{i=1}^{k} U_f |x_i⟩ |0⟩ = \sum_{i=1}^{k} |x_i⟩ |f(x_i)⟩.$$ 

This counts as one query!
Simon’s Problem: Specification

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\[
\begin{align*}
|x\rangle & \quad \quad \quad \quad |x\rangle \\
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Partial measurements / “non-demolition” measurements

Suppose we are given a Boolean function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) and have prepared the state

\[
|\psi\rangle = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle |f(x)\rangle.
\]

\[
= \frac{1}{2^{n/2}} \left( \sum_{x : f(x) = 0} \alpha_x |x\rangle |0\rangle + \sum_{x : f(x) = 1} \alpha_x |x\rangle |1\rangle \right).
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Measuring the last qubit gives a random variable \( X \).

\[
\Pr(X = 0) = \sum_{x : f(x) = 0} |\alpha_x|^2, \quad \Pr(X = 1) = \sum_{x : f(x) = 1} |\alpha_x|^2.
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This talk: We only consider von Neumann measurements of some (or all) of the qubits.
Measuring the State of the System

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We can still work with the collapsed state! For instance if measuring the last qubit of

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yields the result \( X = "1" \), then the collapsed state is

\[ |\psi_1\rangle = \frac{1}{\sqrt{s_1}} \sum_{x : f(x) = 1} \alpha_x |x\rangle , \quad \text{where } s_1 = |\{ x : f(x) = 1 \}|. \]

- Note: cannot be used to find solutions to \( f(x) = 1 \). Why?
- Further reading: the most general operation which can be applied to a quantum system in order to obtain classical information is a POVM (positive operator valued measure).
Measuring the State of the System

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Simon’s Problem: Preparing Useful States

Creating the uniform superposition

The basic idea is to prepare

\[ |0\rangle |0\rangle \xrightarrow{H^\otimes n} \frac{1}{\sqrt{2^n}} \sum_{x \in \mathbb{F}_2^n} |x\rangle |0\rangle \]

\[ U_f \xrightarrow{} \frac{1}{\sqrt{2^n}} \sum_{x \in \mathbb{F}_2^n} |x\rangle |f(x)\rangle. \]

Collapsing the uniform superposition

Now, measuring the second register will yield a random \( s \in \mathbb{F}_2^n \) in the image of \( f \). The state collapses to

\[ |\varphi_{x_0,s}\rangle = \frac{1}{\sqrt{2}} \left( |x_0\rangle + |x_0 \oplus s\rangle \right). \]
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Computing the Hadamard Transform

We apply $H_2^\otimes n$ to the collapsed states $|\varphi_{x_0, s}\rangle$:

$$H_2^\otimes n |\varphi_{x_0, s}\rangle = \left(\frac{1}{2^{n/2}} \sum_{x, y \in F_2^n} (-1)^{x \cdot y} |x\rangle \langle y|\right) \left(\frac{1}{\sqrt{2}} (|x_0\rangle + |x_0 \oplus s\rangle)\right)$$

$$= \frac{1}{\sqrt{2^{n+1}}} \left(\sum_{x \in F_2^n} (-1)^{x \cdot x_0} |x\rangle + \sum_{x \in F_2^n} (-1)^{x \cdot (x_0 \oplus s)} |x\rangle\right)$$

$$= \frac{1}{\sqrt{2^{n+1}}} \sum_{x \in F_2^n} ((-1)^{x \cdot x_0} |x\rangle + (-1)^{x \cdot (x_0 \oplus x \cdot s)}) |x\rangle$$

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$$= \frac{1}{\sqrt{2^{n+1}}} \sum_{x \in F_2^n} (-1)^{x \cdot x_0} (1 + (-1)^{x \cdot s}) |x\rangle$$

Martin Rötteler, NEC Laboratories America

Introduction to Quantum Algorithms
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We apply $H_2^{\otimes n}$ to the collapsed states $|\varphi_{x_0,s}\rangle$:

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Simon’s Problem: Destructive Interference

What have we gained by doing this?

\[ H_2^\otimes n |\varphi_{x_0,s}\rangle = \frac{1}{\sqrt{2^{n+1}}} \sum_{x \in F_2^n} (-1)^{x \cdot x_0} (1 + (-1)^{x \cdot s}) |x\rangle \]

\[ = \frac{1}{\sqrt{2^{n+1}}} \sum_{\substack{x \in F_2^n \\ x \cdot s = 0}} (-1)^{x \cdot x_0} |x\rangle \]

Hence measuring this state yields a random element in

\[ \langle s \rangle^\perp = \{ x \in F_2^n | x \cdot s = 0 \}. \]

What we really want...

... are elements from \( \langle s \rangle \) itself (there is only 0 and \( s \) itself since \( \langle s \rangle \) is one-dimensional). How can we compute \( \langle s \rangle \) from \( \langle s \rangle^\perp \)?
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Solution to Simon’s Problem

Quantum algorithm

Given: Function \( f : \{0, 1\}^n \rightarrow \{0, 1\}^n \) with Simon promise, i.e., preimages of a fixed image have the form \( x_0 \) and \( x_0 \oplus s \).

Task: Find the unknown bit-string \( s \in \mathbb{F}_2^n \).

Repeat the following steps \( n - 1 \) times:

1. Initialize two quantum registers:
   \[ |0\rangle |0\rangle \]

2. Equal distribution on first register:
   \[ \frac{1}{\sqrt{2^n}} \sum_{x \in \{0, 1\}^n} |x\rangle |0\rangle \]

3. Compute \( f \) in superposition:
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4. Measure second register:
   \[ \frac{1}{\sqrt{2}} (|x_0\rangle + |x_0 \oplus s\rangle) = |\varphi_{x_0, s}\rangle \]

5. Compute \( H_2^\otimes n \) on first register:
   \[ \frac{1}{\sqrt{2^{n+1}}} \sum_{x \in \mathbb{F}_2^n} (-1)^{x \cdot x_0} |x\rangle \]

6. Measure first register:
   Sample \( y \in \mathbb{F}_2^n \) with \( y \cdot s = 0 \).

Further classical post-processing is necessary!
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   \[
   \frac{1}{\sqrt{2^{n+1}}} \sum_{x \in \mathbb{F}_2^n} (-1)^{x \cdot x_0} |x\rangle
   \]
6. Measure first register: Sample $y \in \mathbb{F}_2^n$ with $y \cdot s = 0$.

Further classical post-processing is necessary!
Solution to Simon’s Problem

Quantum algorithm

Given: Function \( f : \{0, 1\}^n \rightarrow \{0, 1\}^n \) with Simon promise, i.e., preimages of a fixed image have the form \( x_0 \) and \( x_0 \oplus s \).

Task: Find the unknown bit-string \( s \in \mathbb{F}_2^n \).

Repeat the following steps \( n - 1 \) times:

1. Initialize two quantum registers: \( |0\rangle |0\rangle \)

2. Equal distribution on first register:
\[
\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |0\rangle
\]

3. Compute \( f \) in superposition:
\[
\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |f(x)\rangle
\]

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\[
\frac{1}{\sqrt{2}} (|x_0\rangle + |x_0 \oplus s\rangle) = |\varphi_{x_0,s}\rangle
\]

5. Compute \( H_2^{\otimes n} \) on first register:
\[
\frac{1}{\sqrt{2^{n+1}}} \sum_{\substack{x \in \mathbb{F}_2^n \\text{ s.t. } \ x \cdot s = 0}} (-1)^{x \cdot x_0} |x\rangle
\]

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Martin Rötteler, NEC Laboratories America

Introduction to Quantum Algorithms
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Further classical post-processing is necessary!
Classical post-processing

- After \( n - 1 \) iterations: \( y_1, \ldots, y_{n-1} \in F_2^n \) with \( y_i \cdot s = 0 \).
- We have to infer \( s \) by a purely classical computation.
- Show high probability of success over the choice of \( y_i \).

Linear algebra over \( F_2 \)

We are given the linear system of equations

\[
A \cdot s = \begin{pmatrix}
y_1 \\
\vdots \\
y_{n-1}
\end{pmatrix} \cdot s = 0
\]

Hence, we have to compute the kernel of \( A \in F_2^{(n-1) \times n} \). If the kernel is one-dimensional, then \( s \) is uniquely determined.
Classical Postprocessing in Simon’s Algorithm

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The probability of success

Since $\dim(\langle s \rangle) = 1$ we have that $\dim(\langle s \rangle^\perp) = n - 1$. Hence we have to bound the probability that $n - 1$ random vectors in $F_2^n$ are linear independent:

$$
\Pr(rk(A) = n-1) = \left(\frac{2^{n-1} - 1}{2^{n-1}}\right)\left(\frac{2^{n-1} - 2}{2^{n-1}}\right) \cdots \left(\frac{2^{n-1} - 2^{n-2}}{2^{n-1}}\right)
$$

$$
= \left(1 - \frac{1}{2^{n-1}}\right)\left(1 - \frac{1}{2^{n-2}}\right) \cdots \left(1 - \frac{1}{4}\right) \cdot \frac{1}{2}
$$

$$
\geq \left(1 - \left(\frac{1}{2^{n-1}} + \cdots + \frac{1}{4}\right)\right)\frac{1}{2} \geq \left(1 - \frac{1}{2}\right) \cdot \frac{1}{2} \geq \frac{1}{4}
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Complexity of the quantum algorithm

- We have used $n$ iterations and each individual run uses one query, $2n$ Hadamard transforms, and $n + 1$ single qubit measurements.

- Postprocessing: Computing the kernel of a matrix of size $n \times n$ is linear algebra and can be solved in time $O(n^3)$.

- Overall we have found a polynomial-time quantum algorithm.
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Simon’s Problem: Classical Lower Bound

Theorem
Let $A$ be a classical probabilistic algorithm which determines $s$ using $k$ queries to the function $f$. Then $k = \Omega(2^{n/2})$.

Computational complexity theory lingo
Hence there exists an oracle $\mathcal{O}$ with respect which we have a separation between classical and quantum computation:

$$\text{BPP}^\mathcal{O} \neq \text{BQP}^\mathcal{O}$$

- This is a so-called “relativized result”, i.e., it holds in a world in which calls to the oracle cost only one query.
- Whether this also holds for our world, i.e., without oracles, is a major open problem in theoretical computer science.
- Looking ahead: Why does the fact that FACTORING is in BQP, whereas no classical algorithm is known for it, does not imply that BPP $\neq$ BQP?
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\[
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Suppose \( A \) makes \( k \) queries \( x_1, \ldots, x_k \), where \( x_i \neq x_j \). Let \( \mathcal{F}_k := \{ f(x_i) : i = 1, \ldots, k \} \) and \( \mathcal{E}_k := \{ x_i \oplus x_j : i \neq j \} \).

- If \( |\mathcal{F}_k| < k \) then we have found a collision, i.e. a pair \((i_0, j_0)\) with \( f(x_{i_0}) = f(x_{j_0}) \). Then \( s = x_{i_0} \oplus x_{j_0} \).
- Suppose there was no collision. Then \( s \notin \mathcal{E}_k \) and \( |\mathcal{E}_k| = \binom{k}{2} \) candidates have been eliminated.
- However, there are \( 2^n - 1 - \binom{k}{2} \) candidates for \( s \). We show that they are equally likely for a given \( \mathcal{F}_k \). Then \( k = \Omega(2^n) \).
- Bayes rule:

\[
\Pr(s = s_0 | \mathcal{F}_k = k) = \frac{\Pr(|\mathcal{F}_k| = k | s = s_0) \cdot \Pr(s = s_0)}{\Pr(|\mathcal{F}_k| = k)}.
\]

The a priori probabilities \( \Pr(s = s_0) \) are equal and by symmetry \( \Pr(|\mathcal{F}_k| = k | s = s_0) \) is independent of \( s_0 \).
Sketch of proof of the $\Omega(2^{n/2})$ lower bound

Suppose $A$ makes $k$ queries $x_1, \ldots, x_k$, where $x_i \neq x_j$. Let $\mathcal{F}_k := \{f(x_i) : i = 1, \ldots, k\}$ and $\mathcal{E}_k := \{x_i \oplus x_j : i \neq j\}$.

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Suppose there was no collision. Then $s \neq \mathcal{E}_k$ and $|\mathcal{E}_k| = \left(\begin{array}{c} k \\ 2 \end{array}\right)$ candidates have been eliminated.

However, there are $2^n - 1 - \left(\begin{array}{c} k \\ 2 \end{array}\right)$ candidates for $s$. We show that they are equally likely for a given $\mathcal{F}_k$. Then $k = \Omega(2^n)$.

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$$\Pr(s = s_0 \mid |\mathcal{F}|_k = k) = \frac{\Pr(|\mathcal{F}_k| = k \mid s = s_0) \cdot \Pr(s = s_0)}{\Pr(|\mathcal{F}_k| = k)}.$$

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Martin Rötteler, NEC Laboratories America
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Martin Rötteler, NEC Laboratories America  Introduction to Quantum Algorithms
Sketch of proof of the $\Omega(2^{n/2})$ lower bound

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Conclusions

- Elementary quantum gates:
  - Controlled NOT gate
  - Local unitary transformations

- A simple quantum algorithm on two qubits which distinguishes constant from balanced functions.

- Separation: 1 query (quantum) vs 2 queries (classical)

- Quantum algorithm for Simon’s problem based on:
  - Computing with superpositions
  - Interference of computational paths
Introduction to Quantum Algorithms
Part II: The Algorithms of Shor and Grover

Martin Rötteler

NEC Laboratories America, Inc.
4 Independence Way, Suite 200
Princeton, NJ 08540, U.S.A.

International Summer School on Quantum Information,
Max-Planck-Institut für Physik komplexer Systeme
Dresden, September 1, 2005
Overview

Today:
- Shor’s algorithm
  - Modular exponentiation
  - Period extraction via Quantum Fourier Transform
  - Classical post-processing
- Generalizations of Shor’s algorithm
- Grover’s algorithm for searching a list
- Universal quantum gates

Outlook:
- On September 19, Markus Grassl will continue with an introduction to quantum error-correcting codes.
The Integer Factorization Problem

Basic problem

Given a natural number $N$. Find a (prime) factor of $N$.

Best known classical algorithm

The number field sieve has a complexity of

$$\exp \left( (1.923 + o(1))(\log N)^{1/3}(\log \log N)^{2/3} \right)$$

which is (sub)exponential in the number $n = \log N$ of bits of $N$.

Making money with factoring

The company RSA offers $200,000 for anybody who can factor a certain 617 digit number $N$. This number is known to be of the form $N = pq$ but finding $p$ and $q$ is infeasible using the best known classical algorithms.
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RSA-200 is factored!

In May 2005 a small team of people has factored RSA-200. At 663 bits, this is the largest RSA Challenge Number factored. The classical effort undertaken

Sieving equivalent of 55 years on a single 2.2 GHz Opteron CPU. The matrix step took about 3 months on a cluster of 80 2.2 GHz Opterons. Computed from late 2003 to May 2005.

RSA-200 and its factors

\[ N = 27997833911221327870829467638722601621070446786955 \]
\[ \quad + 42853756000992932612840010760934567105295536085606 \]
\[ \quad + 18223519109513657886371059544820065767750985805576 \]
\[ \quad + 13579098734950144178863178946295187237869221823983 \]
\[ p = 35324619344027701212726049781984643686711974001976 \]
\[ \quad + 25023649303468776121253679423200058547956528088349 \]
\[ q = 792586995447833303334708584148005968773797758573642 \]
\[ \quad + 19960734330341455767872818152135381409304740185467 \]
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RSA-200 and its factors

\[ N = \begin{array}{c}
27997833911221327870829467638722601621070446786955 \\
42853756000992932612840010760934567105295536085606 \\
18223519109513657886371059544820065767750985805576 \\
13579098734950144178863178946295187237869221823983
\end{array} \]

\[ p = \begin{array}{c}
35324619344027701212726049781984643686711974001976 \\
25023649303468776121253679423200058547956528088349
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79258699544783330333470858414800596877379758573642 \\
19960734330341455767872818152135381409304740185467
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The RSA Factoring Challenge

**RSA-200 is factored!**

In May 2005 a small team of people has factored RSA-200. At 663 bits, this is the largest RSA Challenge Number factored.

**The classical effort undertaken**

Sieving equivalent of 55 years on a single 2.2 GHz Opteron CPU. The matrix step took about 3 months on a cluster of 80 2.2 GHz Opterons. Computed from late 2003 to May 2005.

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Reformulating the factoring problem

We can factor $N$ if the following problem can be solved:

- **Input**: A number $a$ with $1 < a < N$.
- **Output**: The order $r$ of $a$ modulo $N$, i.e., the smallest integer $r > 0$ such that $a^r \equiv 1 \pmod{N}$.

Why is this a reduction?

Suppose we want to find a divisor of $N$ different from $+1$ or $-1$.

- Pick a random $a$ with $1 < a < N$ and find its order $r$.
- Suppose that $r$ is even (happens with high probability):
  \[
  0 = (a^r - 1) = (a^{r/2} - 1)(a^{r/2} + 1) \pmod{N}.
  \]
- If $a^{r/2} \neq \pm 1$ then $\gcd(a^{r/2} - 1, N)$ and $\gcd(a^{r/2} + 1, N)$ yield at least one nontrivial divisor of $N$. 

Shor’s Algorithm: Reduction to Order Finding

Martin Rötteler, NEC Laboratories America  
Introduction to Quantum Algorithms
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Introduction to Quantum Algorithms
Remarks about this reduction

- Note that gcd$(a, b)$ of two $n$-bit integers can be computed in $\text{poly}(\log(n))$ time.
- There were two events in which the reduction fails: (i) we pick $a$ with an odd order $r$ and (ii) $a^{r/2} = \pm 1$. We have to bound the probability for one of these events to occur.

Theorem

Let $N = p_1^{\mu_1} \ldots p_m^{\mu_m}$ with $m \geq 2$ and $p_i > 2$. Then

$$\Pr(\text{neither (i) nor (ii) occurs}) \geq 1 - \frac{1}{2^m}$$

The big question

How can we efficiently determine the multiplicative order of a random element $a$ modulo $N$?
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The big question

How can we efficiently determine the multiplicative order of a random element $a$ modulo $N$?
Defining a Period Function

The modular exponentiation map

Let \( N \) be an integer and let \( a \in \mathbb{Z}_N \).

- Let \( M \) be an integer. The modular exponentiation is the map \( f : x \mapsto (a^x \mod N) \) from \( \mathbb{Z}_M \) to \( \mathbb{Z}_N \).

Result: The map \( f \) can be implemented efficiently using standard arithmetic in \( O(poly(\log N)) \) operations.

- Hence also the map \( U_f : |x\rangle |y\rangle \mapsto |x\rangle |a^x \mod N\rangle \) can be implemented efficiently.

- Recall that the order of \( a \) is defined as the smallest integer \( r \) such that \( a^r = 1 \mod N \).

Observation

The function \( f : x \mapsto (a^x \mod N) \) is periodic and has period length \( r \), i.e., \( f(x) = f(x + r) \) for all inputs \( x \).
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Introduction to Quantum Algorithms
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Setting up a Periodic State

Observation

The function $f : x \mapsto a^x \mod N$ is periodic and has period length $r$, i.e., $f(x) = f(x + r)$ for all inputs $x$.

The graph of the function $f(x) = 2^x \mod 165$

$|y = f(x)\rangle$

$|x\rangle$
Creating the graph of $f$

Let $f(x) = a^x \mod N$ be the modular exponentiation and let $U_f : |x\rangle |y\rangle \mapsto |x\rangle |y \oplus f(x)\rangle$ be as usual. We compute (letting $M = 2^m$)

$$|0\rangle |0\rangle \xrightarrow{H_2^\otimes m} \frac{1}{\sqrt{M}} \sum_{x \in \mathbb{Z}_M} |x\rangle |0\rangle \xrightarrow{U_f} \frac{1}{\sqrt{M}} \sum_{x \in \mathbb{Z}_M} |x\rangle |f(x)\rangle.$$

Collapsing the graph of $f$

Now, measuring the second register will yield a random $s \in \mathbb{Z}_N$ in the image of $f$. The state collapses to

$$\frac{1}{\sqrt{N/r}} \sum_{k=0}^{N/r-1} |x_0 + k \cdot r\rangle.$$
Shor’s Problem: Preparing Useful States

Creating the graph of $f$

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Introduction to Quantum Algorithms
An Application of the DFT: Period Extraction

Motivation

We would like to apply the trick from Simon’s algorithm:

$$|\varphi_{x_0,s}\rangle = \frac{1}{\sqrt{2}}(|x_0\rangle + |x_0 \oplus s\rangle) \mapsto \frac{1}{\sqrt{2^{n+1}}} \sum_{x \in \mathbb{F}_2^n \atop x \cdot s = 0} (-1)^{x \cdot x_0} |x\rangle.$$ 

The unknown offset $x_0$ is transferred into the phases.

The analogue of $|\varphi_{x_0,s}\rangle$ in case of the cyclic group $\mathbb{Z}_N$ is

$$|\psi_{x_0,r}\rangle = \frac{1}{\sqrt{N/r}} \sum_{k=0}^{N/r-1} |x_0 + k \cdot r\rangle.$$ 

Again, we would like to transfer $x_0$ into the phases.
An Application of the DFT: Period Extraction

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- Again, we would like to transfer \(x_0\) into the phases.
The Discrete Fourier Transformation (DFT)

**Definition of the DFT**

\[
\text{DFT}_N := \frac{1}{\sqrt{N}} \left[ \omega_{N}^{k\cdot\ell} \right]_{k,\ell=0...N-1}, \quad \omega_{N} = e^{2\pi i/N}
\]

**Example**

![Image of Fourier Transform](image-url)
Useful trick in quantum computing (Character Lemma)

Lemma: For all $i = 0, \ldots, n - 1$ the following holds:

$$
\sum_{j=0}^{n-1} \omega_n^{ij} = n \cdot \delta_{i,0}
$$

Proof: Let $S := \sum_{j=0}^{n-1} \omega_n^{ij}$. Then

$$
\omega_n^i S = \sum_{j=0}^{n-1} \omega_n^i \omega_n^{ij} = \sum_{j=0}^{n-1} \omega_n^{i(j+1)} = \sum_{j=0}^{n-1} \omega_n^{ij} = S
$$

- If $i \neq 0$ then $\omega_n^i \neq 1$, i.e., $(1 - \omega_n^i) \neq 0$. Hence $S = 0$.
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Introduction to Quantum Algorithms
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Extracting the Period of a Function

Theorem (Fourier duality)

Let $N \in \mathbb{N}$ and let $r \in \mathbb{Z}_N$ be a divisor of $N$, and let $x_0 \in \mathbb{Z}_N$. Then

$$\text{DFT}_N |\psi_{x_0,r}\rangle = \text{DFT}_N \left( \frac{1}{\sqrt{N/r}} \sum_{k=0}^{N/r-1} |x_0 + k \cdot r\rangle \right) = \frac{1}{\sqrt{r}} \sum_{\ell=0}^{r-1} \omega_{N/r}^{\ell x_0} N \left| \ell \frac{N}{r} \right\rangle$$

Proof:

$$\text{DFT}_N |\psi_{x_0,r}\rangle = \left( \frac{1}{\sqrt{N}} \sum_{i,j=0}^{N-1} \omega_N^{ij} |i\rangle \langle j| \right) \left( \frac{1}{\sqrt{N/r}} \sum_{k=0}^{N/r-1} |x_0 + k \cdot r\rangle \right)$$

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Proof:

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$$= : \alpha_i :$$
Constructive / Destructive Interference

Computing the coefficients $\alpha_i$

For each $i = 0, \ldots, N - 1$ we have to compute $\alpha_i = \sum_{k=0}^{N/r-1} \omega_N^{ikr}$. 

- Case 1: $i = \frac{N}{r}\ell$ for some $\ell = 0, \ldots, r - 1$. Then

$$\alpha_i = \sum_{k=0}^{N/r-1} \omega_N^{\ell kr} = \sum_{k=0}^{N/r-1} 1 = \frac{N}{r}.$$ 

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by the Character Lemma.
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Constructive / Destructive Interference

End of proof (Fourier duality)

\[
\text{DFT}_N |\psi_{x_0, r}\rangle = \frac{\sqrt{r}}{N} \sum_{i=0}^{N-1} \omega_N^{ix_0} \left( \sum_{k=0}^{N/r-1} \omega_N^{ikr} \right) |i\rangle = \frac{\sqrt{r}}{N} \sum_{i=0}^{N-1} \omega_N^{ix_0} \alpha_i |i\rangle
\]

What happens if \( r \) does not divide \( N \)?

In this case the state can be approximated very accurately by

\[
\text{DFT}_N \left( \sum_k |x_0 + k \cdot r\rangle \right) \approx \sum_k \omega_N^{\ell \mu x_0} |\ell \mu\rangle
\]

with an element \( \mu \in \mathbb{Z}_N \) such that \( \mu r \approx N \).
Constructive / Destructive Interference

End of proof (Fourier duality)

\[ \text{DFT}_N |\psi_{x_0,r}\rangle = \frac{\sqrt{r}}{N} \sum_{i=0}^{N-1} \omega^{ix_0} \left( \sum_{k=0}^{N/r-1} \omega^{ikr} \right) |i\rangle = \frac{\sqrt{r}}{N} \sum_{i=0}^{N-1} \omega^{ix_0} \alpha_i |i\rangle \]

\[ = \frac{\sqrt{r}}{N} \sum_{\ell=0}^{r-1} \omega^{\frac{N}{r} x_0 N} \frac{N}{r} |\ell \frac{N}{r}\rangle = \frac{1}{\sqrt{r}} \sum_{\ell=0}^{r-1} \omega^{\frac{N}{r} x_0 N} |\ell \frac{N}{r}\rangle \]

What happens if \( r \) does not divide \( N \)?

In this case the state can be approximated very accurately by

\[ \text{DFT}_N \left( \sum_k |x_0 + k \cdot r\rangle \right) \approx \sum_k \omega^{\ell \mu x_0} |\ell \mu\rangle \]

with an element \( \mu \in \mathbb{Z}_N \) such that \( \mu r \approx N \).
Constructive / Destructive Interference

End of proof (Fourier duality)

\[
\text{DFT}_N |\psi_{x_0,r}\rangle = \sqrt{r} \frac{N}{N-1} \sum_{i=0}^{i=N-1} \omega^{ix_0} \left( \sum_{k=0}^{k=N-1} \omega^{ikr} \right) |i\rangle = \sqrt{r} \frac{N}{N} \sum_{i=0}^{i=N-1} \omega^{ix_0} \alpha_i |i\rangle
\]

\[
= \sqrt{r} \frac{N}{N} \sum_{\ell=0}^{\ell=r-1} \omega^{\ell/N} x_0 \frac{N}{r} \left| \ell \frac{N}{r} \right\rangle = \frac{1}{\sqrt{r}} \sum_{\ell=0}^{\ell=r-1} \omega^{\ell/N} x_0 \left| \ell \frac{N}{r} \right\rangle
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**Solution to the Period Extraction Problem**

**Quantum algorithm**

Given: Modular exponentiation function $U_a : |x\rangle |0\rangle \mapsto |x\rangle |a^x \mod N\rangle$.

Task: Find the order $r$ of $a$ modulo $N$.

Repeat the following steps one time: (w/o normalizations, $M = 2^m >> N$)

1. Initialize two quantum registers: $|0\rangle |0\rangle$

2. Equal distribution on first register: $\sum_{x=0}^{M-1} |x\rangle |0\rangle$

3. Compute $f$ in superposition: $\sum_{x=0}^{M-1} |x\rangle |a^x \mod N\rangle$

4. Measure second register: $\sum_{k=0}^{M/r-1} |x_0 + k \cdot r\rangle$

5. Compute DFT$_M$ on first register: $\approx \sum_{\ell=0}^{r-1} \omega_M^{\ell N} x_0 \left| \ell N \right>^r$

6. Measure first register: Sample a rational number $\frac{p}{q}$ which is very close to $\frac{\ell_0}{r}$.

How can we classically reconstruct $r$ from $\frac{p}{q}$?
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Given: Modular exponentiation function \( U_a : |x\rangle |0\rangle \mapsto |x\rangle |a^x \text{ mod } N\rangle \).

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$$\left| \frac{p}{q} - \frac{\ell_0}{r} \right| < \frac{1}{2r^2}$$

Diophantine approximation

We apply the continued fractions algorithm to $\frac{p}{q}$. This will lead to several principal fractions and actually $\frac{\ell_0}{r}$ will be one of them. Note that we can check whether a candidate $r$ is indeed the order.

Theorem (Shor ’94)

\[ \text{FACTORIZING} \in \text{BQP}. \]
Classical Post-Processing

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Introduction to Quantum Algorithms
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Introduction to Quantum Algorithms
The continued fractions algorithm

Input: $x \in \mathbb{R}$. Output: Sequence of $b_i \in \mathbb{Z}$ (possibly infinite) which represents $x$. Let $b_0 := \lfloor x \rfloor$, $x_1 := \frac{1}{x - b_0}$, $b_1 := \lfloor x_1 \rfloor$, $x_2 := \frac{1}{x_1 - b_1}$, \ldots. Then

$$x = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \ldots}}$$

Example with rational input

Suppose that $x = \frac{5021264471}{8589934592}$. The algorithm results in (vector of $b_i$’s):

$$[1, 1, 2, 2, 5, 3, 1, 1, 3, 1, 11, 1, 1, 21, 1, 2, 5, 1, 1, 1, 1, 1, 2, 1, 2, 1, 3]$$

Consider the convergents $C_n$ which are obtained by truncating after $n$ steps:

\[
\begin{array}{|c|cccccccccc|}
\hline
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
C_n & 1 & \frac{1}{2} & \frac{3}{5} & \frac{7}{12} & \frac{38}{65} & \frac{121}{207} & \frac{159}{272} & \frac{280}{479} & \frac{999}{1709} & \frac{1279}{2188} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|cccccc|}
\hline
n & 11 & 12 & 13 & 14 & \ldots & 27 \\
\hline
C_n & \frac{15608}{25777} & \frac{16347}{27965} & \frac{31415}{53742} & \frac{676062}{1156547} & \ldots & \frac{5021264471}{8589934592} \\
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Diophantine Approximation

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<table>
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<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</tr>
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<tbody>
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<td>$C_n$</td>
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Martin Rötteler, NEC Laboratories America
Lagrange’s Theorem

Let \( x \in \mathbb{Q} \) and assume that we are given \( \frac{p}{q} \in \mathbb{Q} \) such that

\[
\left| x - \frac{p}{q} \right| \leq \frac{1}{2q^2}
\]

Then \( x \) is a convergent \( C_n \) of \( \frac{p}{q} \), namely that for which \( |C_n - \frac{p}{q}| < \frac{1}{2q^2} \).

Example (cont’d)

Let \( y = \frac{31415}{53742} \) be the inverse period. Since denominator of \( y \) is \( \leq 2^{16} \) we can work with precision \( \leq 2^{33} \). Suppose we measure \( \frac{p}{q} = \frac{5021264471}{8589934592} \):

\[
\begin{array}{cccccccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
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 & & & 25777 & 27965 & 53742 & 1156547 & \cdots & 8589934592 \\
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Parallels between DSP and QC

Digital Signal Processing
- Signal $f(x)$
- Sampling
- Transform
- Result

Quantum Computing
- Quantum information
- Modelling
- Unitary transform & measurement
- Probabilities

$f = \sum_x f(x) \delta_x$
Example: Fast Fourier Transform (FFT)

Cooley-Tukey FFT

The matrix \( \text{DFT}_N = \frac{1}{\sqrt{N}} \left[ \omega_N^{k\cdot\ell} \right]_{k,\ell=0\ldots N-1} \), where \( \omega_N = e^{2\pi i/N} \), can be written as a short product of sparse matrices.

\[
\text{DFT}_4 = \Pi_{\text{rev}} \cdot (1_2 \otimes \text{DFT}_2) \cdot \text{diag} \cdot (\text{DFT}_2 \otimes 1_2)
\]

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & i & -1 & -i \\
1 & -i & 1 & -1 \\
i & -1 & -1 & i
\end{bmatrix} = 
\begin{bmatrix}
1 & 1 \\
1 & -1 \\
i & -1 \\
i & -i
\end{bmatrix} \cdot 
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
i & -1 & 1 & -1 \\
i & -i & -1 & i
\end{bmatrix}
\]

FFT Theorem

Multiplication with \( \text{DFT}_N \) can be performed classically in \( O(N \log N) \) elementary operations.

We can do much better on a quantum computer!

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Introduction to Quantum Algorithms
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1 & 1
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$$DFT_4 = \prod_{\text{rev}} \cdot (1_2 \otimes DFT_2) \cdot \text{diag} \cdot (DFT_2 \otimes 1_2)$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ i \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

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We can do much better on a quantum computer!
Fast Fourier Transform

Cooley-Tukey Formula

$$\Pi_n \text{DFT}_{2^n} = \begin{pmatrix} \text{DFT}_{2^{n-1}} & \text{DFT}_{2^{n-1}} \\ \text{DFT}_{2^{n-1}} D_n & -\text{DFT}_{2^{n-1}} D_n \end{pmatrix}$$

$$= (\mathbf{1}_2 \otimes \text{DFT}_{2^{n-1}}) \cdot (\mathbf{1}_{2^{n-1}} \oplus D_n) \cdot (\text{DFT}_2 \otimes \mathbf{1}_{2^{n-1}})$$

Factorization of the twiddle factors

$$D_n := \begin{pmatrix} 1 \\ \omega_{2^n} \\ \omega_{2^n}^2 \\ \vdots \\ \omega_{2^n}^{2^{n-1}} \\ \omega_{2^n}^{2^{n-1}-1} \end{pmatrix} = \left( \begin{pmatrix} 1 \\ \omega_{2^n}^{2^{n-2}} \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 1 \\ \omega_{2^n} \end{pmatrix} \right)$$
Fast Fourier Transform

Cooley-Tukey Formula

$$\Pi_n \text{DFT}_{2^n} = \begin{pmatrix} \text{DFT}_{2^{n-1}} & \text{DFT}_{2^{n-1}} \\ \text{DFT}_{2^{n-1}} D_n & -\text{DFT}_{2^{n-1}} D_n \end{pmatrix}$$

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Cooley-Tukey Realization of DFT

Quantum circuit for DFT\(_N\)

Cost

Classical Computer
\[ T(N) = 2 \, T(N/2) + O(N) \]
\[ T(N) = O(N \log N) \]

Quantum Computer
\[ T(N) = T(N/2) + O(\log N) \]
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Introduction to Quantum Algorithms
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**Modular exponentiation**

\[ U : |x\rangle |0\rangle \mapsto |x\rangle |a^x \mod N\rangle , \] where \( N \) is the \( k \)-bit number to be factored, and \( x \) is a \( 2k \)-bit number. Implementation using \( 396k(k^2 + O(k)) \) elementary operations [Beckman et al.].

**Quantum Fourier Transform:**

\( \text{DFT}_{2^n} \) needs \( \frac{1}{2} n(n - 1) \) two-qubit gates and \( n \) one-qubit gates.

**An upper bound on the resources for \( k \)-bit number \( N \)**

About \( 400k^3 \) operations are needed (can be improved to \( O(k^2 \log k \log \log k) \)). The space needed is \( 5k + 1 \) qubits.

**Example: Factoring 128-bit and 1024-bit numbers**

For 128-bit we need \( 840 \cdot 10^6 \) operations and 641 qubits. For 1024-bit number \( 429 \cdot 10^9 \) operations and 5121 qubits.
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Hidden Subgroups in \((Z_2)^n\).

**P. Shor, 1994**

- Factoring
- Discrete Logarithm

\{ Hidden Subgroups in \(Z_M\) resp. \(Z_M \times Z_M\) \}

**Kitaev ’95, Brassard & Høyer ’97, Mosca & Ekert ’98**

Generalization to arbitrary abelian groups.

### Open Problems

Can the Hidden Subgroups Problem be solved for

- Any non-abelian group? (yes!)
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The Hidden Subgroup Problem (HSP)

Definition of the problem

**Given:** Finite group $G$, finite set $S$, map $f : G \rightarrow S$

**Promise:** There exists $H \subseteq G$ where

- $f$ constant on $G/H$,
- $g_1H \neq g_2H$ implies $f(g_1) \neq f(g_2)$.

**Problem:** Find generators for $H$.

Note

- This is a natural generalization of Simon’s problem.
- There $G = F_2^n$ and in addition we know $|H| = |\langle s \rangle| = 2$.
- In fact, the HSP for any subgroup $H \leq F_2^n$ can be solved efficiently.

Mathematically

\[ G \xrightarrow{f} S \]
\[ \pi \downarrow \]
\[ G/H \]

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Introduction to Quantum Algorithms
**The Hidden Subgroup Problem (HSP)**

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### Mathematically

![Mathematical Diagram](image)
HSP: Separation of Pre-images

Visualization of the cosets of $G$
Examples for Hidden Subgroup Problems

Simon’s Problem
Find hidden subgroup in $G = \mathbb{Z}_2^n$ of a black-box function $f : G \rightarrow \{0, 1\}^m$, where $m \leq n$ and $x \in y + H \iff f(x) = f(y)$.

Factoring
Find hidden subgroup in $G = \mathbb{Z}_M$ with respect to the function $f(x) := a^x \mod N$.

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Find hidden subgroup in $G = \mathbb{Z}_M \times \mathbb{Z}_M$ with respect to the function $f(x, y) := a^x b^{-y} \mod p$.

The graph isomorphism problem
Can be reduced to the problem of finding certain hidden subgroups of order 2 in the non-abelian group $G = S_n \wr S_2$. 

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Introduction to Quantum Algorithms
Basic identity

\[
\text{DFT}_A\left(\frac{1}{\sqrt{|U|}} \sum_{x \in U + c} |x\rangle \right) = \frac{1}{\sqrt{|U^\perp|}} \sum_{y \in U^\perp} \varphi_{c,y} \cdot |y\rangle
\]

Geometric interpretation

Shifted Line \hspace{2cm} \text{DFT} \hspace{2cm} \text{Amplitude} \hspace{2cm} \text{Phase}
Duality of the Fourier Transform

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**Geometric interpretation**

- **Shifted Line**
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- **Phase**
Definition of $\text{DFT}_G$

Any isomorphism $\Phi : \mathbb{C}[G] \rightarrow \bigoplus_{k=1}^{m} \mathbb{C}^{d_k \times d_k}$ of the group algebra and a direct sum of irreducible matrix algebras.

Some properties of $\text{DFT}_G$

- Defined for arbitrary finite groups.
- The isomorphism $\Phi$ is realized by a unitary matrix

$$\text{DFT}_G = \frac{1}{\sqrt{|G||H|}} \sum_{\rho, i,j} \sqrt{d_{\rho}} \sum_{h \in H} \rho_{ij}(gh) |\rho, i, j\rangle \langle g|.$$ 

- $\text{DFT}_G$ decomposes the regular representation $\phi$ of $G$:

$$\phi^{\text{DFT}_G} = \text{DFT}_G^\dagger \phi \text{ DFT}_G = \bigoplus_{k=1}^{m} 1_{d_{\rho_k}} \otimes \rho_k.$$
Generalized Fourier Transforms

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Introduction to Quantum Algorithms
Hidden Subgroup Problems: Standard Algorithm

Quantum algorithm

Given: finite group $G$ with hidden subgroup $H \leq G$.
Task: Find a set of generators for $H$.

Repeat the following steps $\text{poly}(n)$ many times:

1. Initialize two quantum registers: $|0\rangle |0\rangle$
2. Equal distribution on first register: $\frac{1}{\sqrt{|G|}} \sum_{x \in G} |x\rangle |0\rangle$
3. Compute $f$ in superposition: $\frac{1}{\sqrt{|G|}} \sum_{x \in G} |x\rangle |f(x)\rangle$
4. Measure second register: $\frac{1}{\sqrt{|H|}} \sum_{x \in cH} |x\rangle |f(c)\rangle$
5. Compute $\text{DFT}_G$ on first register: $\frac{1}{\sqrt{|G||H|}} \sum_{\rho,i,j} \sqrt{d_{\rho}} \sum_{h \in H} \rho_{ij}(ch) |\rho, i, j\rangle$
6. Measure first register: Sample $(\rho, j)$ with probability $\sum_{i} \frac{d_{\rho}}{|G|} \sum_{h \in H} \rho_{ij}(ch)^2$.

Further classical post-processing necessary!
Grover’s Algorithm for Searching a List

Searching for a satisfying assignment

Given a Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$, find an $x \in \{0, 1\}^n$ such that $f(x) = 1$. Such an $x$ is also called “satisfying assignment” and $f$ itself is also called “predicate”. Note that this search problem includes NP-complete problems such as 3-SAT.

How the search problem is specified

Given: List $X$ of $N = 2^n$ items and a predicate $f$ on $X$ which is given by the operator

$$V_f : |x\rangle \mapsto (-1)^{f(x)} |x\rangle.$$ 

Task: Find satisfying element $x$, i.e., $f(x) = 1$ (we assume precisely one such element exists).
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Given a Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$, the following operation computes $f$ into the phases:

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Question: What is the relation between $U_f$ and $V_f$?

Realizing $V_f$ from $U_f$

$$U_f |x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}} U_f |x\rangle |0\rangle - \frac{1}{\sqrt{2}} U_f |x\rangle |1\rangle$$

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The diffusion operator

\[ D_n := \begin{pmatrix} -1 + \frac{2}{2^n} & \frac{2}{2^n} & \cdots & \frac{2}{2^n} \\ \frac{2}{2^n} & -1 + \frac{2}{2^n} & \cdots & \frac{2}{2^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{2^n} & \frac{2}{2^n} & \cdots & -1 + \frac{2}{2^n} \end{pmatrix} \]

Inversion about the average

One application of the operator \(-D_n V_f\) when applied to the equal superposition of basis states does the following:
Grover’s Algorithm

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First prepare the equal superposition. Then iterate the operator $-D_nS_f$ a number of $O(\sqrt{2^n})$ times. Afterwards measure the system in the computational basis. With high probability the result will be the solution $x$ for which $f(x) = 1$.

Success probability after several iterations
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Two Different Types of Quantum Algorithms

Factoring

classical: \( O(e^{(c+o(1)) \sqrt[3]{\log n(\log \log n)^2}}) \)
quantum: \( O(\text{poly}(\log n)) \)

- feature extraction using signal transforms
- leads to the idea of “hidden subgroup problems”
- highly regular, in general huge speed-ups can be expected

Searching

classical: \( O(N) \)
quantum: \( O(\sqrt{N}) \) (and this is optimal)

- increase the amplitude of target states via correlations
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Two Different Types of Quantum Algorithms

**Factoring**

- Classical: $O(e^{(c+o(1))^{3\sqrt{\log n(\log \log n)^2}}})$
- Quantum: $O(\text{poly}(\log n))$

- Discrete FFT
- Feature extraction using signal transforms
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Quantum Gates and Circuits

Elementary quantum gates

\[ U^{(i)} = \begin{array}{c}
\vdots \\
\text{U} \\
\vdots \\
\end{array} \quad i \quad \begin{array}{c}
\vdots \\
\text{U} \\
\vdots \\
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\[ \text{CNOT}^{(i,j)} = \begin{array}{c}
\vdots \\
\text{U} \\
\vdots \\
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\vdots \\
\text{U} \\
\vdots \\
\end{array} \]

Universal set of gates

Theorem (Barenco et al., 1995):

\[ \mathcal{U}(2^n) = \langle U^{(i)}, \text{CNOT}^{(i,j)} : i, j = 1, \ldots, n, \quad i \neq j \rangle \]

How to prove this result?

Next: breaking down the proof into several small steps
Realizing a cyclic shift

How to realize $P_n : x \mapsto x + 1 \mod 2^n$, which cyclically shifts the basis states of an $n$ qubit register?

Solution 1

Purely classical realization

Solution 2

Genuinely quantum realization
Examples for quantum circuits

Realizing a cyclic shift

How to realize \( P_n : x \mapsto x + 1 \ \text{mod} \ 2^n \), which cyclically shifts the basis states of an \( n \) qubit register?

Solution 1

Purely classical realization

Solution 2

Genuinely quantum realization
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Universality of CNOT and Local Gates

Proof outline

- Given a unitary matrix \( U \in \mathcal{U}(2^n) \).
- Write \( U = U_1 \cdot \ldots \cdot U_M \), where \( U_i \) acts on pairs of states.
- Factorize each \( U_i \) using multiply-controlled \( \Lambda_n(V) \) gates, where \( V \in \mathcal{U}(2) \).
- Write each \( V \) in the form \((A^\dagger \sigma_x A)(B^\dagger \sigma_x B)\) with \( A, B \in \mathcal{U}(2) \).
- Use this to write each \( \Lambda_n(V) \) in terms of local gates and \( \Lambda_n(\sigma_x) \) (i.e., generalized CNOTs).
- Implement \( \Lambda_n(\sigma_x) \) recursively using \( \Lambda_k(\sigma_x) \), where \( k < n \).

Literature

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Literature

Operations on Subspaces

Rotation on a subspace spanned by the states $s_1$ and $s_2$

$$T(s_1, s_2) = \begin{pmatrix} 1 & \cdots & 1 & \ast & 1 \\ \vdots & \ddots & \ast & \ast & \ast \\ \ast & \cdots & 1 & \ast & \ast \\ \ast & \ast & \ast & 1 & \ast \\ \ast & \ast & \ast & \ast & \ast \end{pmatrix}$$

Theorem

Every $U \in \mathcal{U}(2^n)$ can be written in the form

$$U = \prod_{s_1, s_2 \in \{0,1\}^n} T(s_1, s_2).$$
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Conditional gates with multiple controls

Let $U \in \mathcal{U}(2)$. Then $\Lambda_k(U) \in \mathcal{U}(2^{k+1})$ is defined by

$$\Lambda_k := \begin{pmatrix} 1 \\ \vdots \\ 1 \\ U \end{pmatrix} = 1_{2^{k+1}-2} \oplus U.$$
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$$\Lambda_k := \begin{pmatrix} 1 & \cdots & 1 \\ & & \left[\begin{array}{c} U \end{array}\right] \end{pmatrix} = 1_{2k+1-2} \oplus U.$$ 

Alternative description of $\Lambda_k(U)$

$$\Lambda_k(U) |x_1, \ldots, x_n\rangle |y\rangle = \begin{cases} |x_1, \ldots, x_n\rangle |y\rangle & \text{if } \exists i : x_i \neq 1 \\ |x_1, \ldots, x_n\rangle U |y\rangle & \text{if } \forall i : x_i = 1 \end{cases}$$
Elementary Quantum Gates

Realizing a singly controlled $\Lambda_1(U)$ gate

$$\Lambda_1(U) = A \cdot B \cdot C \cdot E$$

Description of the local gates used in this circuit

- $U = \exp(i\phi)W$, $W \in SU(2)$,
- $E = \begin{pmatrix} 1 & 0 \\ 0 & \exp(i\phi) \end{pmatrix}$,
- $ABC = I$, $A\sigma_x B\sigma_x C = W$.
- The matrices $A, B, C$ are obtained from the decomposition $U = (A^\dagger \sigma_x A)(B^\dagger \sigma_x B)$. 
Realizing a singly controlled $\Lambda_1(U)$ gate

$\Lambda_1(U) = A \otimes B \otimes C$

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Realizing a two-fold controlled gate $\Lambda_2(U)$

$$U \cdot \cdot = V \cdot i \cdot V^\dagger \cdot i \cdot V$$

Comments

- Here we have to find a unitary $V$ with $V^2 = U$.
- This idea can be generalized to arbitrary $\Lambda_k(U)$, however, the complexity obtained from this factorization scales exponentially with $k$.
- Need better break-down strategy.
Elementary Quantum Gates

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Efficient Break-Down Strategies

\[ \Lambda_{n-2}(\sigma_x) \]

\[ = \]

\[ \Rightarrow \text{linear in } n \]

\[ \Lambda_{n-1}(W) \text{ with } W \in SU(2) \]

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Important result about multiply-controlled gates

Let \( U \in U(2) \). Then any \( \Lambda_{n-1}(U) \) gate operating on \( n \) qubits can be implemented using at most \( O(n) \) elementary gates.
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Important result about multiply-controlled gates

Let $U \in U(2)$. Then any $\Lambda_{n-1}(U)$ gate operating on $n$ qubits can be implemented using at most $O(n)$ elementary gates.
Operations on a two-dimensional subspace

Consider a subspace with basis $|i\rangle$, $|j\rangle$:
- $\Lambda_{n-1}(U)$, if $i$ and $j$ differ in only one bit
- Use Gray code sequence to connect $i$ and $j$.

Example: $n = 7$, $i = 5$, $j = 100$

$\begin{align*}
0000101 & \\
0000100 & \\
0100100 & \\
1100100 & \\
\end{align*}$

$\implies$ Complexity is $O(n^3)$ elementary gates.
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The Cybenko trick (2001)

We can save almost all of the permutation gates necessary for the Gray code by using CNOT gates at the different positions:

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1000101 \\
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\]

\[\Rightarrow\] Complexity of \( O(n) \) gates for \( W \in SU(n) \)

Theorem

Any unitary transformation on \( n \) qubits can be implemented using at most \( 4^n \) elementary gates. This bound is tight and for almost all elements of \( U(2^n) \) we need \( \Theta(4^n) \) gates.
Saving Even More Gates and the Final Result

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\[=\]

Complexity of \( O(n) \) gates for \( W \in SU(n) \)

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Conclusions

- Shor’s algorithm for factoring integers
  - Reducing factoring to order finding
  - Setting up a periodic state
  - Efficient extraction of the period by computing a QFT
- Grover’s algorithm: searching $N$ items in time $O(\sqrt{N})$.
- Elementary quantum gates:
  - Controlled NOT gate
  - Local unitary transformations
Dedicated to Thomas Beth, *16.11.1949, † 17.08.2005