

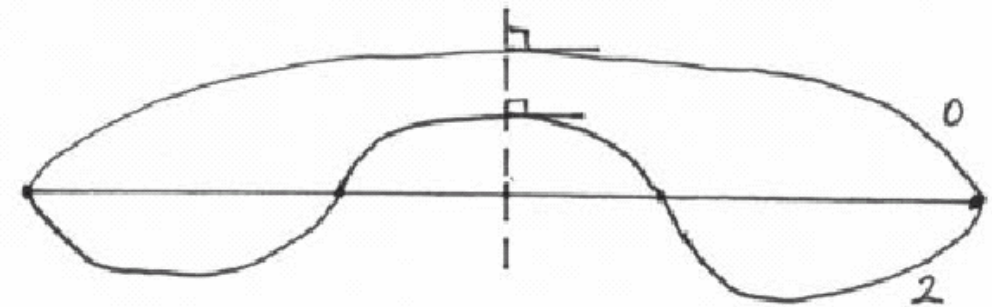
# ENTANGLEMENT IN PHASE SPACE

1. Entanglement and classical physics
2. Classical and quantum correspondence
3. Semiclassical quantum states
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5. The Wigner function and the chord function
6. The partial trace
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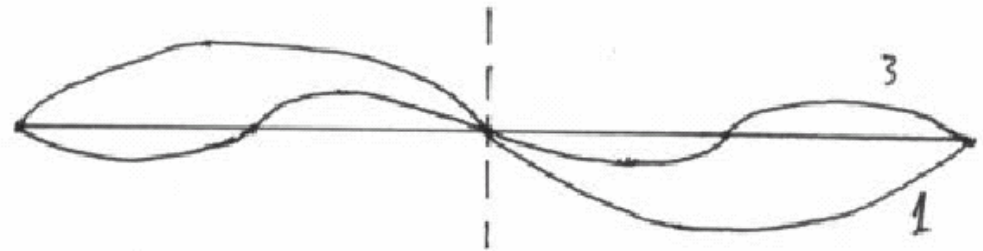
# 1. Classical physics

- PARTICLES
- Measure position, or momentum
- WAVES
- Measure spectrum, or parity:

even



odd



## 2. Classical-quantum correspondence

A single degree of freedom:

Classical states described by points in phase space,

$$x=(p, q)$$

Motion generated by Hamiltonian,  $H(x)$ ,  
and Hamilton's equations:

$$\dot{p} = -\frac{\partial H}{\partial q} \quad , \quad \dot{q} = \frac{\partial H}{\partial p} \quad , \quad \text{or}$$

$$\dot{x} = \mathbf{J} \frac{\partial H}{\partial x} \quad , \quad \text{with} \quad \mathbf{J} = \left[ \begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array} \right]$$

Corresponding quantum operators,  $\hat{q}$  and  $\hat{p}$ , which do not commute,  $[\hat{p}, \hat{q}] = i\hbar$ ,

Motion of states,  $|\psi\rangle$   
determined by Schroedinger's equation:

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H(\hat{x}) |\psi\rangle,$$

Entanglement only for more degrees of freedom.

Basis of product states:

$$|\psi\rangle = |\psi_1\rangle \otimes \dots |\psi_t\rangle \otimes \dots |\psi_L\rangle$$

The corresponding product phase space is

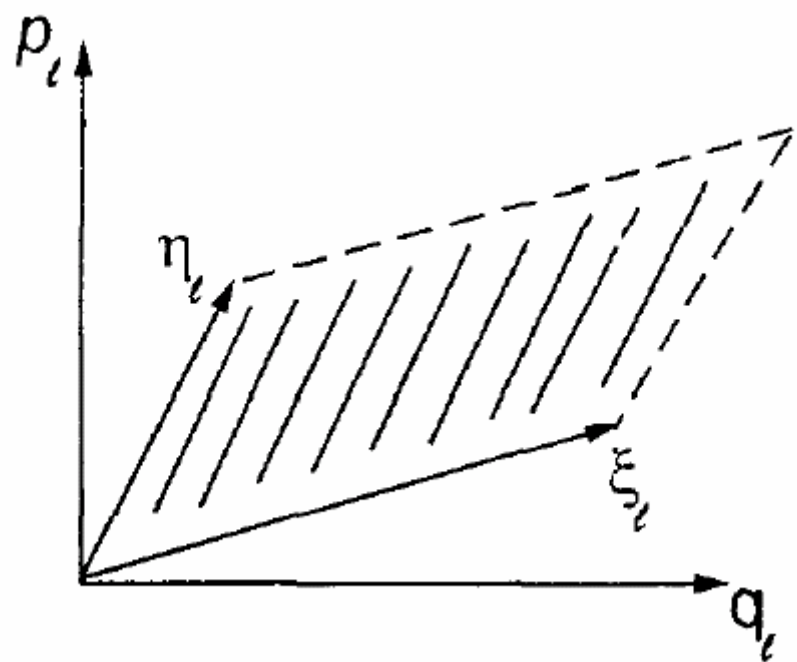
$$x = x_1 \otimes \dots \otimes x_l \otimes \dots \otimes x_L$$

now  $\partial H / \partial x$  is a  $(2L)$ -dimensional vector

and  $\mathbf{J}$  is a block matrix.

We will need the skew product:

$$x \wedge x' = \sum_{n=1}^L (p_n q'_n - q_n p'_n) = \mathbf{J} x \cdot x'$$



$$\sum_{\ell=1}^L (\xi_{p_\ell} \eta_{q_\ell} - \xi_{q_\ell} \eta_{p_\ell}) = \xi \wedge \eta$$

Approximate correspondance of quantum states with probability distributions,  $f(\mathbf{x})$

If the degrees of freedom are decoupled,  
The distribution is just a product:

$$f(\mathbf{x}) = f_1(x_1) \dots f_L(x_L).$$

In general the single distributions are reobtained by tracing over the other variables:

$$f_1(x_1) = \int f(\mathbf{x}) dx_2 \dots dx_L.$$

The product nature of the classical distribution is preserved by an additive Hamiltonian:

$$H(x) = H_1(x_1) + \dots H_l(x_l) + \dots H_L(x_L),$$

Hamilton's equations decouple,

$$\dot{x}_l = \mathbf{J}_l \frac{\partial H_l}{\partial x_l}.$$

so that, if  $k \neq l$ , then  $x_l(t; x_l(0))$  does not depend on  $x_k$ .

Furthermore,

$$0 \downarrow \\ f_l(t; x_l) = f_l(t; x_l(-t; x_l))$$



Examples:

-A pair of particles, each moving in one dimension.

-A single particle moving in two dimensions.

All that is demanded is that  $[\hat{P}_x, \hat{Q}_y] = i\hbar\delta_{xy}$

Or that the classical Poisson bracket  $\{P_x, Q_y\} = \delta_{xy}$ .

What about other variables  
obtained through canonical transformations?

Consider a model for the piano string as  $L$  masses  
connected by  $L$  strings. The transformation to the  $L$   
normal modes of vibration describes  
collective motions of the masses.

This linear canonical transformation  
(i.e. symplectic transformation)

substitutes the original  $L$  conjugate planes,  $x_i = (p_i, q_i)$ ,  
by new conjugate planes,  $x'_i = (p'_i, q'_i)$

This is also a proper phase space to be quantized,  $x'_i \rightarrow \hat{x}'_i$ .

However, general nonlinear canonical transformations  
are not matched exactly by quantum unitary  
transformations:

At best a semiclassical approximation

-Another example is a particle with internal structure, e.g. angular momentum.

The Stern-Gerlach experiment is on such a system. Here the spin angular momentum is intrinsically quantum.

A similar coupling between the orbital angular momentum of a Rydberg atom (large dipole moment) and its translation degrees of freedom would result from a homogeneous electric field.

Basic differences between classical and quantum systems:

- i) The nature of the initial state;
- ii) The nature of the evolution;
- iii) The effect of experiments.

Objective:

To cast the quantum mechanical description of i) and ii) in the most classical form possible, so as to highlight the truly inovitative elements of the quantum theory.

### 3. Semiclassical quantum states

Consider a momentum eigenstate  $|p'\rangle$  for  $L = 1$ .

In the momentum representation,

$$\langle p|p'\rangle = \delta(p' - p).$$

No good for semiclassical extrapolation.

Use alternative representation:

$$\langle q|p'\rangle = \exp\left(\frac{iqp'}{\hbar}\right) = \exp\left(\frac{iS_{p'}(q)}{\hbar}\right).$$

Arbitrary phase is determined by choice of initial point in the action integral:

$$S(q) = \int_{q_0}^q p(q) dq.$$

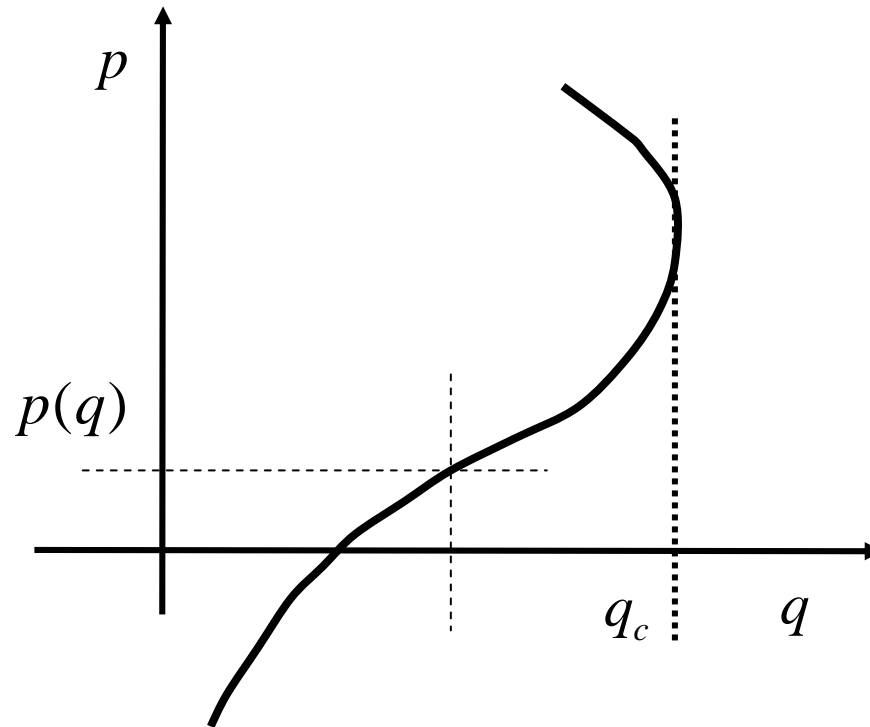
Consider now a general observable,  $K(\hat{p}, \hat{q})$ .  
Eigenstates correspond to  
curves in phase space,  $\gamma : K(p, q) = k$ .  
Given the (multivalued) functions,  $p_j(q)$   
The semiclassical approximation is

$$\langle q|k\rangle = \sum_j A_j(q) \exp i \left[ \frac{S_j(q)}{\hbar} + \nu_j \right].$$

In the case of bound states,  
eigenvalues determined by the  
Bohr-Sommerfeld condition:

$$\oint_{\gamma} p dq = \left(n + \frac{1}{2}\right) h.$$

The different branches of the function  
 $p(q)$  are joined at *caustics* where the  
classical curve is vertical:



The vertical tangent  
 would correspond to the state,  $(p|q_c\rangle$   
 The corresponding semiclassical approximation  
 superposes terms like

$$(p|k\rangle = B(p) \exp i \left[ \frac{S(p)}{\hbar} + \nu \right]$$



Near the caustic, the semiclassical approximation is defined by the Fourier transform:

$$\langle q|k\rangle = \frac{1}{(2\pi\hbar)^{1/2}} \int dp \langle p|k\rangle \exp\left(\frac{iqp}{\hbar}\right)$$

(Airy functions instead of complex exponentials)

Let us now consider a product state for  $L > 1$ . Then,

$$\langle q | \mathcal{P}' \rangle = \exp\left(\frac{i q_1 \mathcal{P}'_1}{\hbar}\right) \dots \exp\left(\frac{i q_L \mathcal{P}'_L}{\hbar}\right) = \exp\left(\frac{i q \cdot \mathcal{P}'}{\hbar}\right)$$

and we can generalize the definition of action,

$$S(q) = \int_{q_0}^q p(q) \cdot dq.$$

Independent of the path, so  $p(q)$  is  
a *Lagrangian surface* :

$$S(q) = S_1(q_1) + \dots + S_L(q_L) = \oint p \cdot dq = 0$$

for any (reducible) circuit.

arbitrary eigenstates of  $L$  observables,

$\widehat{K} = \widehat{K}_1 \widehat{K}_2 \dots \widehat{K}_L$ , each in its own Hilbert space:

$$\langle q|k\rangle = \langle q_1|k_1\rangle \dots \langle q_L|k_L\rangle.$$

Considering just one branch of each function,  $p_t(q_t)$ , then the wave function will be a superposition of terms with the form

$$\langle q|k\rangle = \prod_t A_t(q_t) \exp \frac{i}{\hbar} [S_1(q_1) + \dots + S_L(q_L)].$$

$K(p, q) = k$ , is also Lagrangian.

If the surface is the product of  $L$  quantized circles  
it will be an  $L$ -torus,  $\tau$ .

Each of the  $L$  irreducible circuits,  $\gamma_i$ ,  
must then satisfy the Bohr-Sommerfeld conditions,

$$\oint_{\gamma_i} p_i \cdot dq_i = (n_i + \frac{1}{2})\hbar,$$

# SEMICLASSICAL EVOLUTION

Van Vleck: classical and quantum evolutions commute.

Each evolved observable then corresponds to

$$K_t(x_t, t) = K_t(x_t(x_{cl}, t), 0)$$

With the quantization condition

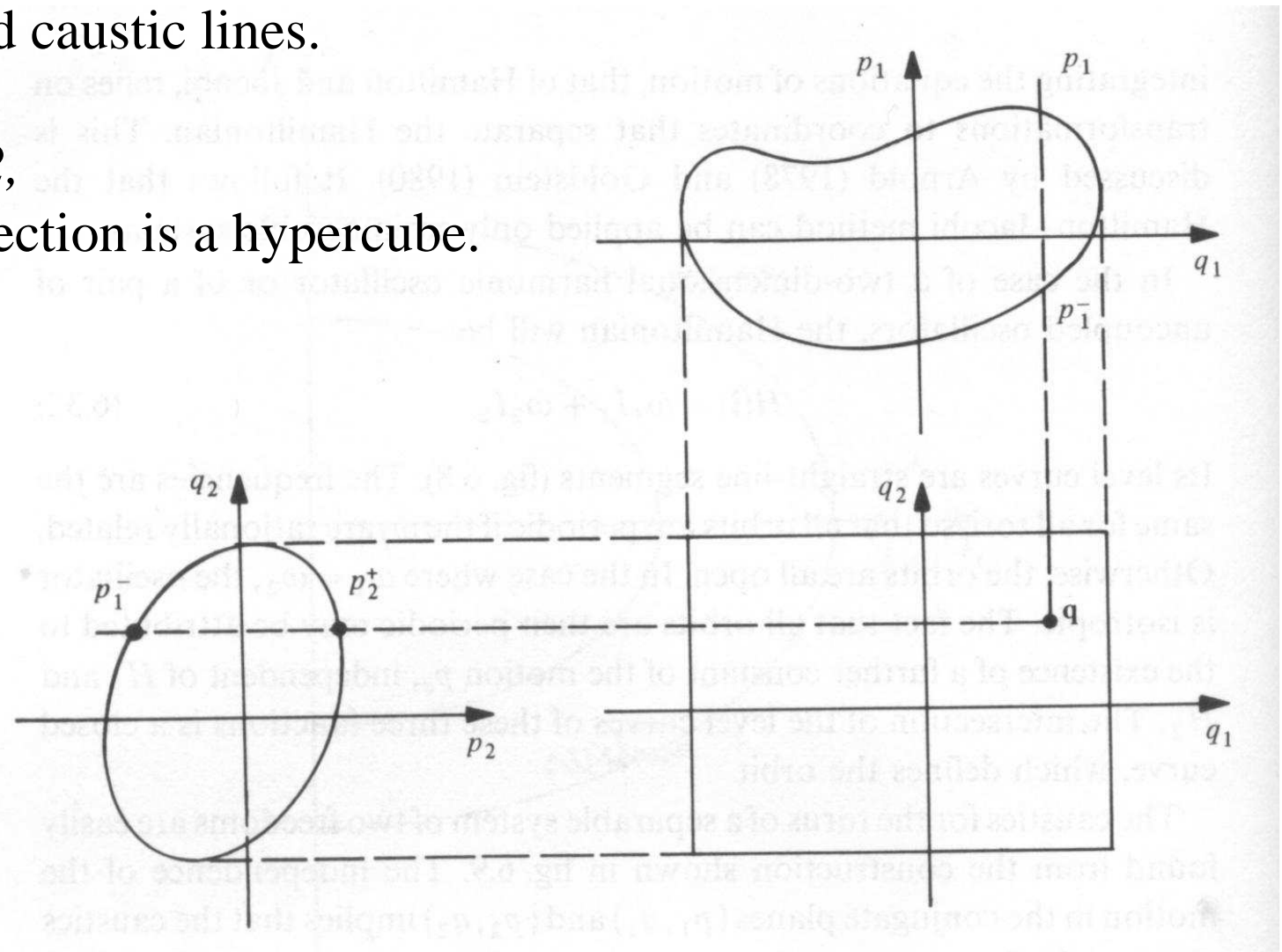
$$K_t(x_t, t) = k_{cl}.$$

The evolved classical surface is still Lagrangian because

$$\oint_{\gamma_0} p_0 \cdot dq_0 = \oint_{\gamma_t} p_t \cdot dq_t,$$

Example: 2-D separable torus,  $\tau = \gamma_1 \otimes \gamma_2$ ,  
Combining the projections onto each position axis,  
the projection onto the position plane is a rectangle,  
bordered caustic lines.

For  $L > 2$ ,  
the projection is a hypercube.



## COHERENT STATE BASIS

labeled by the phase space vector,  $\eta = (\eta_p, \eta_q)$ ,

$$\langle q|\eta\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\omega}{2\hbar}(q - \eta_q)^2 + i\frac{\eta_p}{\hbar}\left(q - \frac{\eta_q}{2}\right)\right)$$

Overcomplete and not orthogonal, but

$$|\psi\rangle = \frac{1}{\pi} \int d\eta |\eta\rangle \langle \eta|\psi\rangle$$

General *phase space translations*:

$$\hat{T}_\eta = \exp\left[\frac{i}{\hbar}(\eta_p \cdot \hat{q} - \eta_q \cdot \hat{p})\right] = \exp\left(\frac{i}{\hbar}\eta \wedge \hat{x}\right)$$

of the HO ground state:

$$\langle q|0\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\omega}{2\hbar}q^2\right).$$

In quantum optics, switch to creation and annihilation operators,  $(\hat{q} \pm i\hat{p})/\sqrt{2\hbar}$ .  
Then, have complex vectors,  $(\xi_p \pm i\xi_q)/\sqrt{2\hbar}$

By working directly with the representation of operators, orthogonality is regained.

We will again use the translation operator, but it will not be biased by acting on any particular state.



## 4. Operator representations and double phase space

The linear operators,  $\hat{A}$ ,  
form a vector space  $|A\rangle\rangle$ .

Defining the Hilbert-Schmidt product,

$$\langle\langle A|B\rangle\rangle = \text{tr } \hat{A}^\dagger \hat{B},$$

dyadic operators  $|Q\rangle\rangle = |q_-\rangle\langle q_+|$  form a complete basis,

$$\langle\langle Q|A\rangle\rangle = \langle q_+|\hat{A}|q_-\rangle = \text{tr } |q_-\rangle\langle q_+|\hat{A}$$

Note similarity between dyadic basis,  $|q_-\rangle\langle q_+|$ ,

and product state basis,  $|q_1\rangle \otimes |q_2\rangle$ .

Then, natural to relate double Hilbert space

to double phase space:  $X = x_- \otimes x_+$ .

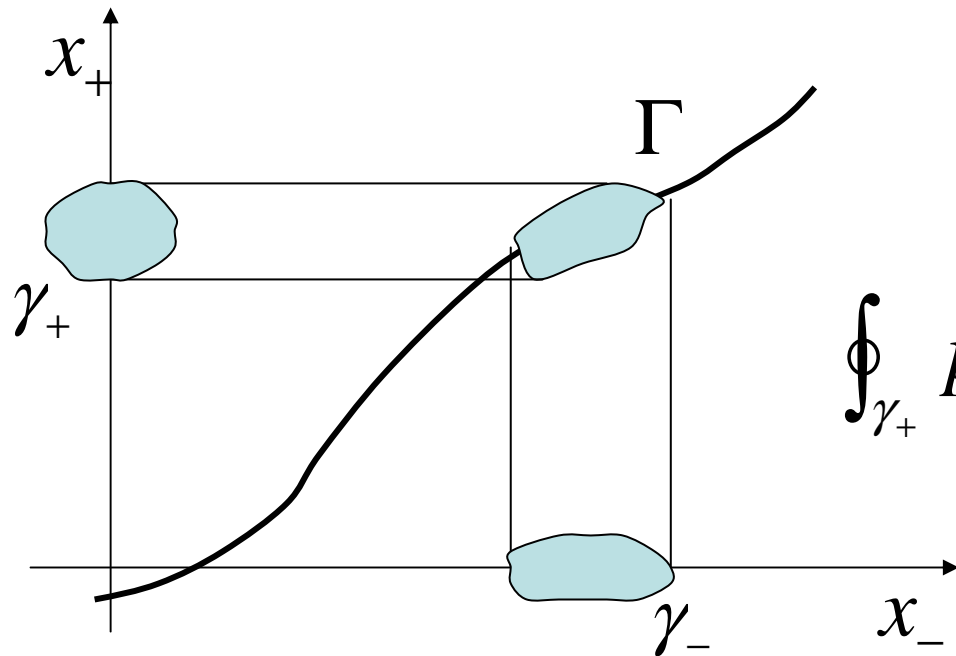
The operator  $|Q\rangle\rangle$  corresponds to the

Lagrangian plane,  $Q = \text{constant}$ ,

in double phase space. But adapt coordinates:

$$Q = (q_-, q_+), \text{ but } P = (-p_-, p_+) \quad X = (P, Q).$$

Canonical transformations,  $C : x_- \rightarrow x_+ = (q_+, p_+)$



$$\oint_{\gamma_+} p_+ \cdot dq_+ - \oint_{\gamma_-} p_- \cdot dq_- = 0$$

Define:  $Q = (q_-, q_+)$

$P = (-p_-, p_+) \Rightarrow \oint_{\Gamma} P \cdot dQ = 0$

$\Gamma = (\gamma_-, \gamma_+)$

Notice the richness of structures.

The canonical transformations define  $x_+$  ( $x_-$ )  
a one-to-one function.

The product of a Lagrangian surface,  $\lambda_-$  in  $x_-$   
with another surface  $\lambda_+$  in  $x_+$ ,  $\Lambda = \lambda_- \otimes \lambda_+$ ,  
is also Lagrangian in double phase space,

but projects singularly onto either of the factor spaces.

If both surfaces are tori,  $\tau = \tau_- \otimes \tau_+$ ,  
if  $L=1$ , a 2-D product torus,  
but with  $p_- \rightarrow -p_-$

If each Lagrangian surface in single phase space corresponds to a state,  
 i.e.  $|\psi_-\rangle$  and  $|\psi_+\rangle$ ,  
 then we represent  $|\Psi\rangle\rangle = |\psi_+\rangle\langle\psi_-|$  in the  $|Q\rangle\rangle$  representation

$$\langle\langle Q|\Psi\rangle\rangle = \langle q_+|\psi_+\rangle\langle\psi_-|q_-\rangle.$$

the semiclassical approximation is just a superposition

$$\langle\langle Q|\Psi\rangle\rangle = A_J(Q) \exp[iS_J(Q)/\hbar]$$

with

$$A_J(Q) = A_{j-}(q_-)^* A_{j+}(q_+)$$

and

$$S_J(Q) = \int_0^Q P_J(Q') \cdot dQ'.$$

Just like product states: projects as a rectangle onto  $P$ , or  $Q$ .

The semiclassical approximation for the unitary operators,  $\hat{U}$ , that correspond to a canonical transformation,  $\mathbf{C} : x_- \rightarrow x_+$ , has exactly the same form,

$$\langle\langle Q|U\rangle\rangle = \langle q_+|\hat{U}|q_-\rangle = U_J(Q) \exp[iS_J(Q)/\hbar],$$

for each branch of the function  $P_J(Q)$

- i. Note that the Lagrangian surface is not a product.
- ii. Note that projections onto  $P$  and  $Q$  may be singular.

$S(Q)$  is the generating function of a canonical transformation:

$$\frac{\partial S}{\partial Q} = P_C(Q), \quad \text{or} \quad \frac{\partial S}{\partial q_+} = p_+, \quad \frac{\partial S}{\partial q_-} = -p_-.$$

For symplectic transformations (linear canonical),  
 $S(Q)$  is quadratic and the semiclassical propagator is exact.

Legendre transforms create new generating functions:

$\pi/2$  rotation,  $q_+ \rightarrow p_+, p_+ \rightarrow -q_+$ ,

Then  $Q' = (q_-, p_+)$  is also a good Lagrangian plane  
the new coordinate plane for the description of  $\Lambda_C$ .

$$\frac{\partial S'}{\partial Q'} = P'_C(Q'), \quad \text{or} \quad \frac{\partial S'}{\partial p_+} = -q_+, \quad \frac{\partial S'}{\partial q_-} = -p_-.$$

Nontrivial change of coordinates in double phase space:

$$\begin{aligned} x &= (p, q) = \frac{x_+ + x_-}{2} \\ y &= J \xi = J(x_+ - x_-) \end{aligned} \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

New Lagrangian coordinate planes  
correspond to unitary operators:

$$\begin{aligned} y = \xi = 0 &\Rightarrow I \quad (\text{identity operator}) \quad x_+ = x_- \\ \xi \neq 0 &\Rightarrow T_\xi \quad (\text{phase space translation}) \quad x_+ = x_- + \xi \end{aligned}$$

Phase space translations form a group.



Exact correspondence to quantum operators:

each plane  $\mathbf{y} = \text{constant}$  corresponds precisely to the translation operator,  $\hat{T}_{\mathbf{y}}$ , previously defined.

we could also use  $\hat{T}_{\mathbf{y}} = \exp[i\mathbf{y} \cdot \hat{\mathbf{x}}/\hbar]$

---

previous dyadic  $|Q\rangle\rangle$  basis,

$$\hat{T}_{\mathbf{y}} = \int d\mathbf{q} \left| \mathbf{q} + \frac{\xi_{\mathbf{q}}}{2} \right\rangle \left\langle \mathbf{q} - \frac{\xi_{\mathbf{q}}}{2} \right| e^{i\mathbf{y} \cdot \mathbf{q}/\hbar}$$

---

The transformation from *horizontal* to the *vertical* basis is given by the full Fourier transform ( as with states):

$$2^L \hat{R}_{\mathbf{x}} = \int \frac{d\xi}{(2\pi\hbar)^L} \hat{T}_{\xi} \exp\left(\frac{i}{\hbar} \mathbf{x} \wedge \xi\right), \quad \text{The Reflection Operator}$$

---

previous dyadic  $|Q\rangle\rangle$  basis,

$$2^L \hat{R}_{\mathbf{x}} = \int d\xi_{\mathbf{q}} \left| \mathbf{q} + \frac{\xi_{\mathbf{q}}}{2} \right\rangle \left\langle \mathbf{q} - \frac{\xi_{\mathbf{q}}}{2} \right| e^{i\mathbf{x} \cdot \xi_{\mathbf{q}}/\hbar}.$$

---

Switch to the unitary operator basis:  $|y\rangle\rangle = \hat{T}_\xi$ :

$$\langle\langle y|A\rangle\rangle = \text{tr} \hat{T}_{-\xi} \hat{A} = A(\xi),$$

where we use  $\hat{T}_{-\xi} = \hat{T}^\dagger$ .

This is the expansion coefficient in

$$\hat{A} = \int \frac{d\xi}{(2\pi\hbar)^L} A(\xi) \hat{T}_\xi,$$

because

$$\begin{aligned} \text{tr}(\hat{T}_{-\xi} \hat{A}) &= \text{tr} \int \frac{d\xi'}{(2\pi\hbar)^L} A(\xi') \hat{T}_{-\xi} \hat{T}_{\xi'} \\ &= \int \frac{d\xi'}{(2\pi\hbar)^L} A(\xi') \exp\left[\frac{i}{2\hbar} \xi' \wedge \xi\right] \text{tr} \hat{T}_{\xi' - \xi} = A(\xi) \end{aligned}$$

using

$$\text{tr} \hat{T}_\xi = (2\pi\hbar)^L \delta(\xi) = \langle\langle y|I\rangle\rangle$$

We have also used the quantum version of the group property:

$$\hat{T}_{\xi_2} \hat{T}_{\xi_1} = \hat{T}_{\xi_1 + \xi_2} \exp\left[\frac{-i}{2\hbar} \xi_1 \wedge \xi_2\right].$$

The other products of the (quantum) affine group are:

$$\hat{R}_x \hat{T}_\xi = \exp\left[-\frac{i}{\hbar} x \wedge \xi\right] \hat{R}_{x-\xi/2},$$

$$\hat{T}_\xi \hat{R}_x = \exp\left[-\frac{i}{\hbar} x \wedge \xi\right] \hat{R}_{x+\xi/2}$$

and

$$\hat{R}_{x_2} \hat{R}_{x_1} = \exp\left[\frac{2i}{\hbar} x_1 \wedge x_2\right] \hat{T}_{2(x_2 - x_1)}.$$

Note that  $\hat{R}_x^2 = \hat{I}$ , the identity,

hence the (degenerate) eigenvalues of  $\hat{R}_x$  must be either +1, or -1.

Therefore these operators are Hermitian, as well as unitary.



Are the reflection operators true observables?

The parity,  $+1$ , or  $-1$ , around the origin is an observable wave property.

This is currently measured in quantum optics.

There, the natural basis are the even and odd states of the Harmonic Oscillator.

For reflections around other centres,  $x$ , translate the whole HO basis, just as the translation of the ground state generates coherent states.

Now represent arbitrary operators in terms of reflection centres. The assumption that

$$\hat{A} = \int d\mathbf{x} A(\mathbf{x}) 2^L \hat{R}_{\mathbf{x}},$$

leads to

$$\langle\langle \mathbf{x} | A \rangle\rangle = \text{tr} (2^L \hat{R}_{\mathbf{x}}) \hat{A} = \text{tr} \int \frac{d\mathbf{x}'}{(2\pi\hbar)^L} A(\mathbf{x}') (2^L \hat{R}_{\mathbf{x}}) (2^L \hat{R}'_{\mathbf{x}'}) = A(\mathbf{x}).$$

## THE WEYL REPRESENTATION

Again, we use half the coordinates of double phase space, inside a Lagrangian plane that is *a phase space on its own*.

(Balazs and Jennings + geometry)

Semiclassical form of representations of unitary operators in terms of centres or chords is the same as for other Lagrangian planes.

The Weyl representation is a superposition of

$$U_J(\mathbf{x}) = A_J(\mathbf{x}) \exp[iS_J(\mathbf{x})/\hbar],$$

with the centre action is defined as

$$S(\mathbf{x}) = \int_0^{\mathbf{x}} \mathbf{y}(\mathbf{x}') \cdot d\mathbf{x}' = \int_0^{\mathbf{x}} \xi(\mathbf{x}') \wedge d\mathbf{x}',$$

For a symplectic transformation there is only one branch of  $\mathbf{y}(\mathbf{x})$  and the semiclassical form is exact.

In general there may be caustics, where the Lagrangian surface projects singularly onto the  $\mathbf{y}=0$  plane. (the identity plane)

Turning on a Hamiltonian for a small time:

$$S_\epsilon(\mathbf{x}, t = \epsilon) \rightarrow -\epsilon H(\mathbf{x}) + \mathcal{O}(\epsilon^2). \quad \text{No caustics!}$$

The Weyl representation of the Hamiltonian, or any smooth observable, coincides with the corresponding classical function within first order in Planck's constant.

(Not so with the reflection operator)

Their chord representation is not smooth:

$$A(\xi) = \frac{1}{(2\pi\hbar)^L} \int d\mathbf{x} \exp\left(-\frac{i}{\hbar}\xi \wedge \mathbf{x}\right) A(\mathbf{x})$$

This Fourier transform takes the symbol for the identity,  $I(\mathbf{x}) = 1$ , into  $I(\xi) = \delta(\xi)$  and a Taylor series in  $\mathbf{x}$  into a series of derivatives of  $\delta$ -functions.



Some general formulae:

For the trace of an operator, we have the alternative forms:

$$\text{tr } \hat{A} = \text{tr } \hat{I} \hat{A} = \langle\langle \hat{T}_{\xi=0} | A \rangle\rangle = A(\xi = 0) = \int d\mathbf{x} A(\mathbf{x}).$$

The adjoint operator,  $\hat{A}^\dagger$ , is represented by

$$A^\dagger(\mathbf{x}) = [A(\mathbf{x})]^*, \text{ or } A^\dagger(\xi) = [A(-\xi)]^*.$$

Thus, if  $\hat{A}$  is Hermitian,  $A(\mathbf{x})$  is real, though  $A(\xi)$  may well be complex.

The Weyl or chord representation for the product of operators is not obvious, but

$$\text{tr } \hat{A}_2 \hat{A}_1 = \int \frac{d\xi}{(2\pi\hbar)^L} A_2(\xi) A_1(-\xi) = \int \frac{d\mathbf{x}}{(2\pi\hbar)^L} A_2(\mathbf{x}) A_1(\mathbf{x}).$$

# 6. The partial trace

Recall that the representation of operators,  $\hat{A} = |A\rangle\rangle$ , in a given basis, such as  $\langle\langle Q|A\rangle\rangle$ , corresponds to the foliation of the double phase space,  $X = (P, Q)$ , by a set of Lagrangian planes,  $Q = \text{constant}$ . Admitting linear canonical transformations in double phase space, we are free to choose  $Q = (q_-, q_+)$ , or  $Q = \mathbf{x}$ , or  $Q = \mathbf{y} = J\xi$  among others.

In all cases,

$$\langle\langle Q'|Q\rangle\rangle = \delta(Q' - Q)$$

which permits us to identify the expansion coefficient in

$$\hat{A} = |A\rangle\rangle = \int dQ A(Q) |Q\rangle\rangle$$

with  $\langle\langle Q|A\rangle\rangle$ .

the phase space is a product of a pair of phase spaces,  $X = X_1 \otimes X_2$ , each with  $2L_j$  dimensions,

Then decompose the Lagrangian planes chosen as a basis for double phase space as  $Q = Q_1 \otimes Q_2$ , corresponding to, operators  $|Q\rangle\rangle = |Q_1\rangle\rangle \otimes |Q_2\rangle\rangle$ .

Thus the complete  $|Q\rangle\rangle$  representation becomes

$$\hat{A} = |A\rangle\rangle = \int dQ_1 dQ_2 A(Q_1, Q_2) |Q_1\rangle\rangle \otimes |Q_2\rangle\rangle.$$

The definition of the partial trace is then

$$\langle\langle I_2 | A \rangle\rangle = \text{tr}_2 \hat{A} = \text{tr}_2 \hat{I}_2 \hat{A} = \int dQ_1 dQ_2 A(Q_1, Q_2) |Q_1\rangle\rangle \langle\langle I_2 | Q_2 \rangle\rangle,$$

so that

$$A_1(Q_1) = \int dQ_2 A(Q_1, Q_2) \langle\langle I_2 | Q_2 \rangle\rangle$$

defines the  $|Q_1\rangle\rangle$  representation of a reduced operator  $\hat{A}_1$

The different forms of the partial trace depend on the Hilbert –Schmidt product of each basis with the identity.

For the position basis:

$$\langle\langle I | Q \rangle\rangle = \text{tr} \hat{I} |q_-\rangle \langle q_+| = \delta(q_- - q_+),$$

so that

$$\begin{aligned} A_1(Q_1) &= \int dq_- dq_+ A(Q_1, Q_2 = (q_-, q_+)) \delta(q_- - q_+) \\ &= \int dq A(Q_1, (q, q)). \end{aligned}$$

Recall the matrix notation:

$$A(Q_1, (q, q)) = \langle q_{1-}, (q_{2-} = q) | A | q_{1+}, (q_{2+} = q) \rangle$$

In the centre representation, we have simply

$$\langle\langle I | \mathbf{x} \rangle\rangle = \text{tr} \hat{I} (2^L \hat{R}_{\mathbf{x}}) = 1,$$

leading to the phase space projection:

$$A_1(\mathbf{x}_1) = \int d\mathbf{x}_2 A(\mathbf{x}_1, \mathbf{x}_2).$$

The simplest choice turns out to be the chord representation.

Then,  $|I\rangle\rangle = T_{\xi=0}$  is an element of the operator basis, so that

$$\langle\langle I | \mathbf{y} \rangle\rangle = \delta(\mathbf{y}) = \delta(\xi).$$

instead of projecting, we obtain the reduced operator merely by slicing through the chord symbol:

$$A_1(\xi_1) = A(\xi_1, \xi_2 = 0).$$

In the case of the chord function,

$$\chi_1(\xi_1) = (2\pi\hbar)^{L_0} \chi(\xi_1, \xi_2 = 0).$$

Notice that the operators

$$|y_1\rangle\rangle_1 = \hat{T}'_{\xi_1} = \hat{T}_{\xi_1} \otimes \hat{I}_2$$

are a subset of the translation operators  
 a representation in terms of the restricted translation  
 operators,  $\hat{T}'_{\xi_1}$ , would not be complete.

Likewise, we may define the restricted unitary reflections

$$|x_1\rangle\rangle_1 = 2^{L_1} \hat{R}'_{x_1} = \hat{R}_{x_1} \otimes \hat{I}_2,$$

but these do not belong to the centre basis

Even so, we may also define

directly the reduced operator  $\hat{A}_1$  as

$$\hat{A}_1 = \int dx_1 A_1(x_1) |x_1\rangle\rangle_1, \quad \text{with} \quad A_1(x_1) = \text{tr} \hat{A} (2^{L_1} \hat{R}'_{x_1}).$$

## PRODUCT STATES

Let us now specialize to density operators. All the representations that we have been discussing will factor in the case that  $\hat{\rho} = (|\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|)$  is a product pure state. For instance,  $W(\mathbf{x}) = W_1(\mathbf{x}_1) W_2(\mathbf{x}_2)$  and  $\chi(\xi) = \chi_1(\xi_1) \chi_2(\xi_2)$ . These relations may be interpreted in terms of average values of the basis operators, i.e.  $\langle \hat{R}_{\mathbf{x}} \rangle = \langle \hat{R}'_{\mathbf{x}_1} \rangle \langle \hat{R}'_{\mathbf{x}_2} \rangle$  and  $\langle \hat{T}_{\xi} \rangle = \langle \hat{T}'_{\xi_1} \rangle \langle \hat{T}'_{\xi_2} \rangle$ . Thus, a sufficient criterion for the existence of entanglement is that either of these equalities not hold for some centre,  $\mathbf{x}$ , or some chord  $\xi$ .

$$\langle \hat{R}_{\mathbf{x}} \rangle = \langle \hat{R}'_{\mathbf{x}_1} \rangle \langle \hat{R}'_{\mathbf{x}_2} \rangle$$

$$\langle \hat{T}_{\xi} \rangle = \langle \hat{T}'_{\xi_1} \rangle \langle \hat{T}'_{\xi_2} \rangle$$

*Cross correlations imply entanglement*

It is more usual to measure the entanglement by the purity of just one of the partial traces:

$$\text{tr} \hat{\rho}_1^2 = \int d\mathbf{x}_1 [W_1(\mathbf{x}_1)]^2 = \int d\xi_1 |\chi_1(\xi_1)|^2.$$

Why is this the same measure as for subsystem-2 ?

*Use Fourier invariance of quantum correlations:*

$$\begin{aligned} \text{tr} \hat{\rho}_1^2 &= \int d\xi_1 |\chi(\xi_1, \xi_2 = 0)|^2 = \int d\xi_1 \int \frac{d\tau_1}{(2\pi\hbar)^L} |\chi(\tau_1)|^2 e^{i\tau_1 \wedge \xi_1} \\ &= \int d\tau_1 |\chi(\tau_1)|^2 \delta(\tau_1) = \text{tr} \hat{\rho}_2^2. \end{aligned}$$

Reinterpreted in terms of Wigner functions,

$$\int d\mathbf{x}_1 [W_1(\mathbf{x}_1)]^2 = \int d\mathbf{x}_2 [W_2(\mathbf{x}_2)]^2,$$

*Equality is not expected for the second moment of marginal distributions.*



# 7. *Classical* entanglement

We now entangle a simple,  $L=2$ , product state,

$$W_0(\mathbf{x}) = W_0(\mathbf{x}_1)W_0(\mathbf{x}_2) \text{ and } \chi_0(\xi) = \chi_0(\xi_1)\chi_0(\xi_2).$$

This is acted on by the Hamiltonian,

$$H = p_1q_2 - p_2q_1 \text{ (angular momentum } L_3).$$

(classical, or Weyl representation)

Being quadratic, this merely rotates both the  $p$  and the  $q$  coordinates in the argument of  $W(\mathbf{x})$  and  $\chi(\xi)$ .

Then, after a  $\pi/4$  rotation,

The reduced density is just a section, so

$$\chi_1(\xi_1) = \left[ \chi_0\left(\frac{\xi_{p1}}{\sqrt{2}}, \frac{\xi_{q1}}{\sqrt{2}}\right) \right]^2 .$$

To show how *classical* an entanglement can be, choose a simple Gaussian state, the product of HO ground states:

$$W_0(\mathbf{x}_j) = \frac{1}{\pi \hbar} \exp \left[ -\frac{\omega}{\hbar} \mathbf{q}_j^2 - \frac{1}{\hbar \omega} \mathbf{p}_j^2 \right],$$

$$\chi_0(\xi_j) = \frac{1}{2\pi \hbar} \exp \left[ -\frac{\omega}{\hbar} \left( \frac{\xi_{q_j}}{2} \right)^2 - \frac{1}{\hbar \omega} \left( \frac{\xi_{p_j}}{2} \right)^2 \right].$$

Thus, the probability distribution for positions,

$$f(\mathbf{q}) = \int d\mathbf{p} W(\mathbf{x})$$

is also a Gaussian, *with elliptic level curves that are also rotated.*

After rotation and the partial trace:

$$\chi_1(\xi_1) = \chi_0(\sqrt{2}\xi_{p_1}, \sqrt{2}\xi_q) = \chi_0(\sqrt{2}\xi_1).$$

The narrowing of the Gaussian shows that the state is not pure.

*The Wigner function is more intuitive:*

*Obtained by rescaling the symmetric chord function,*

$$W_1(\mathbf{x}_1) = \frac{1}{2} W_0\left(\frac{\mathbf{x}_1}{\sqrt{2}}\right).$$

This broader Gaussian still integrates to one.

It could be obtained as an average over pure Gaussian Wigner functions.

Another confirmation that this is not a pure state is that

$$\text{tr } \hat{\rho}_1^2 = \int d\mathbf{x}_1 [W_1(\mathbf{x}_1)]^2 = \int d\xi_1 |\chi_1(\xi_1)|^2 < 1.$$

Is this a freak?

Nothing could be more *classical* for a start (positive Wigner function) and then a *classical* rotation produces entanglement!

Analyze the parity:

Both the initial state and the rotated density operator commute with  $\hat{A}_0$ . However,  $W_1(0) < \pi\hbar$ , so it does not have pure parity.

There is a finite probability to obtain negative (odd) parity, if such a measurement is performed on subsystem-1.

The same holds for subsystem-2.

All the deductions proceed as before.

In both cases, all the pure states, into which the reduced density operator can be decomposed, have pure parity, but they are not all even.

How have we generated the probability of negative parity measurements in both subsystems?

The crucial point is that the rotated state  $\hat{\rho}'$ , does not commute with either of the partial reflections,  $\hat{R}'_0(1)$  or  $\hat{R}'_0(2)$ , even though it commutes with their product:

$$\hat{R}_0 = \hat{R}'_0(1)\hat{R}'_0(2) = \hat{R}'_0(2)\hat{R}'_0(1).$$

To understand the measurement of either of the partial reflections, we need a common basis for all these operators.

This is just the product of an even-odd basis for subsystem-1 and subsystem-2

For this we have the table:

Since  $\hat{\rho}'$  is even, it must be a superposition of basis states:  $even \otimes even$ , or,  $odd \otimes odd$ .

Evolved state has

**FULL PARITY CORRELATION.**

$even \otimes even \rightarrow even$

$even \otimes odd \rightarrow odd$

$odd \otimes even \rightarrow odd$

$odd \otimes odd \rightarrow even$

*A true entangled state?*

It is correctly described as entangled,  
with all the nonclassicality that this implies.

The secret lies in choosing the property to be measured:  
A position measurement on one of the subsystems  
would not distinguish between this pure quantum state  
and a classical distribution.

Entanglement is not an intrinsic property.  
It is only meaningful for a specific basis.

*like a semiclassical caustic*

If the physical realization of the system  
were the ground state of a 2-D HO,  
then the entanglement producing rotation  
would merely produce a coordinate transformation  
to a completely equivalent system.

It would make no sense to say  
that the original system became entangled,  
while the original system was a mere product,  
unless all measurements be restricted  
to the original coordinate axes.



# 8. Entanglement and decoherence

Now generalize the previous example to a *small* system, with  $L = 1$  and phase space  $x_1$ , and a *large* environment, represented by  $\tilde{x}$ , a  $2(L - 1)$ -dimensional vector.

*Simplify with some drastic assumptions:*

(i) The full Hamiltonian,  $H(x_1, \tilde{x})$  is quadratic.

This means that the interaction,

$H_{\text{int}}$  is at most bilinear in the components of  $x_1$  and  $\tilde{x}$ ,

(ii) No movement in the environment,

except that driven by the interaction,  $H_{\text{int}}(x_1, \tilde{x})$ .

But we allow an internal Hamiltonian,  $H_1(x_1) = x_1 \mathbf{H}_1 x_1$ ,

where  $\mathbf{H}_1$  is a symmetric matrix

- (iii) The full Hamiltonian is periodic in time:  
 The Hamiltonian  $H_1$  is switched  
 on its own for a time  $t_1$ ,  
 then it is substituted by  $H_{\text{int}}$  for a time and so on...

The internal evolution within a single period  
 is given by  $x_1 \rightarrow \mathbf{M}_1 x_1$ , where

$$\mathbf{M}_1 = \exp(t_1 \mathbf{J} H_1)$$

propagates the argument of either the Wigner function,  
 or the chord function.

The environment remains constant in this step,  
 but moves together with  $x_1$  when the interaction is turned on.

Take the limit of short interaction time:

$$x_1 \rightarrow x_1 + \epsilon \dot{x}_1 \text{ and } \tilde{x} \rightarrow \tilde{x} + \epsilon \dot{\tilde{x}}.$$

Then  $\dot{x}_1(\tilde{x})$  does not depend on  $x_1$  and  $\dot{\tilde{x}}(x_1)$  is independent of  $\tilde{x}$ .

a choice of initial state:

$$W_+(\mathbf{x}_1) = \frac{1}{2\pi\hbar(1 + e^{-\eta_1^2/\hbar})} \left[ e^{-(x_1 - \eta_1)^2/\hbar} + e^{-(x_1 + \eta_1)^2/\hbar} + 2e^{-x_1^2/\hbar} \cos \frac{2}{\hbar} x_1 \wedge \eta_1 \right],$$

or

$$\chi_+(\xi_1) = \frac{1}{4\pi\hbar(1 + e^{-\eta_1^2/\hbar})} \left[ e^{-(\xi_1/2 - \eta_1)^2/\hbar} + e^{-(\xi_1/2 + \eta_1)^2/\hbar} + 2e^{-\xi_1^2/4\hbar} \cos \frac{1}{\hbar} \xi_1 \wedge \eta_1 \right]$$

*an even Schroedinger cat*

the full state being its product  
with a multidimensional coherent state,  
centred on the unknown vector,  $\tilde{\eta}$  :

$$W_{\tilde{\eta}}(\tilde{\mathbf{x}}) = \left( \frac{1}{\pi\hbar} \right)^{(L-1)} e^{-(\tilde{\mathbf{x}} - \tilde{\eta})^2/\hbar},$$

$$\chi_{\tilde{\eta}}(\tilde{\xi}) = \left( \frac{1}{2\pi\hbar} \right)^{L-1} e^{i\tilde{\eta} \wedge \tilde{\xi}/\hbar} e^{-\tilde{\xi}^2/4\hbar},$$

i.e the initial state is represented by

$$W(\mathbf{x}) = W_+(\mathbf{x}_1) W_{\tilde{\eta}}(\tilde{\mathbf{x}}), \quad \text{or} \quad \chi(\xi) = \chi_+(\xi_1) \chi_{\tilde{\eta}}(\tilde{\xi}).$$

The state is still a product after the first internal motion, but now

$$W_+(\mathbf{x}_1) \rightarrow W_+(\mathbf{M}_1^{-1}\mathbf{x}_1) \quad \text{and} \quad \chi_+(\xi_1) \rightarrow \chi_+(\mathbf{M}_1^{-1}\xi_1).$$

Turning on the interaction, we now obtain the chord function after a full period

$$\begin{aligned} \chi_\epsilon(\xi) = & \frac{e^{i\bar{\eta} \wedge (\bar{\xi} - \epsilon \dot{\xi}) / \hbar} e^{-(\bar{\xi} - \epsilon \dot{\xi})^2 / 4\hbar}}{2(2\pi\hbar)^L (1 + e^{-\eta_1^2 / \hbar})} \\ & \left[ e^{-(\mathbf{M}_1^{-1}\xi_1 - \epsilon \dot{\xi}_1 / 2 - \eta_1)^2 / \hbar} + e^{-((\mathbf{M}_1^{-1}\xi_1 - \epsilon \dot{\xi}_1 / 2 + \eta_1)^2 / \hbar} \right. \\ & \left. + 2e^{-(\mathbf{M}_1^{-1}\xi_1 - \epsilon \dot{\xi}_1)^2 / 4\hbar} \cos \frac{1}{\hbar} (\mathbf{M}_1^{-1}\xi_1 - \epsilon \dot{\xi}_1) \wedge \eta_1 \right]. \end{aligned}$$

The focus on the chord function now provides the reduced density as merely the section of the chord function for the entire system, but it should be recalled that  $\dot{x}_1(\tilde{x})$  does not depend on  $x_2$  and  $\dot{\tilde{x}}(x_1)$  is independent of  $\tilde{x}$ , so we only keep the latter in the above formula:

$$\chi_\epsilon(\xi_1) = \frac{e^{i\tilde{\eta} \wedge \epsilon \dot{\tilde{\xi}}/\hbar} e^{-\epsilon \dot{\tilde{\xi}}^2/4\hbar}}{4\pi\hbar(1 + e^{-\eta_1^2/\hbar})}$$

$$\left[ e^{-(M_1^{-1}\xi_1/2 - \eta_1)^2/\hbar} + e^{-(M_1^{-1}\xi_1/2 + \eta_1)^2/\hbar} + 2e^{-(M_1^{-1}\xi_1)^2/4\hbar} \cos \frac{1}{\hbar} \xi_1 \wedge \eta_1 \right]$$

Thus the three terms in the square bracket are identical to that in the original product state, the only novelty being the prefactor which leaves the trace invariant. Recall that  $\text{tr } \tilde{\rho}_\epsilon^2$  is proportional to the integral of  $|\chi_\epsilon(\xi)|^2$ , so that the Gaussian envelope leads to  $\text{tr } \tilde{\rho}_\epsilon^2 < 1$ , by reducing the contribution of both the classical Gaussians centred at  $\pm\eta_1$ .

Missing element of decoherence:

*Our ignorance as to the initial state of the environment.*

Assume a Gaussian distribution  
for the coherent states:

$$\widehat{\rho}_{\bar{\sigma}} = \left(\frac{\pi}{\sigma}\right)^L \int d\tilde{\eta} e^{-\tilde{\eta}^2/\sigma^2} \widehat{\rho}_{\tilde{\eta}},$$

so that

$$\chi_{\bar{\sigma}}(\tilde{\xi}) = \left(\frac{\pi}{\sigma}\right)^L \int d\tilde{\eta} e^{-\tilde{\eta}^2/\sigma^2} \chi_{\tilde{\eta}}(\tilde{\xi})$$

Since this will be just a Fourier integral,  
the effect on the reduced chord function  
is to substitute the phase factor by a further Gaussian envelope.

Partial tracing and the initial average combine  
in the **decoherence**:

$$\chi_{\sigma,\varepsilon}(\xi_1) = \chi_+(\xi_1) \exp[-\varepsilon(1 + \varepsilon\sigma)\dot{\xi}^2]$$

The Wigner function is a convolution of a widening Gaussian window with the internally evolved pure state Wigner function.

After a finite time, we obtain the pure state Husimi function:  
**the Wigner function becomes positive.**

This is the same result as integrating the Lindblad equation for the density operator in the chord representation:  
A link to the general theory for Markovian evolution of open quantum systems.

Not so surprising: no need of memory in a single step,  
*but our approximations were not excessively severe.*

Notice that the classical and the quantum motions are not dissipative. Quantum Markovian theory accommodates dissipation, so why is our periodic Hamiltonian conservative of  $\langle \hat{H}_1 \rangle$ ?

Consequence of the fact that  $x_1 \rightarrow x_1 + \varepsilon \dot{x}_1(\tilde{x})$  is a canonical transformation for each  $\tilde{x}$ .

Thus,

$$\begin{aligned} \langle \hat{H}_1 \rangle_\varepsilon &= \int dx W_\varepsilon(x) H_1(x_1) = \int d\tilde{x} \int dx_1 W[x_1(-\varepsilon, \tilde{x}), \tilde{x}] H_1(x_1) \\ &= \int d\tilde{x} \int dx_1 W(x_1, \tilde{x}) H_1[x_1(\varepsilon, \tilde{x})] = \langle \hat{H}_1 \rangle. \end{aligned}$$



Second period:

*Adjust notation:*  $\mathbf{M}_1^{-1} \rightarrow \mathbf{M}$

It is just as easy to evolve a pure state as a mixture (unitarily).

So, at first, we carry on with the evolved pure state, i.e.

We evolve the argument of  $\chi_\epsilon(\xi)$  with the same sequence of symplectic transformations as before.

*But remember: the velocities  $\dot{\xi}$  and  $\dot{\xi}_1$  switch subspaces with respect to  $\tilde{\xi}$  and  $\xi_1$*

Then the next internal propagation leads to

$$\frac{\exp\{i\tilde{\eta} \wedge (\tilde{\xi} - \epsilon\dot{\xi}(\mathbf{M}\xi_1))/\hbar\} \exp\{-(\tilde{\xi} - \epsilon\dot{\xi}(\mathbf{M}\xi_1))^2/4\hbar\}}{2(2\pi\hbar)^L(1 + \exp(-\eta_1^2/\hbar))}$$

$$\left[ \exp\{-[\mathbf{M}(\mathbf{M}\xi_1 - \epsilon\dot{\xi}_1(\tilde{\xi}))/2 - \eta_1]^2/\hbar\} + \exp\{-[(\mathbf{M}(\mathbf{M}\xi_1 - \epsilon\dot{\xi}_1(\tilde{\xi}))/2 + \eta_1]^2/\hbar\} \right. \\ \left. + 2 \exp\{-[\mathbf{M}(\mathbf{M}\xi_1 - \epsilon\dot{\xi}_1(\tilde{\xi}))]^2/4\hbar\} \cos\{\mathbf{M}(\mathbf{M}\xi_1 - \epsilon\dot{\xi}_1(\tilde{\xi})) \wedge \eta_1/\hbar\} \right]$$

Then, turning on the interaction, we have

$$\begin{aligned} \chi_{2\epsilon}(\xi) = & \frac{\exp\{i\tilde{\eta} \wedge [\tilde{\xi} - \epsilon(\dot{\xi}(\xi_1) + \dot{\xi}(\mathbf{M}(\xi_1 + \epsilon\dot{\xi}_1(\tilde{\xi}))))]/4\hbar}}{2(2\pi\hbar)^L(1 + e^{-\eta_1^2/\hbar})} \\ & \exp\{-[\tilde{\xi} - \epsilon(\dot{\xi}(\xi_1) + \dot{\xi}(\mathbf{M}(\xi_1 + \epsilon\dot{\xi}_1(\tilde{\xi}))))]^2/4\hbar\} \\ & \left[ \exp\{-[\mathbf{M}(\mathbf{M}\xi_1 - \epsilon\dot{\xi}_1(\tilde{\xi}))/2 - \epsilon\dot{\xi}_1(\tilde{\xi} - \epsilon\dot{\xi}(\xi_1))/2 - \eta_1]^2/\hbar\} \right. \\ & + \exp\{-[\mathbf{M}(\mathbf{M}\xi_1 - \epsilon\dot{\xi}_1(\tilde{\xi}))/2 - \epsilon\dot{\xi}_1(\tilde{\xi} - \epsilon\dot{\xi}(\xi_1))/2 + \eta_1]^2/\hbar\} \\ & + 2 \exp\{-[\mathbf{M}(\mathbf{M}\xi_1 - \epsilon\dot{\xi}_1(\tilde{\xi})) - \epsilon\dot{\xi}_1(\tilde{\xi} - \epsilon\dot{\xi}(\xi_1))/2]^2/4\hbar\} \\ & \left. \cos\left\{\frac{1}{\hbar}[\mathbf{M}(\mathbf{M}\xi_1 - \epsilon\dot{\xi}_1(\tilde{\xi})) - \epsilon\dot{\xi}_1(\tilde{\xi} - \epsilon\dot{\xi}(\xi_1))/2] \wedge \eta_1\right\} \right]. \end{aligned}$$

and the partial trace slices through the full chord function to obtain:

$$\chi_{2\epsilon}(\xi_1) = \frac{\exp\{i\tilde{\eta} \wedge [-\epsilon(\dot{\xi}(\xi_1) + \dot{\xi}(\mathbf{M}(\xi_1)))]/\hbar\} \exp\{-[-\epsilon(\dot{\xi}(\xi_1) + \dot{\xi}(\mathbf{M}(\xi_1)))]^2/4\hbar\}}{4\pi\hbar(1 + e^{-\eta_1^2/\hbar})} \left[ \begin{aligned} & \exp\{-[\mathbf{M}(\mathbf{M}\xi_1)/2 - \epsilon\dot{\xi}_1(-\epsilon\dot{\xi}(\xi_1))/2 - \eta_1]^2/\hbar\} \\ & + \exp\{-[\mathbf{M}(\mathbf{M}\xi_1)/2 - \epsilon\dot{\xi}_1(-\epsilon\dot{\xi}(\xi_1))/2 + \eta_1]^2/\hbar\} \\ & + 2 \exp\{-[\mathbf{M}(\mathbf{M}\xi_1) - \epsilon\dot{\xi}_1(-\epsilon\dot{\xi}(\xi_1))/2]^2/4\hbar\} \\ & \cos\left\{\frac{1}{\hbar}[\mathbf{M}(\mathbf{M}\xi_1) - \epsilon\dot{\xi}_1(-\epsilon\dot{\xi}(\xi_1))/2] \wedge \eta_1\right\} \end{aligned} \right].$$

Note that the linear phase factor is altered by the displacement of the single environmental coherent state, caused by the system during the first interaction.

Averaging over the same ensemble of initial states eliminates the phase factor, while reinforcing the Gaussian:

$$\begin{aligned}
2\epsilon, \sigma(\xi_1) = & \frac{\exp\{-(1 + \sigma^2)[- \epsilon(\dot{\xi}(\xi_1) + \ddot{\xi}(\mathbf{M}(\xi_1)))]^2 / 4\hbar\}}{4\pi\hbar(1 + e^{-\eta_1^2/\hbar})} \\
& \left[ \exp\{-[\mathbf{M}(\mathbf{M}\xi_1)]/2 - \epsilon\dot{\xi}_1(-\epsilon\ddot{\xi}(\xi_1))/2 - \eta_1]^2/\hbar\} \right. \\
& + \exp\{-[\mathbf{M}(\mathbf{M}\xi_1)]/2 - \epsilon\dot{\xi}_1(-\epsilon\ddot{\xi}(\xi_1))/2 + \eta_1]^2/\hbar\} \\
& + 2 \exp\{-[\mathbf{M}(\mathbf{M}\xi_1) - \epsilon\dot{\xi}_1(-\epsilon\ddot{\xi}(\xi_1))/2]^2/4\hbar\} \\
& \left. \cos\left\{\frac{1}{\hbar}[\mathbf{M}(\mathbf{M}\xi_1) - \epsilon\dot{\xi}_1(-\epsilon\ddot{\xi}(\xi_1))/2] \wedge \eta_1\right\} \right].
\end{aligned}$$

Note the non-Markovian,  $\epsilon\dot{\xi}_1(-\epsilon\ddot{\xi}(\xi_1))$ , term which is of second order in the coupling coefficient: *Markovian theory results from a weak coupling approximation.*

*This kind of deduction as a weak coupling approximation is well known in quantum optics. In which case the internal Hamiltonian is the HO.*

Markovian theory is developed abstractly and is independent of any particular model of the environment.

The motion of a quadratic system that is linearly coupled to the environment is purely classical, *even when dissipative.*

The memory of previous motion of the system, which is relayed back by the environment, can not be accommodated by a differential equation.

The Feynman-Vernon functional is the most natural instrument.

*(Caldeira-Leggett)*

*Chaotic decoherence?*

Generally the quadratic form  $\mathbf{H}_1$  may be elliptic, or hyperbolic (real eigenvalues).

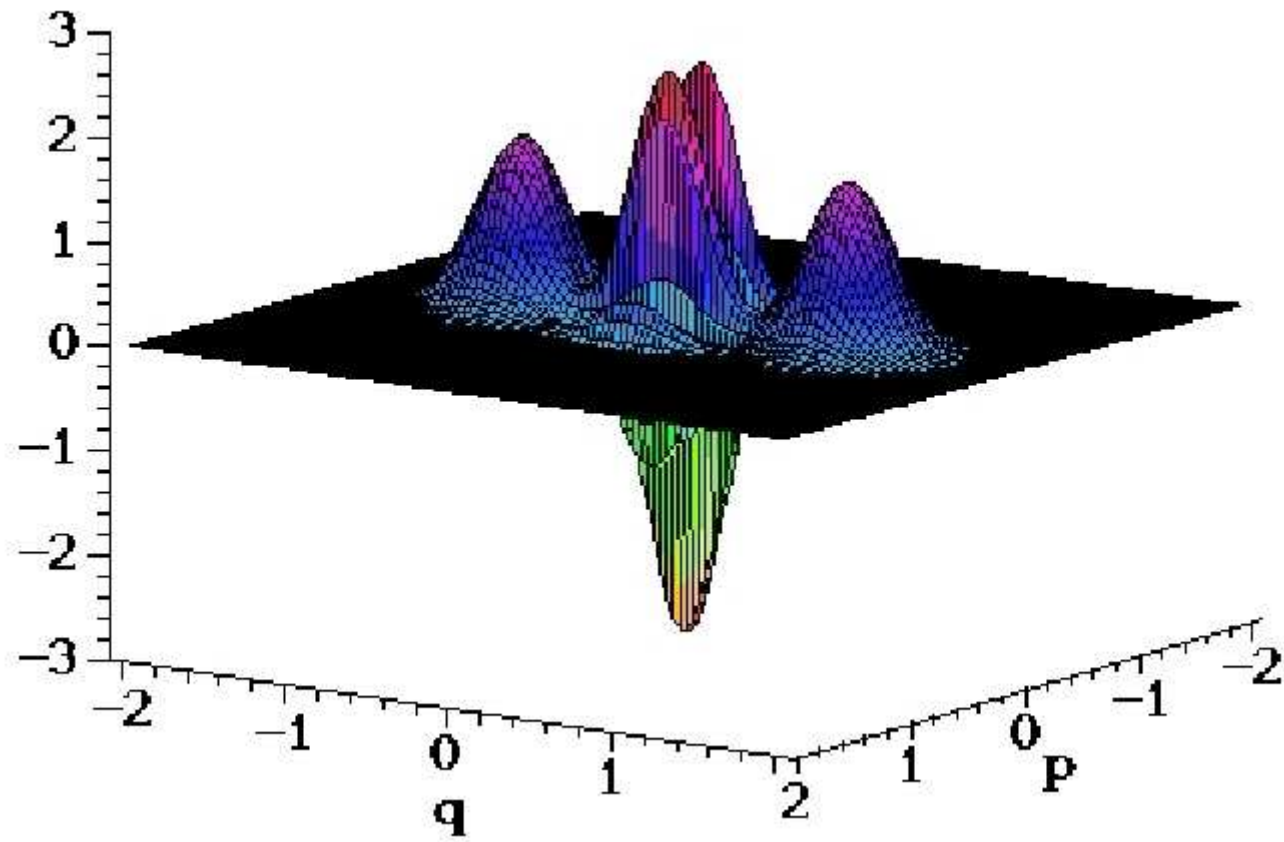
In the latter case, decoherence proceeds much faster.

General hyperbolicity of chaotic systems indicates that they are a bad option for preserving quantum coherence.

What happens if we measure the parity of a very *decohered* density operator?

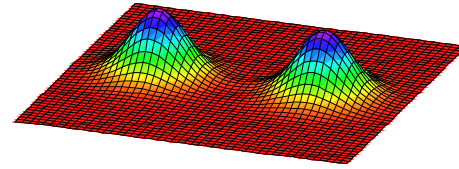
$$W_{\pm}^{\mathbf{X}}(\mathbf{x}) = \frac{1}{2} \frac{W(\mathbf{x}) + W(2\mathbf{X} - \mathbf{x}) \pm 4\Re\left(\tilde{W}(2(\mathbf{x} - \mathbf{X}))\exp\left(-\frac{2i}{\hbar}\mathbf{x} \cdot \mathbf{J}\mathbf{X}\right)\right)}{1 \pm \pi\hbar W(\mathbf{X})}$$

The original state:

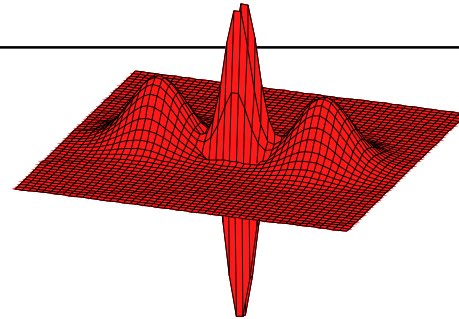




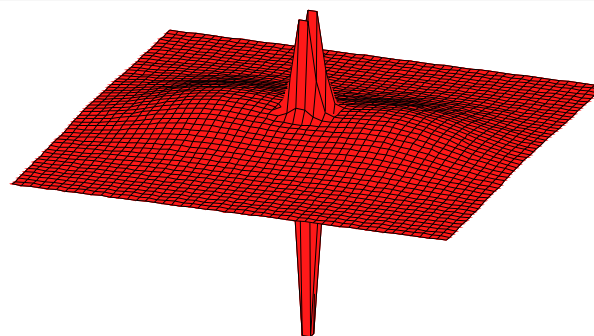
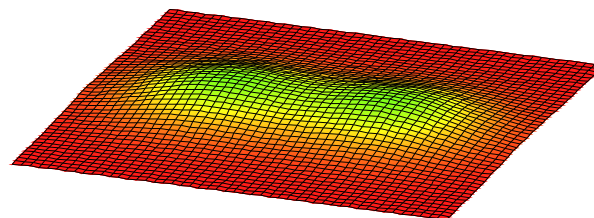
Schroedinger cat,  
mixed by decoherence.



spiky, but mixed cat,  
after parity measurement.



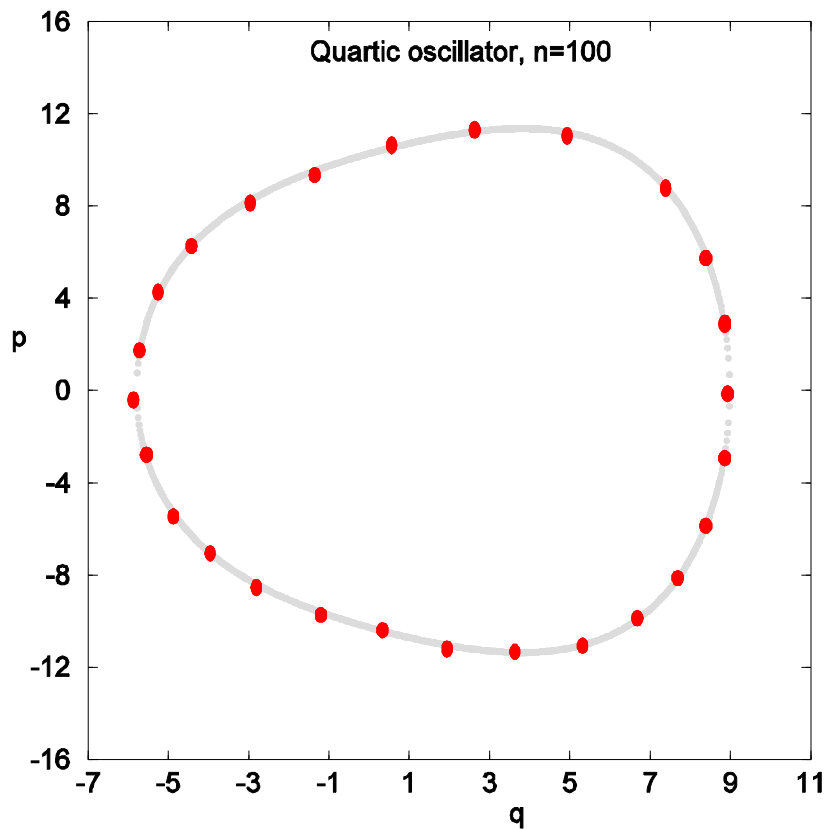
*Parity resurrection  
of a dissolved cat.*



# 9. A semiclassical picture of entanglement

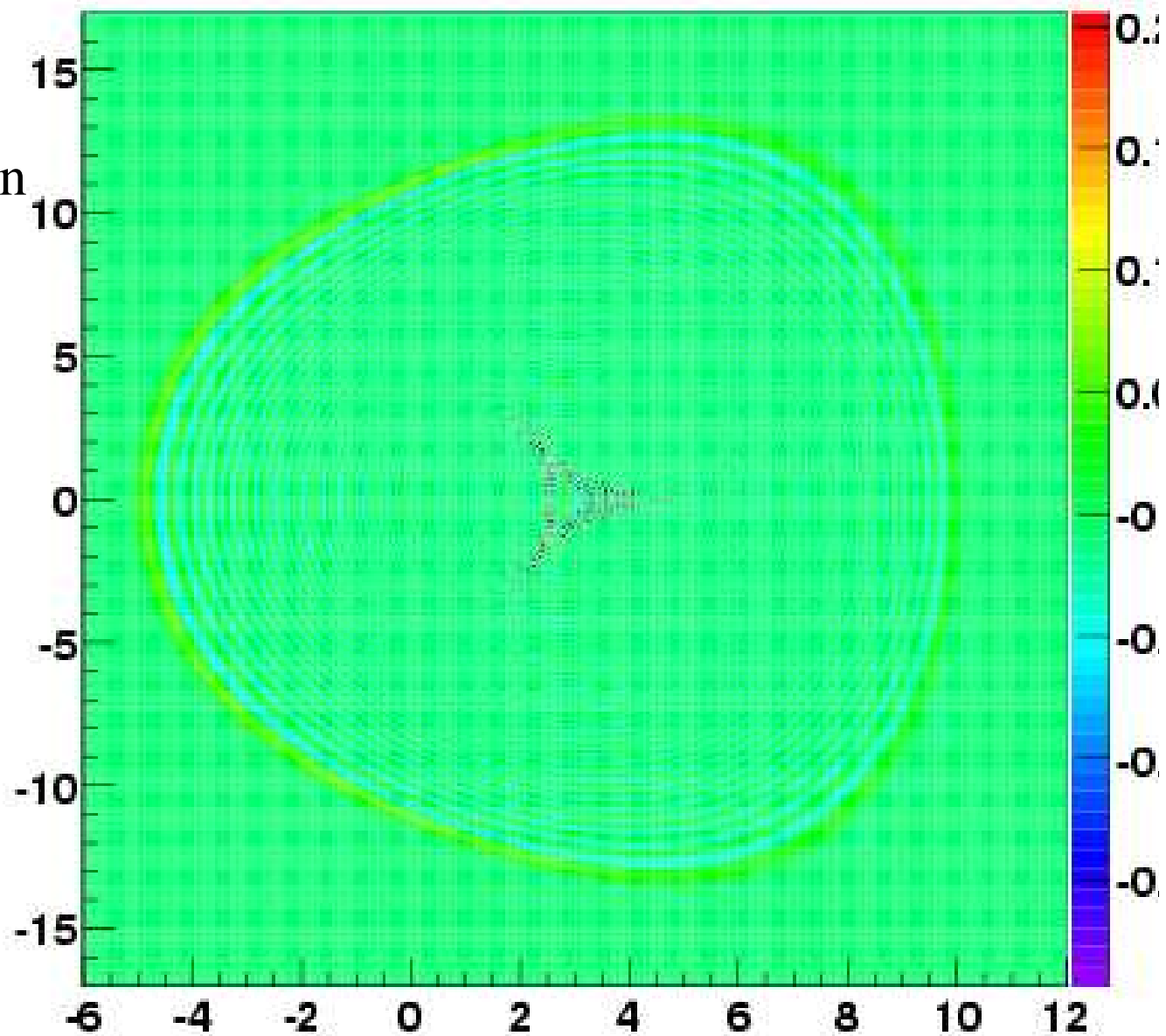
A full theory lies in the future.

Note that the possibility of fitting even the caustics of semiclassical states with Gaussians shows that the examples with Schroedinger cats were not irrelevant.



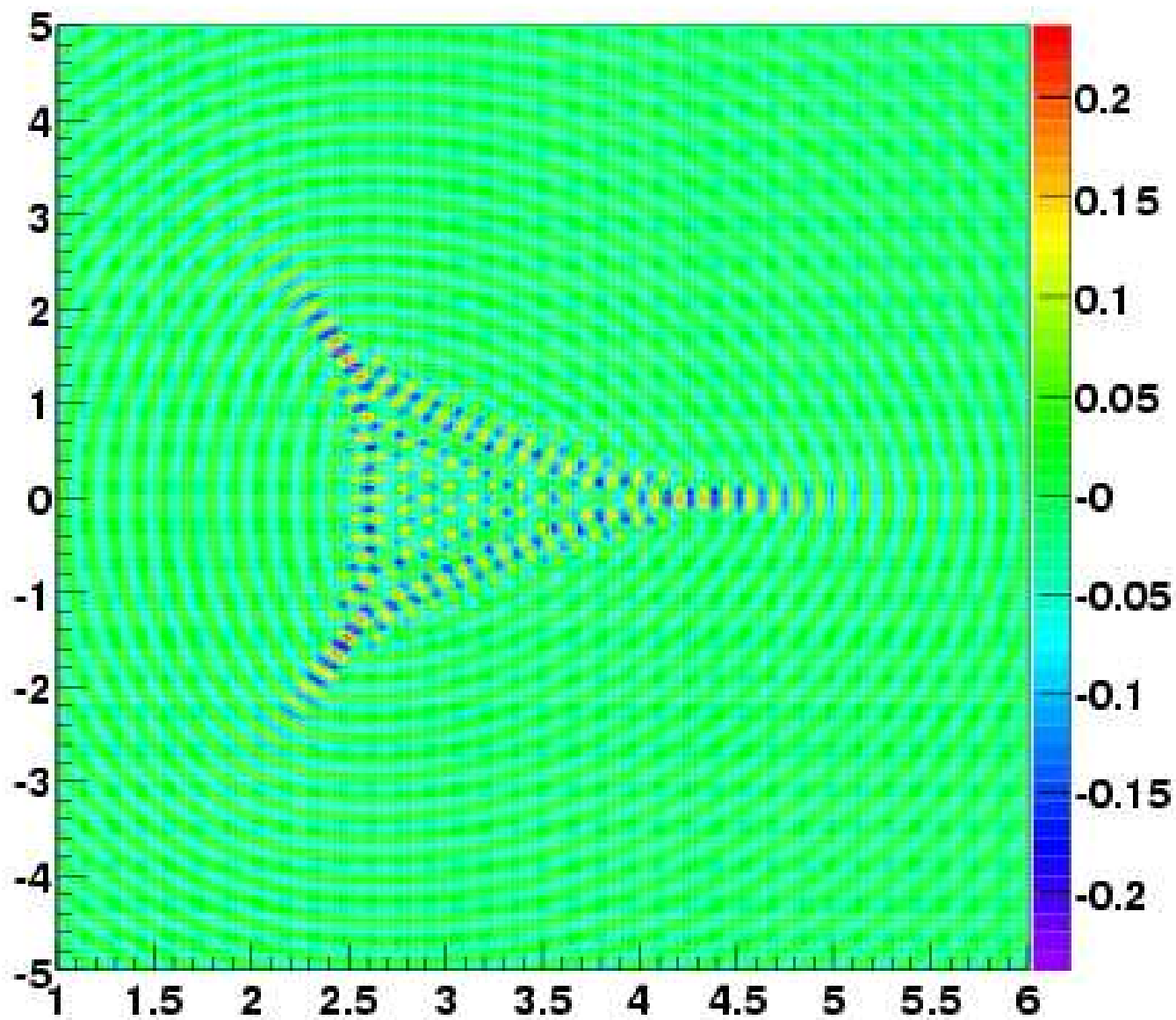
fitted Wigner function  
*72 Gaussians used*

$|n = 100\rangle$



zoom of  
caustic

$|n = 100\rangle$



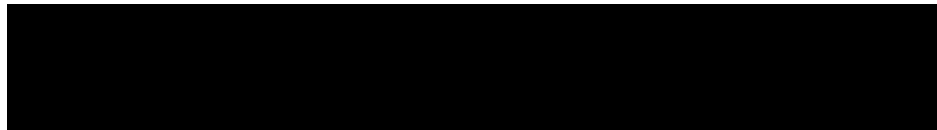
## Semiclassical Wigner and chord functions

Recall that dyadic operators,  $|\psi\rangle\langle\phi|$ , correspond to the Lagrangian surface,  $\lambda_\psi \otimes \lambda_\phi$ , in double phase space.

The pure state density operator,  $\hat{\rho} = |\psi\rangle\langle\psi|$ , is a particular case.

If the state corresponds to an L-D quantized torus in  $(2L)$ -D phase space, then the density operator corresponds to a  $(2L)$ -D torus in  $(4L)$ -D double phase space.

Using centre coordinates,  $\mathbf{x}$ , the Wigner function is obtained just as the centre representation of a unitary operator:



*The problem is to relate the amplitudes and the phases of each branch of the Lagrangian surface to the torus in single phase space.*

Consider again the Fock states:



introducing the asymptotic expression for Laguerre polynomials:

$$\lim_{n \rightarrow \infty} L_n \left( \frac{z^2}{2n} \right) = J_0 \left( \sqrt{2}z \right) ,$$

together with the large argument expansion,

$$J_0(y) \approx \frac{2}{\sqrt{\pi y}} \cos \left( y - \frac{\pi}{4} \right) ,$$

which is already in semiclassical form.

Every point on the double torus represents a pair of points on the quantized curve.

The conjugate variable to  $\mathbf{x} = (x_+ + x_-)/2$  is  $\mathbf{y} = \mathbf{J}\xi$ , where  $\xi = x_+ - x_-$  is a geometrical torus chord.

the exchange of  $x_+$  with  $x_-$  produces a new chord

Thus, the 2-D Lagrangian surface is symmetric with respect to the  $y=0$  plane:  
the semiclassical Wigner function is always a cosine.

*Close to a convex quantized curve there is a single pair of chords.*

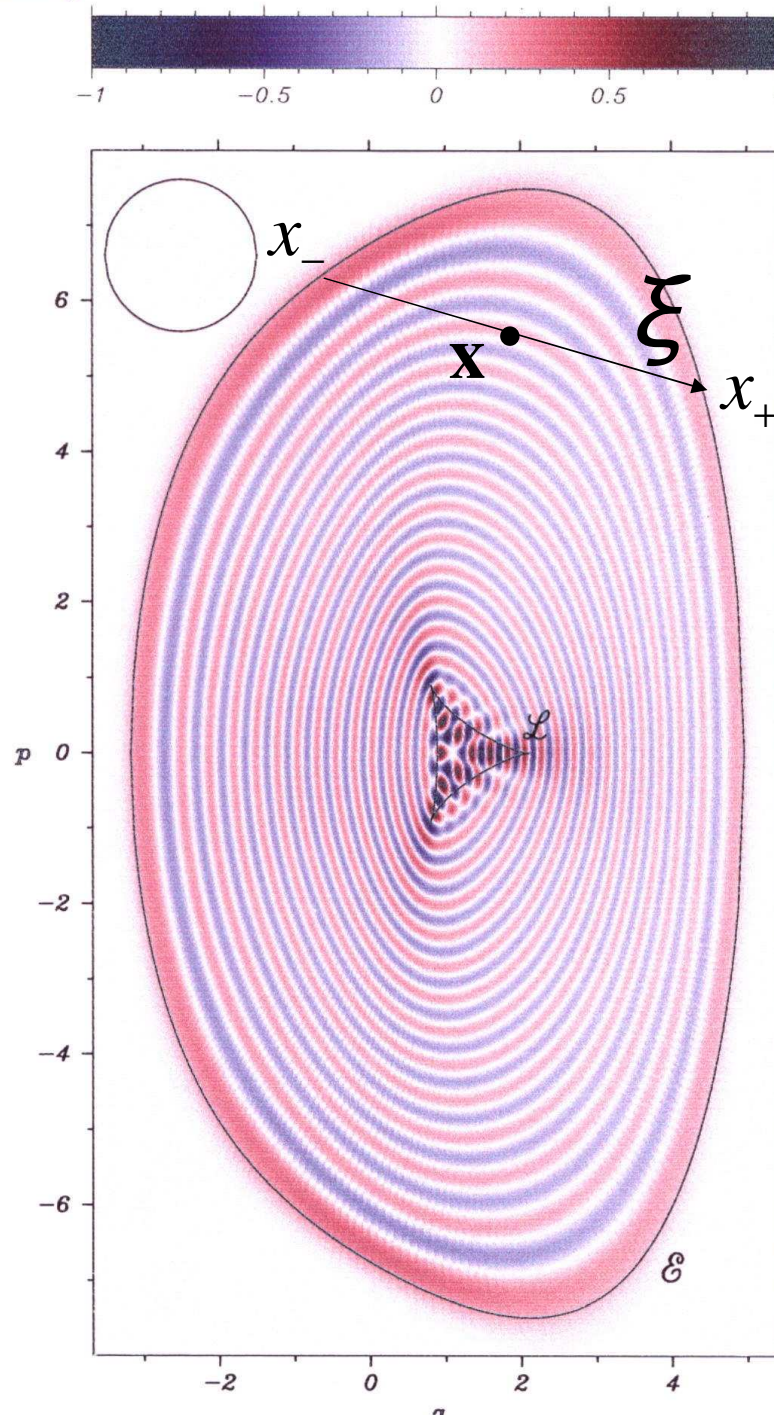


This is the  $y=0$  plane.

We can think of the quantized curve as lying in the  $x_-$  space, the  $x_+$  space, or in the centre,  $\mathbf{x}$  space, i.e. as the intersection of the Lagrangian surface with the  $y=0$  plane .

A pair of Lagrangian sheets are joined along this curve, so it is a **caustic** of the Wigner function.

The L curve is also a caustic, where chords coalesce:  
A remnant of the central Fock peak.



*How can we construct the chords centred on a known centre,  $\mathbf{x}$  ?*

Recall that, for a pure state,

$$W_\psi(\mathbf{x}) = (\pi\hbar)^{-1} \langle \psi | (\hat{R}_\mathbf{x} | \psi \rangle) \rangle$$

*so, reflect the classical torus that corresponds to  $(\hat{R}_\mathbf{x} | \psi \rangle)$ .*

The intersections of the quantized curve,  $\lambda_\psi$ ,  
with its reflection,  $R_\mathbf{x}(\lambda_\psi)$ ,  
determine all chords that are centred on  $\mathbf{x}$ .

Likewise, the fact that

$$\chi_\psi(\xi) = (2\pi\hbar)^{-1} \langle \psi | (\hat{T}_\xi | \psi \rangle) \rangle$$

relates to the fact that all the centres, corresponding to a given chord,  
are specified by the classical translation,  $T_\xi(\lambda_\psi)$ ,  
of the quantized curve.

## *Phase of the semiclassical Wigner function?*

General principle:

The phase difference between a pair of semiclassical contributions to the overlap,  $\langle \psi | \phi \rangle$ , of quantum states is the area sandwiched between  $\lambda_\psi$  and  $\lambda_\phi$ , divided by  $\hbar$ .

*(Littlejohn)*

**All these constructions can be generalized to higher dimensions.**

Caustics of the Wigner function, where chords coalesce;  
caustics of the chord function, where centres coalesce.

*For open curves, the chord function may be simpler:*

*A single branch, no interference, no caustics.*

## Evolution of the doubled torus

The classical Hamiltonian,  $H(\mathbf{x})$ , accounts for the classical evolution of both  $x_-$  and  $x_+$ . Due to the change of sign,  $p_- \rightarrow -p_-$ , the double phase space Hamiltonian has to be

$$\mathbb{H}(X) = H(x_+) - H(x_-) = H(\mathbf{x} - \mathbf{J}\mathbf{y}/2) - H(\mathbf{x} + \mathbf{J}\mathbf{y}/2).$$

This Hamiltonian will always preserve the product form of the geometric structures in each of the phase spaces  $x_+$ , but it will not preserve initial products within each of these in the general case that the single Hamiltonian  $H(x)$  has coupling terms between different degrees of freedom. The semiclassical theory of evolution in double phase space has recently led to the definition of caustic-free propagators for the Wigner and chord functions. These are defined in terms of the propagation of the reflection operators, or the translation operators, instead of directly evolving the density operator itself.

## Classical separability

*disentangling...*

Any quantum state in a product Hilbert space can be made separable by a unitary transformation.

Can any torus in a product phase space be separated by a canonical transformation?

Yes, by transforming to **action-angle variables**, though this may not be exactly equivalent to any quantum transformation.

What about *ergodic eigenstates* of a chaotic Hamiltonian?

These are states that correspond to an entire energy shell (3-D), instead of a Lagrangian torus (2-D) in a 4-D phase space.

No complete semiclassical theory, but  
**Shnirelman's theorem:**

**For most of the chaotic eigenstates,  
averages of smooth observables are obtained  
as if the Wigner function were  $\delta(H(\mathbf{x}) - E_n)$ .**

No canonical transformation will transform  
a 3-D sphere into a 2-D torus. (*let alone separable*)  
Is the corresponding quantum state *harder to disentangle*?

QUANTUM CHAOLGY (M.V. Berry)

*The search for quantum effects of classical chaos...*