Bose-Einstein condensation and a two-dimensional walk model

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Motivation

- If we consider:
  1) Driven-Diffusive Systems
  2) Zero-Range Processes
  3) Lattice Path Models

  are these three systems related? Sharing a common partition function, observable ...

- What is the role of the Matrix Product Ansatz?
I) Open boundary conditions

- PASEP
- Steady-state of PASEP as a superposition of single-shock measures
- Steady-state of PASEP as a superposition of multiple-shock measures
- Mapping onto a lattice path model
I) Open boundary conditions

- PASEP
- Steady-state of PASEP as a superposition of single-shock measures
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- Mapping onto a lattice path model

II) Periodic boundary conditions

- A simple driven-diffusive system → Mapping onto a zero-range process → Mapping onto a lattice path model
- A generalized driven-diffusive system → Mapping onto a zero-range process → Mapping onto a lattice path model
Partially Asymmetric Simple Exclusion Process with open boundaries:
Partially Asymmetric Simple Exclusion Process with open boundaries:

\[ x^{1-d} = \kappa_+ (\beta, \delta) \kappa_+ (\alpha, \gamma), \quad d = 1, 2, 3, \ldots \]

\[ \kappa_+ (u, v) = \frac{-u + v + 1 + \sqrt{(u - v - 1)^2 + 4uv}}{2u} \]
Steady-state of PASEP

For $d = 2$ the steady-state can be written as a linear superposition of single-shock measures with random walk dynamics:

$$|k\rangle = \left( \begin{array} { c c } 1 - \rho_1 & \rho_1 \\ \rho_1 & \rho_2 \end{array} \right)^{\otimes k} \otimes \left( \begin{array} { c c } 1 - \rho_2 & \rho_2 \\ \rho_2 & \rho_1 \end{array} \right)^{\otimes N-k}$$

A shock:
Steady-state of PASEP

For $d = 2$ the steady-state can be written as a linear superposition of single-shock measures with random walk dynamics:

$$|k⟩ = \left( \frac{1 - \rho_1}{\rho_1} \right)^k \otimes \left( \frac{1 - \rho_2}{\rho_2} \right)^{N-k}$$

A shock:

The steady-state can also be obtained using the matrix product method:

$$|P^*⟩ = \frac{1}{Z_N} \sum_{k=0}^{N} c_k |k⟩$$
Steady-state of PASEP

For $d = 2$ the steady-state can be written as a linear superposition of single-shock measures with random walk dynamics:

$$|k⟩ = \left( \begin{array}{c} 1 - \rho_1 \\ \rho_1 \end{array} \right) \otimes k \left( \begin{array}{c} 1 - \rho_2 \\ \rho_2 \end{array} \right)^{\otimes N-k}$$

A shock:

$$\begin{array}{c}
\rho_1 \\
1 \\
k \\
N \\
\rho_2
\end{array}$$

The steady-state can also be obtained using the matrix product method:

$$|P^*⟩ = \frac{1}{Z_N} \sum_{k=0}^{N} c_k |k⟩ = \frac{1}{Z_N} \langle ⟨W| \left( \begin{array}{c} E \\ D \end{array} \right)^{\otimes N} |V⟩\rangle$$
Steady-state of PASEP

The partition function of the system:

\[ Z_N = \sum_{k=0}^{N} c_k = \langle \langle W | (D + E)^N | V \rangle \rangle \]
Steady-state of PASEP

The partition function of the system:

\[ Z_N = \sum_{k=0}^{N} c_k = \langle \langle W | (D + E)^N | V \rangle \rangle \]

The matrix representation for \( d = 2 \):

\[
E = \begin{pmatrix}
(1 - \rho_1) & d_0 \\
\delta_l & \delta_r (1 - \rho_2)
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
\rho_1 & -d_0 \\
0 & \delta_l \delta_r \rho_2
\end{pmatrix}
\]

Question!
The lattice path model is defined on a rotated square lattice as follows:

- Assign the weight $\frac{\delta r}{\delta l}$ to each upward step
- Assign the weight 1 to each downward step except those steps which end on the horizontal axis.

![Diagram of the lattice path model](image)
PASEP mapped onto a lattice path model

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$$Z_N = \langle L | T^N | R \rangle$$
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$$Z_N = \langle L | T^N | R \rangle$$

$$Z_N = \langle \langle W | (D + E)^N | V \rangle \rangle$$

For an arbitrary $d$:

$$x^{1-d} = \kappa_+ (\beta, \delta) \kappa_+ (\alpha, \gamma)$$

one can define a multiple-shock measure:

which shares a common partition function with a multiple-transit lattice path:

A simple disordered DDS

A disordered Driven Diffusive System (DDS) defined on a lattice of length $N$ consisting of $M - 1$ first-class particles in the presence of a second-class particle.

\[
1 \emptyset \rightarrow \emptyset 1 \text{ with rate } 1
\]

\[
2 \emptyset \rightarrow \emptyset 2 \text{ with rate } p
\]
A simple disordered DDS

A disordered Driven Diffusive System (DDS) defined on a lattice of length $N$ consisting of $M - 1$ first-class particles in the presence of a second-class particle.

A Zero Range Process (ZRP) defined on a lattice of length $M$ consisting of $N - M$ particles. The particles in the first box leave it with the rate $p$.

1 $\emptyset$ $\rightarrow$ $\emptyset$ 1 with rate 1
2 $\emptyset$ $\rightarrow$ $\emptyset$ 2 with rate $p$

$\emptyset$ $\rightarrow$ $\emptyset$ 1 with rate 1
$\emptyset$ 2 with rate $p$

A simple disordered DDS

Steady-state as a matrix product state

\[
P(\{n_1, n_2, \cdots, n_M\}) = \frac{1}{Z_{N,M}} \text{Tr}(D' E^{n_1} D E^{n_2} \cdots D E^{n_M})
\]

\[
D' \rightarrow 2 , \quad D \rightarrow 1 , \quad E \rightarrow \emptyset
\]
A simple disordered DDS

Steady-state as a matrix product state

\[
\begin{array}{cccccccc}
2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
& n_1 & & & & n_2 & & \\
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& n_3 & & & & \cdots & & \\
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
& n_M & & & & & & \\
\end{array}
\]

\[P(\{n_1, n_2, \cdots, n_M\}) = \frac{1}{Z_{N,M}} \text{Tr}(D'E^{n_1}DE^{n_2} \cdots DE^{n_M})\]

\[D' \rightarrow 2 , \ D \rightarrow 1 , \ E \rightarrow \emptyset\]

These operators satisfy a quadratic algebra:

\[pD'E = D', \ DE = D.\]

The matrix representation of this algebra is:

\[D' = \sum_{i=0}^{\infty} p^{-i} |0\rangle\langle i|, \ D = \sum_{i=0}^{\infty} |0\rangle\langle i|, \ E = \sum_{i=0}^{\infty} |i+1\rangle\langle i|\]
A simple disordered DDS

Canonical partition function

\[ Z_{N,M}(p) = \sum_{\{n_i\}} \delta\left( \sum_{i=1}^{M} n_i - N + M \right) \text{Tr}(D' E^{n_1} D E^{n_2} \cdots D E^{n_M}) \]

\[ = \sum_{i=0}^{N-M} \binom{N-i-2}{M-2} p^{-i} \]

Two phases: Bose-Einstein condensation!

Depending on the values of \( p \) and \( \rho = \frac{M}{N} \), the system has two phases:

\[ \langle n_1 \rangle \sim \begin{cases} \mathcal{O}(N) & \text{for } p < 1 - \rho, \\ \mathcal{O}(1) & \text{for } p > 1 - \rho. \end{cases} \]

A simple disordered DDS

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A simple disordered DDS

Question!
Is there an equivalent lattice path model?
A simple disordered DDS

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Is there an equivalent lattice path model?

**An equivalent lattice path model**

A lattice path model defined on \( \mathbb{Z}^2_+ = \{(i,j) : i, j \geq 0 \text{ are integers}\} \).

- From \((i,j)\) to \((i+1,j+1)\) with a weight \(\frac{1}{p}\).
- From \((i,j)\) to \((i+1,0)\) with a weight \(zp^j\).
A simple disordered DDS

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- From \((i,j)\) to \((i+1,0)\) with a weight \(zp^j\).

A transfer matrix \(T\) can be defined:

\[
T|j\rangle = zp^j|0\rangle + \frac{1}{p}|j+1\rangle
\]

\[
T^{N-1}|0\rangle = \sum_{j=0}^{N-2} \frac{z(z+1)^{N-j-2}}{p^j} |j\rangle + \frac{1}{p^{N-1}}|N-1\rangle
\]
A simple disordered DDS

The grand-canonical partition function of the model:

\[ Z_N(p, z) = \sum_{j=0}^{\infty} \langle j | T^{N-1} | 0 \rangle = \sum_{i=1}^{N-1} \sum_{j=0}^{N-i-1} \left( \begin{array}{c} N-j-2 \\ i-1 \end{array} \right) p^{-j} z^i + \frac{1}{p^{N-1}} \]
A simple disordered DDS

The grand-canonical partition function of the model:

\[ Z_N(p, z) = \sum_{j=0}^{\infty} \langle j | T^{N-1} | 0 \rangle = \sum_{i=1}^{N-1} \sum_{j=0}^{N-i-1} \binom{N-j-2}{i-1} p^{-j} z^i + \frac{1}{p^{N-1}} \]

The canonical partition function of the model in which the total number of upward steps is exactly \( N - M \). This is given by the coefficient \( z^{M-1} \).

The canonical partition function of the model

\[ Z_{N,M}(p) = \sum_{i=0}^{N-M} \binom{N-i-2}{M-2} p^{-i} \]

\[ P_{N,M}(j) = \frac{1}{Z_{N,M}(p)} \binom{N-j-2}{M-2} p^{-j}. \]
A simple disordered DDS

The mean height after $N - 1$ steps ($\rho = \frac{M}{N}$):

$$\langle h \rangle = \sum_{j=0}^{N-M} j \ P_{N,M}(j) \approx \begin{cases} N(1 - \frac{\rho}{1-p}) & \text{for } p < 1 - \rho, \\ \frac{1-\rho}{p-1+\rho} & \text{for } p > 1 - \rho. \end{cases}$$
A simple disordered DDS

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\end{cases}$$

Grand-canonical partition function of the lattice path model

$$Z_N(p, z) = \sum_{j=0}^{\infty} \langle j | T^{N-1} | 0 \rangle$$
A simple disordered DDS

The mean height after $N - 1$ steps ($\rho = \frac{M}{N}$):

$$\langle h \rangle = \sum_{j=0}^{N-M} j \; P_{N,M}(j) \simeq \begin{cases} N(1 - \frac{\rho}{1-p}) & \text{for } p < 1 - \rho, \\ \frac{1-\rho}{p-1+\rho} & \text{for } p > 1 - \rho. \end{cases}$$

Grand-canonical partition function of the lattice path model

$$Z_N(p, z) = \sum_{j=0}^{\infty} \langle j | T^{N-1} | 0 \rangle$$

Grand-canonical partition function of the disordered DDS

$$Z_N(p, z) = Tr(D'(E + zD)^{N-1}) = \sum_{j=0}^{\infty} \langle j | C^{N-1} | 0 \rangle$$

A generalized disordered DDS defined on a lattice of length $N$ consisting of $M - 1$ first-class particles in the presence of a second-class particle.
A generalized disordered DDS defined on a lattice of length $N$ consisting of $M - 1$ first-class particles in the presence of a second-class particle.

A generalized ZRP defined on a lattice of length $M$ consisting of $N - M$ particles. The particles in the first box leave it with the rate $p$.

\[
\mu \begin{array}{cccc}
0 & \cdots & 0 & \mu' \\
\mu' & \mu' & \mu' & \\
n_{\mu} & n_{\mu} & n_{\mu-1} & \\
\end{array}
\]

A generalized disordered DDS

Steady-state as a matrix product state

\[
P(\{n_1, n_2, \ldots, n_M\}) = \frac{1}{Z_{N,M}} \text{Tr}(D_2 E^{n_1} D_1 E^{n_2} \cdots D_1 E^{n_M})
\]

\[D_2 \rightarrow 2 \ , \ D_1 \rightarrow 1 \ , \ E \rightarrow \emptyset\]
A generalized disordered DDS

Steady-state as a matrix product state

\[
P(\{n_1, n_2, \cdots, n_M\}) = \frac{1}{Z_{N,M}} \text{Tr}(D_2 E^{n_1} D_1 E^{n_2} \cdots D_1 E^{n_M})
\]

\[D_2 \rightarrow 2, \ D_1 \rightarrow 1, \ E \rightarrow \emptyset\]

These operators satisfy an algebra:

\[D_\mu E^{n_\mu} D_{\mu'} = f_\mu(n_\mu) D_{\mu'} \text{ for } \mu, \mu' = 1, 2 \text{ where } f_\mu(n_\mu) = \prod_{m=1}^{n_\mu} \frac{1}{u_\mu(m)}\]

The matrix representation of this algebra is:

\[D_\mu = \sum_{i=0}^{\infty} f_\mu(i) |0\rangle \langle i| \text{ (for } \mu = 1, 2), \ E = \sum_{i=0}^{\infty} |i + 1\rangle \langle i|\]
A generalized disordered DDS

The grand-canonical partition function

\[ Z_N(z) = Tr(D_2 C^{N-1}) = \sum_{i=0}^{N-1} f_2(i) \langle i | C^{N-1} | 0 \rangle \]

where \( C = E + zD_1 \)

and that we have used:

\[ C|i\rangle = zf_1(i)|0\rangle + |i + 1\rangle \]

Question!
Is there an equivalent lattice path model?
A generalized disordered DDS

A generalized WM

A lattice path model defined on \( \mathbb{Z}^2_+ = \{(i, j) : i, j \geq 0 \text{ are integers}\} \).

- From \((i, j)\) to \((i + 1, j + 1)\) with a weight 1.
- From \((i, j)\) to \((i + 1, 0)\) with a weight \(zf_1(j)\).
- A weight \(f_2(j)\) is assigned to the point \((N - 1, j)\).
A generalized disordered DDS

A generalized WM

A lattice path model defined on $\mathbb{Z}_+^2 = \{(i, j) : i, j \geq 0 \text{ are integers}\}$.

- From $(i, j)$ to $(i + 1, j + 1)$ with a weight 1.
- From $(i, j)$ to $(i + 1, 0)$ with a weight $zf_1(j)$.
- A weight $f_2(j)$ is assigned to the point $(N - 1, j)$.

The grand-canonical partition function:

$$Z_N(z) = \sum_{j=0}^{N-1} f_2(j) \langle j | T^{N-1} | 0 \rangle \quad \text{where} \quad T = E + zD_1$$

$$D_1 = \sum_{i=0}^{\infty} f_1(i) | 0 \rangle \langle i | \quad , \quad E = \sum_{i=0}^{\infty} | i + 1 \rangle \langle i |$$

A generalized disordered DDS

Grand-canonical partition function of the lattice path model

\[ Z_N(z) = \sum_{j=0}^{N-1} f_2(j) \langle j | T^{N-1} | 0 \rangle \quad \text{where} \quad T = E + zD_1 \]

Grand-canonical partition function of the disordered DDS

\[ Z_N(z) = \text{Tr}(D_2 C^{N-1}) = \sum_{i=0}^{N-1} f_2(i) \langle i | C^{N-1} | 0 \rangle \quad \text{where} \quad C = E + zD_1 \]

Thank you for your attention!