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MULTIPARTITE STATES UNDER LOCAL UNITARY TRANSFORMATIONS

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Introduction

Any measure of entanglement $E(\cdot)$ on a finite dimensional Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$$

is constant under *local unitary transformations*

$$U_1 \otimes U_2 \otimes \cdots \otimes U_N.$$

Goal: *To find a complete set of invariants under local unitary transformations* $U_1 \otimes U_2 \otimes \cdots \otimes U_N$.

Bipartite case [1]

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim(\mathcal{H}_i) = N_i$, $i = A, B$.

For **pure states**, i.e., states

$$|\psi\rangle = \sum_{j=1}^{N_A} \sum_{k=1}^{N_B} a_{jk} |j\rangle_A \otimes |k\rangle_B,$$

where $\sum_{j=1}^{N_A} \sum_{k=1}^{N_B} a_{jk} a_{jk}^* = 1$ and $\{|l\rangle_i\}_{l=1,2,\ldots,N_i}$ is an orthonormal basis of $\mathcal{H}_i$, $i = A, B$, the functions

$$I_\alpha^A(|\psi\rangle) = \mathrm{Tr}(\mathrm{Tr}_A(|\psi\rangle\langle\psi|))^\alpha, \; I_\alpha^B(|\psi\rangle) = \mathrm{Tr}(\mathrm{Tr}_B(|\psi\rangle\langle\psi|))^\alpha$$

are *invariant under local unitary transformations* $U_A \otimes U_B$, for any power $\alpha \in \mathbb{N}$.

Theorem. *The set of invariants*

$$\{I_\alpha^A(\cdot) \,|\, \alpha = 1, \ldots, \min\{N_A, N_B\}\}$$

is complete.

Remark. The same holds for the set $\{I_\alpha^B(\cdot)\}$.

Consider now **mixed states**, i.e., states described by a density

$$\rho = \sum_{j=1}^{n} \lambda_j |\varphi_j\rangle\langle\varphi_j|,$$

where $n \leqslant N_A N_B$, $0 < \lambda_j \leqslant 1$, and $\{|\varphi_j\rangle\}_{j=1,2,\ldots,n}$ is a set of normalized pure states.

The functions

$$J_\alpha^A(\rho) = \mathrm{Tr}(\mathrm{Tr}_A\,\rho)^\alpha,\ J_\alpha^B(\rho) = \mathrm{Tr}(\mathrm{Tr}_B\,\rho)^\alpha$$

are *invariant under local unitary transformations* $U_A \otimes U_B$. However, these invariants are not sufficient in the case of mixed states.

Let us therefore introduce the following *further invariants*:

$$\begin{aligned} \Omega(\rho)_{jk} &= \mathrm{Tr}(\mathrm{Tr}_B(|\varphi_j\rangle\langle\varphi_j|)\,\mathrm{Tr}_B(|\varphi_k\rangle\langle\varphi_k|)) \\ X(\rho)_{jkl} &= \mathrm{Tr}(\mathrm{Tr}_B(|\varphi_j\rangle\langle\varphi_j|)\,\mathrm{Tr}_B(|\varphi_k\rangle\langle\varphi_k|)\,\mathrm{Tr}_B(|\varphi_l\rangle\langle\varphi_l|)) \end{aligned}$$

where $j, k, l = 1, \ldots, n$.

Ω can be completed to a $(N_A N_B \times N_A N_B)$-matrix by defining $\Omega(\rho)_{jk} = 0$ for j or k bigger than n. A mixed state ρ is called **generic** if $\Omega(\rho)$ is non-degenerate.

Theorem. *For* mixed generic states *the set of invariants*

$$\{J^A_\alpha(\cdot)\,|\,\alpha = 1, \ldots, \min\{N_A, N_B\}\} \cup \{\Omega(\cdot), X(\cdot)\}$$

is complete.

Tripartite case

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ with $\dim(\mathcal{H}_i) = N_i$, $i = A, B, C$.

Consider **pure states**, i.e., states

$$|\psi\rangle = \sum_{j=1}^{N_A} \sum_{k=1}^{N_B} \sum_{l=1}^{N_C} a_{jkl} |j\rangle_A \otimes |k\rangle_B \otimes |l\rangle_C,$$

where $\sum_{j=1}^{N_A} \sum_{k=1}^{N_B} \sum_{l=1}^{N_C} a_{jkl} a_{jkl}^* = 1$ and $\{|m\rangle_i\}_{m=1,2,\ldots,N_i}$ is an orthonormal basis of $\mathcal{H}_i$, $i = A, B, C$.

Remark. A main tool in the bipartite case was the *Schmidt decomposition*. Such decomposition holds however only for bipartite systems. A direct generalization is therefore not possible.

First step: handle $\mathcal{H}$ as a bipartite system A-BC:

$$\left.\begin{array}{c} |\psi'\rangle = U_A \otimes U_{BC}|\psi\rangle \\ \text{with } U_A,\, U_{BC} \text{ unitary} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} I^A_\alpha(|\psi\rangle) = I^A_\alpha(|\psi'\rangle) \\ \text{for } \alpha = 1, \ldots, \min\{N_A, N_B N_C\} \end{array}\right.$$

Remark. U_A and U_{BC} depend on $|\psi\rangle$ and can be found explicitly (via singular value decomposition).

If $|\psi'\rangle = U_A \otimes U_{BC}|\psi\rangle$, the problem can be reduced to a bipartite one via partial trace on $\mathcal{H}_A$. The following lemma is easy to prove.

Lemma. *Let $|\psi'\rangle = U_A \otimes U_{BC}|\psi\rangle$ and define $\rho = \mathrm{Tr}_A(|\psi\rangle\langle\psi|)$. If*

$$U_{BC}\rho U_{BC}^\dagger = (U_B \otimes U_C)\rho(U_B^\dagger \otimes U_C^\dagger),$$

there exist unitary matrices V_A, V_B, V_C such that $|\psi'\rangle = V_A \otimes V_B \otimes V_C|\psi\rangle$.

Problem: In general ρ is a mixed state and we don't have a general solution for such states.

Special case 1: ρ is a generic mixed state [2]

The functions

$$I_{\alpha,\beta}^{A,j}(|\psi\rangle) = \mathrm{Tr}(\mathrm{Tr}_j(\mathrm{Tr}_A(|\psi\rangle\langle\psi|))^{\alpha})^{\beta},$$

where $j \in \{B, C\}$, $\alpha = 1, \ldots, N_B N_C$, and $\beta = 1, \ldots, \{N_k | k \in \{B, C\} \text{ and } k \neq j\}$, are *invariant under local unitary transformations* $U_A \otimes U_B \otimes U_C$.

Remarks.

- I_{α}^{A} is included as $I_{\alpha,1}^{A,B} = I_{\alpha,1}^{A,C}$.
- In general, the functions

$$\mathrm{Tr}(\mathrm{Tr}_{j_1}(\mathrm{Tr}_{j_2}(\ldots(\mathrm{Tr}_{j_N}|\psi\rangle\langle\psi|)^{\alpha_N}\ldots)^{\alpha_3})^{\alpha_2})^{\alpha_1}$$

  are invariant under local unitary transformations $U_1 \otimes U_2 \otimes \cdots \otimes U_N$.

Theorem. *Let $N_A = N_B N_C$.*
For states $|\psi\rangle$ such that $\rho = \mathrm{Tr}_A(|\psi\rangle\langle\psi|)$ is a generic mixed state the set of invariants

$$\{I^{A,j}_{\alpha,\beta}(|\psi\rangle)\} \cup \{\Omega(\rho), X(\rho)\}$$

is complete.

Remarks.

- The invariance of $\{I^{A,j}_{\alpha,\beta}\}$ implies automatically the invariance of Ω_{jk} and X_{jkl} if $\lambda_j \neq \lambda_k \neq \lambda_l$ in $\rho = \sum_{j=1}^{n} \lambda_j |\varphi_j\rangle\langle\varphi_j|$.
- $N_A = N_B N_C$ is needed in order to have ρ generic.

Special case 2: U_{BC} is unitarily decomposable [3]

Definition. Let A be a $m \times m$ block matrix with blocks of dimension $n \times n$. The *realigned matrix* $\tilde{A}$ is defined as

$$\tilde{A} = [\mathrm{vec}(A_{11}), \mathrm{vec}(A_{21}), \ldots, \mathrm{vec}(A_{m1}), \mathrm{vec}(A_{12}), \ldots, \mathrm{vec}(A_{mm})]^T,$$

where
$$\mathrm{vec}(A_{jk}) = [a_{11}^{jk}, a_{21}^{jk}, \ldots, a_{n1}^{jk}, a_{12}^{jk}, \ldots, a_{n2}^{jk}, \ldots, a_{nn}^{jk}]^T$$
for every block $A_{jk} = (a_{pq}^{jk})_{p,q=1,\ldots,n}$.

Consider U_{BC} as a block matrix with blocks of dimension $N_C \times N_C$. If the rank of the realigned matrix $\tilde{U}_{BC}$ is one, there exist two unitary matrices U_B and U_C such that $U_{BC} = U_B \otimes U_C$.

Theorem. *Let* $|\psi\rangle$ *and* $|\psi'\rangle$ *be two tripartite states. If* $I_\alpha^A(|\psi\rangle) = I_\alpha^A(|\psi'\rangle)$ *for* $\alpha = 1, \ldots, \min\{N_A, N_B N_C\}$ *and the rank of* $\tilde{U}_{BC}$ *is one,* $|\psi\rangle$ *and* $|\psi'\rangle$ *are equivalent under local unitary transformations.*

Remarks.

- The decomposition $U_{BC} = U_B \otimes U_C$ is more than what we need: $1 \otimes U_{BC}|\psi\rangle = 1 \otimes U_B \otimes U_C|\psi\rangle$ would be enough.
- Condition on U_{BC}: $|\psi\rangle$ cannot be compared with all $|\psi'\rangle$.
- No conditions on the N_i's $\Rightarrow$ we can also look at B-AC or C-AB.

Open problems

- **non-generic bipartite mixed** states ($\rightarrow$ tripartite pure states);
- **tripartite mixed** states;
- generalization to **multipartite** systems: the functions

$$\mathrm{Tr}(\mathrm{Tr}_{j_1}(\mathrm{Tr}_{j_2}(\ldots(\mathrm{Tr}_{j_N}|\psi\rangle\langle\psi|)^{\alpha_N}\ldots)^{\alpha_3})^{\alpha_2})^{\alpha_1}$$

are invariant under local unitary transformations $U_1 \otimes U_2 \otimes \cdots \otimes U_N$.

$$\boxed{N\text{-partite pure states} \quad \longleftarrow \quad (N\text{-}1)\text{-partite mixed states}}$$

References

[1] S. Albeverio, S.M. Fei, P. Parashar, and W.L. Yang, *Phys. Rev. A* **68**, 010303 (R) (2003).

[2] S. Albeverio, L. C., S.M. Fei, X.H. Wang, to appear in *Rep. Math. Phys.*

[3] S. Albeverio, L. C., S.M. Fei, X.H. Wang, to appear in *I.J.Q.I.*