Multipartite Bell’s Inequalities

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ISSQUI05, Dresden
Bell’s Theorem

✓ Quantum Probabilities

✓ Underlying Theory
  reproduces quantum probabilities as averages over inaccessible variables

✓ Hidden Variable Theories
  hidden variables describe the properties of physical objects

✓ Bell’s Theorem
  it is impossible to explain all quantum mechanical expectations
  with local hidden-variable theories

☐ J. S. Bell, Physics 1, 195 (1964).
Clauser-Horne-Shimony-Holt

- **realism**
  undone measurements have well-defined, yet unknown, results

- **locality**
  no “action at a distance”

- For each experimental run:
  \[ A_1[B_1 + B_2] + A_2[B_1 - B_1] = \pm 2 \]

- Average over many runs:
  \[ -2 \leq \langle A_1B_1 \rangle + \langle A_1B_2 \rangle + \langle A_2B_1 \rangle - \langle A_2B_2 \rangle \leq 2 \]
  \[ |E_{11} + E_{12} + E_{21} - E_{22}| \leq 2 \]

Quantum Prediction

✓ Take the state $|\psi^-\rangle = 1/\sqrt{2}[|0\rangle|1\rangle - |1\rangle|0\rangle]$

✓ Quantum correlation function:

$$E_{kl}^{QM} = -\cos \varphi_{kl}$$

✓ All measurements in one plane, they are determined by the angle relative to some fixed axis

$$a_1 = 0 \quad b_1 = \pi / 4$$
$$a_2 = \pi / 2 \quad b_2 = -\pi / 4$$

✓ CHSH expression:

$$\left| E_{11} + E_{12} + E_{21} - E_{22} \right| \leq 2$$

$$\cos(\pi / 4) + \cos(\pi / 4) + \cos(\pi / 4) - \cos(3\pi / 4) = 2\sqrt{2}$$
Bell’s inequalities = facets of the polytope

N&S condition for local realistic model is a set of inequalities
Two Qubits

✓ For each experimental run:
\[
\sum_{s_1, s_2 = \pm 1} S(s_1, s_2)(A_1 + s_1 A_2)(B_1 + s_2 B_2) = \pm 4
\]

✓ Average over many runs:
\[
\left| \sum_{s_1, s_2 = \pm 1} S(s_1, s_2)[E_{11} + s_2 E_{12} + s_1 E_{21} + s_1 s_2 E_{22}] \right| \leq 4
\]

✓ There are as many inequalities as different sign functions

✓ The complete set is equivalent to the single inequality:
\[
\sum_{s_1, s_2 = \pm 1} \left| E_{11} + s_2 E_{12} + s_1 E_{21} + s_1 s_2 E_{22} \right| \leq 4
\]

✓ Works for arbitrary number of qubits and two measurement settings

Nonclassical States

✓ General N-qubit state:

$$\rho = \frac{1}{2^N} \sum_{x_1, \ldots, x_N=0}^{3} T_{x_1 \ldots x_N} \sigma_{x_1}^{1} \otimes \ldots \otimes \sigma_{x_N}^{N}$$

✓ Quantum correlation function:

$$E_{k_1 \ldots k_N}^{QM} = Tr[\rho (\vec{n}_{k_1} \cdot \vec{\sigma} \otimes \ldots \otimes \vec{n}_{k_N} \cdot \vec{\sigma})] = \sum_{x_1, \ldots, x_N=1}^{3} T_{x_1 \ldots x_N} (\vec{n}_{k_1})_{x_1} \ldots (\vec{n}_{k_N})_{x_N}$$

✓ Condition for the general inequality to hold:

$$\max_{x_1, \ldots, x_N=1} \sum_{x_1, \ldots, x_N}^{2} T_{x_1 \ldots x_N}^2 \leq 1$$

maximization is taken over all local measurement directions

✓ For more than 2 parties this condition is only the necessary one violation of it implies violation of the inequality

✓ There are pure entangled states which do not violate the inequality

✓ Three-particle $4 \times 4 \times 2$ case

\[
A_{12,S'} \equiv \sum_{s_1,s_2 = \pm 1} S'(s_1, s_2)(A_1 + s_1 A_2)(B_1 + s_2 B_2) = \pm 4
\]

\[
A_{34,S''} \equiv \sum_{s_1,s_2 = \pm 1} S''(s_1, s_2)(A_3 + s_1 A_4)(B_3 + s_2 B_4) = \pm 4
\]

\[
\sum_{s_1,s_2 = \pm 1} S(s_1, s_2)(A_{12,S'} + s_1 A_{34,S''})(C_1 + s_2 C_2) = \pm 16
\]

✓ There are $(2^4)^3$ inequalities
   they are generated by one inequality with $S, S', S''$ non-factorable

✓ Incomplete set

✓ Tight inequalities

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More Nonclassical States

✓ N&S for the 4x4x2 inequality to hold:

$$\max \sum_{m=1}^{2} \sum_{k_m, l_m=1}^{2} T_{k_m l_m m}^2 \leq 1$$

✓ Necessary for the 2x2x2 case:

$$\max \sum_{k,l,m=1}^{2} T_{k l m}^2 \leq 1$$

✓ New condition is more demanding
the settings $k_1, l_1$ do not have to be equal to $k_2, l_2$

✓ The generalized GHZ states
violate new inequality for all $\alpha$ and arbitrary number of particles

$$|\psi\rangle = \cos \alpha |0...0\rangle + \sin \alpha |1...1\rangle$$