Aging in the one-dimensional coagulation-diffusion process

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Introduction

- Ageing phenomena: from simple magnets to directed percolation
- Two-time observables, Fluctuation-Dissipation ratio
- Model

One-time quantities

- Training example (Method used)
- Influence of the initial conditions

Two-time functions

- Generalisation of the empty-interval method
- Ageing exponents

Fluctuation-dissipation ratio

Conclusion
I.1 Ageing

Three defining properties of ageing:
1. observe slow relaxation after quenching PVC from melt to low $T$
2. creep curves depend on waiting time $t_e$ (or $s$) and creep time $t$
3. find master curve for all $(t, t_e)$ → dynamical scaling
$t = t_1$

$magnet \ T < T_c \rightarrow \text{ordered cluster}$

$magnet \ T = T_c \rightarrow \text{correlated cluster}$

critical contact process diffusion, $A \rightarrow 2A$, $A \rightarrow \phi$

$\Longrightarrow \text{cluster dilution}$

$voter \ model, \ contact \ process, \ldots$

Characteristic length scale: $L(t) \sim t^{1/z}$
1.2 Two-time observables

Time-dependent order-parameter $\phi(t, r)$

(Directed percolation: $\phi =$ part. density)

two-time **correlator**

$$C(t, s) := \langle \phi(t, r) \phi(s, r) \rangle - \langle \phi(t, r) \rangle \langle \phi(s, r) \rangle$$

two-time **response**

$$R(t, s) := \frac{\delta \langle \phi(t, r) \rangle}{\delta h(s, r)} \bigg|_{h=0}$$

(Directed percolation: $h(t) =$ creation of part.)

t : observation time, s : waiting time

**Scaling regime** : $t \gg s \gg \tau_{\text{micro}}$ (For simple magnets)

$$C(t, s) = s^{-b} f_C \left( \frac{t}{s} \right), \quad R(t, s) = s^{-1-a} f_R \left( \frac{t}{s} \right)$$

**Asymptotics** : $f_{C,R}(y) \sim y^{-\lambda_{C,R}/z}$ for $y \gg 1$

$\lambda_C$ : autocorrelation exponent, $\lambda_R$ : autoresponse exponent,
$z$ : dynamical exponent, $a, b$ : ageing exponents
The fluctuation-dissipation ratio (FDR) \( X(t, s) := \frac{TR(t, s)}{\partial C(t, s)/\partial s} \) measures the distance to the equilibrium: \( X_{eq} = X(t - s) = 1 \).  

\( a = b \) valid when systems satisfy detailed balance.

Contact process

\[ 1 + a = b \quad \iff \quad \text{rapidity-reversal symmetry} \quad \text{of stationary state of CP} \quad \Rightarrow \quad \text{specific property!} \]

\[ \Xi(t, s) := \frac{R(t, s)}{C(t, s)} = \frac{f_R(t/s)}{f_C(t/s)}, \quad \Xi_\infty := \lim_{s \to \infty} \left( \lim_{t \to \infty} \Xi(t, s) \right) \]

 Universality of \( \Xi_\infty \) proven to one-loop order.
AIM: to test these scaling predictions on an exactly solvable model without detailed balance

Model: One dimensional lattice of spacing $a$

(diffusion and coagulation can occur in both directions)

- Space translation invariance
- **Absence of detailed balance**
- Absorbing phase
- Stationary state
II.1 Empty interval method: training example

Particle concentration: \( c(t) = \Pr(\bullet; t) \)

\( n \) empty sites

\( E_n(t) \): time-dependent probability of having an interval of \( n \)
consecutive empty sites at time \( t \)

\[ c(t) = E_1(t) - E_0(t) \quad \text{continuum limit (} x = na \text{)} \quad c(t) = - \frac{\partial_x E(x, t)}{x = 0} \]

Equation of motion

For \( n > 1 \)

\[ \frac{\partial_t E_n(t)}{} = \left(2D/a^2\right)(E_{n-1} - 2E_n + E_{n+1}) \]

For \( n = 1 \)

\[ \frac{\partial_t E_1(t)}{} = \left(2D/a^2\right) \left[ 1 - 2E_1(t) + E_2(t) \right] \]

This gives the constraint: \( E_0(t) = 1 \)

Equation of motion in the continuum limit (\( x = na \))

\[ \frac{\partial_t E(x, t)}{} = 2D\frac{\partial_x E(x, t)}{, \text{ and } E(0, t) = 1}. \]
II.2 Solution by analytical continuation

Assume that the differential equation is valid for $n \leq 0$

$$E(x, t) = \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{\pi \ell_0}} \exp \left[ -\frac{1}{\ell_0^2} (x - x')^2 \right] E(x', 0).$$

where $\ell_0$ is the scaling length $\ell_0 := \sqrt{8Dt}$.

Take into account the constraint: $E_0(t) = 1$.

For $n = 0$

\[
\begin{align*}
\partial_t E_0(t) &= (2D/a^2) (E_{-1} - 2E_0 + E_1) = 0 \\
E_{-1}(t) &= 2E_0(t) - E_1(t) = 2 - E_1(t)
\end{align*}
\]

Redefine the meaning of $E(n, 0)$ for negative $n$ such that

$$E_{-n}(t) = 2 - E_n(t) \quad \text{and} \quad E(-x, t) = 2 - E(x, t)$$
II.3 General expression for the particle concentration

One-empty-interval probability

\[ E(x, t) = \text{erfc}(x/\ell_0) + \int_0^{+\infty} \frac{dx'}{\sqrt{\pi\ell_0}} E(x', 0) \left[ e^{-\frac{1}{\ell_0^2}(x-x')^2} - e^{-\frac{1}{\ell_0^2}(x+x')^2} \right]. \]

Hierarchy

Particle concentration

\[ c(t) = \frac{2}{\sqrt{\pi\ell_0}} \left( 1 - \int_0^{\infty} dx E(x\ell_0, 0) 2xe^{-x^2} \right) \]

\[ c(t) = \frac{2}{\sqrt{\pi\ell_0}} + o(1/\ell_0) \sim t^{-1/2} \]

Independent of initial condition
→ very well known result

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II. 4 One-time Correlation function

Connected correlator: \( C(d, t) = \Pr(\bullet d \bullet, t) - \Pr(\bullet, t)\Pr(\bullet, t) \)

Two-interval probability

\[ \begin{array}{c}
\begin{array}{c}
\text{n} \quad d \quad \text{m}
\end{array}
\end{array} \]

\( E_{n_1,n_2,d}(t) \): time-dependent probability of having two intervals of \( n_1 \) and \( n_2 \) consecutive empty sites distant from \( d \) at time \( t \)

- Continuum limit (\( x = n_1 a, y = n_2 a, z = da \))

\[ C(z, t) = \partial_{xy}^2 E(x, y, z, t) \bigg|_{x=0,y=0} - \partial_x E(x, t) \bigg|_{x=0} \partial_y E(y, t) \bigg|_{y=0} \]
II.5 One-time correlation function

Equation of motion: (only for $x$, $y$ and $z$ positive)

$$
\partial_t E(x, y, z, t) = 2D \left[ \partial_x^2 + \partial_y^2 + \partial_z^2 - \left( \partial_x \partial_z + \partial_y \partial_z \right) \right] E(x, y, z, t)
$$

with compatibility conditions (ex: $E(x, 0, d, t) = E(x, t)$).

Decomposition of the solution as a sum of three terms:

$$
E(x, y, z, t) = E^{(0)}(x, y, z, t) + E^{(1)}(x, y, z, t) + E^{(2)}(x, y, z, t)
$$

1. $E^{(0)}(x, y, z, t)$ is independent of the initial conditions
2. $E^{(1)}(x, y, z, t)$ depends on the initial one-interval probability
3. $E^{(2)}(x, y, z, t)$ depends on the initial two-intervals probability

→ System initially filled with particles

$$
E(x, y, z; t) = E^{(0)}(x, y, z, t) = \text{erfc} \left( \frac{x}{\ell_0} \right) \text{erfc} \left( \frac{y}{\ell_0} \right) + \text{erfc} \left( \frac{z}{\ell_0} \right) \text{erfc} \left( \frac{x + y + z}{\ell_0} \right) - \text{erfc} \left( \frac{x + z}{\ell_0} \right) \text{erfc} \left( \frac{y + z}{\ell_0} \right)
$$
II.6 One-time correlation function

Correlation function

\[ C(z, t) = \partial_{xy}^2 E(x, y, z, t) \bigg|_{x=0, y=0} - \partial_x E(x, t) \bigg|_{x=0} \partial_y E(y, t) \bigg|_{y=0} \]

In the case of an initially completely filled system, \( E(x, 0) = 0 \) and \( E(x, y, z, 0) = 0 \), we obtain dynamical scaling with \( \ell_0 = \sqrt{8Dt} \).

Connected correlator

\[
C(z, t) = \left( \frac{2}{\sqrt{\pi} \ell_0} \right)^2 f\left( \frac{z}{\ell_0} \right)
\]

with \( f(y) = -e^{-2y^2} + \sqrt{\pi} ye^{-y^2} \text{erfc}(y) \)

exact in asymptotic regime for all initial conditions.
II.7 Correction to the leading behaviour

Time evolution of $C(z; t)$ for an initial one-interval probability $E(x; 0) = \exp(-x)$ and $z = 1/2$

- Full black solid line: leading contribution
  - Algebraic behaviour when $t$ large $|C(z, t)| \sim t^{-1}$.
- Red dashed line includes the effect of one-interval contribution
- Dashed-dotted line includes all contributions

Hierarchy: $|C^{(0)}(1/2, t)| \gg |C^{(1)}(1/2, t)| \gg |C^{(2)}(1/2, t)|$
### III.1 Two-times functions

- We want to evaluate connected correlation $C(z; t, s)$ and response $R(z; t, s)$ functions ($t \geq s$) using the interval probability method.

- In the discrete space, their definition are:

$$C(d; t, s) = \Pr(\{\bullet; t\} \ d \ \{\bullet; s\}) - \Pr(\bullet; t)\Pr(\bullet; s)$$

If we add a particle at a given site at time $s$:

$$R(d; t, s) = \Pr(\{\bullet; t\} \ d \ \{\bullet; s\}) - \Pr(\bullet; t)$$

$$= \frac{\delta \langle \phi(t, r) \rangle}{\delta h(s, r)} \bigg|_{h=0}$$

$\phi =$ part. density and $h(t) =$ creation of part.

---

**Mixed-interval probability**

$$F(n, d; t, s) = \Pr(\{n; t\} \ d \ \{\bullet; s\})$$
III.2 Mixed-interval probability

Two-time correlation

\[ C(z; t, s) = \lim_{a \to 0} \frac{C(d; t, s)}{a} = -\partial_x F(x, z; t, s)|_{x=0} \sim 1/L^2 \]

Two-time response, \( G \) has the same definition than \( F \)

\[ R(z; t, s) = \lim_{a \to 0} \frac{R(d; t, s)}{a} = -\partial_x G(x, z; t, s)|_{x=0} \sim 1/L \]

Initial conditions at \( t = s \)

\[ F(x, z; s, s) = \lim_{a \to 0} \frac{1}{a} F(n, d; s, s) = -\partial_y E(x, y, z; s)|_{y=0} \]

\[ G(x, z; s, s) = \lim_{a \to 0} G(n, d; s, s) = E(x; s) \]

\( F \) and \( G \) are different because of their initial conditions at \( t = s \).
In the continuum limit and for $x$ and $z$ positive variables

$$\partial_t F(x, z; t, s) = 2D \left[ \frac{\partial^2}{\partial x^2} + \frac{1}{2} \frac{\partial^2}{\partial z^2} - \partial_x \partial_z \right] F(x, z; t, s)$$

Symmetries between positive and negative variables

$$F(-x < 0, z; s, s) = 2c_0(s) - F(x, z - x; s, s)$$
$$F(x, -z < 0; s, s) = \theta(z - x)F(x, z - x; s, s))$$
$$F(-x < 0, -z < 0; s, s) = 2c_0(s) - F(x, z; s, s)$$

with $c_0(s) = c(s)$ for $F$ and $c_0(s) = 1$ for $G$
III.4 Response function

Expression for an initially filled system

\[
R(z; t, s) = \frac{2}{\pi \ell_1 \ell_0} \int_{\mathbb{R}^+} dx' \left[ \text{erfc} \left( \frac{x' - 2z}{\ell_1} \right) + \text{erfc} \left( \frac{x' + 2z}{\ell_1} \right) \right] \exp \left( - \frac{x'^2 (\ell_0^2 + \ell_1^2)}{\ell_0^2 \ell_1^2} \right) \\
+ \frac{2}{\pi \ell_1^2} e^{-2z^2 / \ell_1^2} \int_{\mathbb{R}^+} dx' \left[ \exp \left( - \frac{2(z - x')^2}{\ell_1^2} \right) + \exp \left( - \frac{2(z + x')^2}{\ell_1^2} \right) \right] \text{erfc} \left( \frac{x'}{\ell_0} \right) - \frac{2}{\sqrt{\pi} \ell_0}
\]

Analogous expression for the correlation function but much more lengthy.
III.5 Response function

Discrete numerical simulations on a chain of 512 sites with $s = 10$ and $D = 1/2$ versus analytical expressions

- Full line: analytical solution; symbols: simulation
- $r = [0, 1, 2, 3]$ from top to bottom
III.6 Auto-correlator and auto-response

- Autocorrelator

\[ C(ys, s) = \frac{4}{\pi \ell_0^2} \left\{ \frac{1}{\sqrt{y^2 - 1}} + \frac{\sqrt{2}(y - 1)}{\pi (1 + y)} \tan^{-1} \left( \frac{\sqrt{2}}{\sqrt{y - 1}} \right) \right\} - \frac{2}{\pi \sqrt{y}} \tan^{-1} \left( \frac{1}{\sqrt{y}} \right) - \frac{1}{1 + y} \] 

\[ = \frac{1}{s} f_C(y) = \frac{1}{s^b} f_C(y) \]

in agreement with P. Mayer and P. Sollich 2007

- Autoresponse

\[ R(ys, s) = \frac{2}{\sqrt{\pi \ell_0}} \left\{ \frac{\sqrt{2}}{\pi \sqrt{y - 1}} \tan^{-1} \left( \frac{\sqrt{2}}{\sqrt{y - 1}} \right) \right\} - \frac{2}{\pi \sqrt{y}} \tan^{-1} \left( \frac{1}{\sqrt{y}} \right) \] 

\[ := \frac{1}{\sqrt{s}} f_R(y) = \frac{1}{s^{1+a}} f_R(y) \]

Exponents

\[ \lambda_C = \lambda_R = 4, \quad a = -1/2 \text{ and } b = 1 \]
Question What’s the relation between $R$ and $C$?

- $a = b$ detailed balance $→$ Usual definition of the fluctuation-dissipation ratio

\[ X(t,s) = \frac{TR(t,s)}{\partial_s C(t,s)} \]

L.F. Cugliandolo and al. 1994

- $1 + a = b$ directed percolation $→$

\[ \Xi(t,s) := \frac{R(t,s)}{C(t,s)} = \frac{f_R(t/s)}{f_C(t/s)} \quad , \quad \Xi_\infty := \lim_{s \to \infty} \left( \lim_{t \to \infty} \Xi(t,s) \right) \]

- here $b = 1$ and $a = -1/2$ $→$ $3/2 + a = b$, therefore we define another ratio

\[ \Xi(t,s) = \frac{R(t,s)}{\sqrt{8D}\partial_s^{-1/2} C(t,s)} \]
Return to the non-equilibrium scaling forms

\[
C(t, s) = L(s)^{-bz} F_C \left( \frac{t}{s} \right) = s^{-b} f_C \left( \frac{t}{s} \right)
\]

\[
R(t, s) = L(s)^{-az-z} F_R \left( \frac{t}{s} \right) = s^{-1-a} f_R \left( \frac{t}{s} \right)
\]

time-dependent length scale: \[ L(s) = (\nu s)^{1/z} \]
IV.3 Generalisation?

\[ \Xi(t, s) := \vartheta^{1+a-b} \frac{R(t, s)}{\partial_s^{1+a-b} C(t, s)} \]

- use Riemann-Liouville fractional derivative \( \partial_s^\alpha \)
- \( \vartheta^{-1} \) is a characteristic time for the passage into the scaling behaviour.
- contains two situations:
  1. critical systems with detailed balance (\( a = b \) holds true and \( \vartheta \mapsto T \) becomes the equilibrium temperature)
  2. critical particle-reaction models with \( a + 1 = b \)
- in the limit \( y = t/s \gg 1 \), the function \( \Xi \) tends towards an universal limit \( \Xi_\infty \).
- for the coagulation diffusion process: \( \Xi_\infty = \frac{3\pi}{6\pi-16} \)

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Conclusion

- Ageing in the 1D coagulation-diffusion process quite analogous to what have been seen in simple magnets
- Exactly solvable model allow us to verify the expected scaling forms
- Tool : generalized empty interval method
- We find the one-time and two-time correlation functions for any given initial conditions
- Proposal of a new FDR
- Physical meaning of $\Xi$?
III.3 Quantum representation

- Empty site $|0 >_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- Filled site $|1 >_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Operators of creation and destruction

$$d_i^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad d_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$d_i|0 >_i = 0, \quad d_i|1 >_i = |0 >_i, \quad d_i^\dagger|0 >_i = |1 >_i, \quad d_i^\dagger|1 >_i = |1 >_i$$

- Particle number operator $n_i = d_i^\dagger d_i$ and hole operator $h_i = 1 - n_i$

- Initial state:

$$|P(0) > = \sum_{\{\alpha_i = 0, 1\}} P(\{\alpha_i\}, t = 0) \otimes_i |\alpha_i >$$

- Ground state:

$$< G | = \otimes_i \{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \}$$
III.4 Quantum representation

- Hamiltonian

\[ H = -\Gamma \sum_i \left( d_{i+1}^+ d_i + d_{i-1}^+ d_i \right) + \mu \sum_i n_i \]

\( \mu = 2\Gamma \) is the chemical potential such that \( \langle G | H = 0 \) and \( P(t) = \langle G | P(t) \rangle = 1 \)

- Empty-interval operator

\[ X(1, n) := \prod_{i=1}^{n} (1 - n_i) = \prod_{i=1}^{n} h_i. \]

- Mixed function is represented by the expectation value

\[ G(n, d; t, s) = \langle G | X(1, n) e^{-H(t-s)} d_{n+d+1}^+ e^{-Hs} | P(0) \rangle \]
III.3 Equations of motion

**Same equation for $F$ and $G$**

\[
\partial_t F(n, d; t, s) = \Gamma \left[ F(n + 1, d; t, s) + F(n + 1, d - 1; t, s) + F(n - 1, d; t, s) + F(n - 1, d + 1; t, s) - 4F(n, d; t, s) \right]
\]

- In the continuum limit

\[
\partial_t F(x, z; t, s) = 2D \left[ \partial_x^2 + \frac{1}{2} \partial_z^2 - \partial_x \partial_z \right] F(x, z; t, s)
\]

- General solution

\[
F(x, z; t, s) = \int_{\mathbb{R}^2} \frac{2dx'dz'}{\pi \ell_1^2} \mathcal{W}_1(x - x', z - z') F(x', z'; s, s),
\]

\[
\mathcal{W}_1(x - x', z - z') = e^{-\frac{2}{\ell_1^2}[(x-x'+z-z')^2-(z-z')^2]}
\]

with $\ell_1 = \sqrt{8D(t-s)}$ and $\ell_0 = \sqrt{8Ds}$
III.4 Symmetries

- We have to rewrite the former integrals in terms of **physical domain** \((x > 0, z > 0)\)
- We obtain, using the constraints imposed by the differential equation of motion

\[
F(-x < 0, z; s, s) = 2c_0(s) - F(x, z - x; s, s)
\]
\[
F(x, -z < 0; s, s) = \theta(z - x)F(x, z - x; s, s))
\]
\[
F(-x < 0, -z < 0; s, s) = 2c_0(s) - F(x, z; s, s)
\]

with \(c_0(s) = c(s)\) for \(F\) and \(c_0(s) = 1\) for \(G\)
Discrete numerical simulations on a chain of 512 sites with $s = 10$ and $D = 1/2$ versus analytical expressions

The full line are the analytical solution while the symbols are the simulations

Black curves are $z = 0$ and red curves are for $z = 3$

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III.9 Response function

Expression for an initially filled system

\[ R(z; t, s) = \frac{2}{\pi \ell_1 \ell_0} \int_{\mathbb{R}^+} dx' \left[ \text{erfc} \left( \frac{x' - 2z}{\ell_1} \right) + \text{erfc} \left( \frac{x' + 2z}{\ell_1} \right) \right] \exp \left( -\frac{x'^2 (\ell_0^2 + \ell_1^2)}{\ell_0^2 \ell_1^2} \right) \]

\[ + \frac{2}{\pi \ell_1^2} e^{-2z^2/\ell_1^2} \int_{\mathbb{R}^+} dx' \left[ \exp \left( -\frac{2(z - x')^2}{\ell_1^2} \right) + \exp \left( -\frac{2(z + x')^2}{\ell_1^2} \right) \right] \text{erfc} \left( \frac{x'}{\ell_0} \right) - \frac{2}{\sqrt{\pi} \ell_0} \]
### III.12 Auto-correlation and auto-response functions

\[
C(0; t, s) = C(t, s) = \frac{1}{s^b} f_C(y = t/s) := \frac{1}{s} f_C(y = t/s)
\]
\[
R(0; t, s) = R(t, s) = \frac{1}{s^{1+a}} f_R(y = t/s) := \frac{1}{\sqrt{s}} f_R(y = t/s)
\]

Asymptotically, \(f_C(y) \simeq (2D)^{-1} \left[ (1 - 8/(3\pi))\pi^{-1}y^{-2} + (\frac{16}{5}\pi - 1/2)\pi^{-1}y^{-3} \right]\) which gives the exponent \(\lambda_C = 4\).

Asymptotically, \(f_R(y) \simeq (2D)^{-1/2} \left[ \frac{4}{3}\pi^{-3/2}y^{-2} + \frac{8}{15}\pi^{-3/2}y^{-3} \right]\) which gives the exponent \(\lambda_R = 4\).
III.13 Asymptotic behavior

- $f_C$ behaves asymptotically like

$$f_C(y) \simeq (2D)^{-1} \left[ (1 - 8/(3\pi))\pi^{-1}y^{-2} + \left(\frac{16}{5}\pi - 1/2\right)\pi^{-1}y^{-3} \right]$$

with the dynamical exponent $z = 2$, which gives the exponent $\lambda_C = 4$.

- $f_R$ behaves asymptotically like

$$f_R(y) \simeq (2D)^{-1/2} \left[ \frac{4}{3}\pi^{-3/2}y^{-2} + \frac{8}{15}\pi^{-3/2}y^{-3} \right]$$

which gives the exponent $\lambda_R = 4$
The behaviour is not monotonous while in models such that $a + 1 = b$ or $a = b$ it seems to be generically the case.
We use the Riemann representation for fractional derivatives with negative arguments:

\[
\frac{\partial^{-\nu}}{\partial s^{-\nu}} \varphi(s) = \frac{1}{\Gamma(\nu)} \int_0^s (s - u)^{\nu-1} \varphi(u) \, du
\]

Asymptotic limits when \( t \gg s \gg 1 \):

\[
C(t, s) \simeq a_C \, t^{-2} s
\]

and

\[
R(t, s) \simeq a_R \, t^{-2} s^{3/2}
\]

with \( a_C = (2D)^{-1}(3\pi - 8)/3\pi^2 \) and \( a_R = (2D)^{-1/2} \frac{4}{3\pi^{3/2}} \)

then the FDR in the asymptotic limit \( t \gg s \gg 1 \) reads

\[
X_\infty = \lim_{y \gg 1} X(t, s) = \frac{3\pi}{6\pi - 16} \simeq 3.307
\]
## IV.2 Models

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<th>reactions</th>
<th>Diffusion</th>
<th>$n_{\text{max}}$</th>
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<tbody>
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<td>BCPD</td>
<td>$A \rightarrow 2A$ $A \rightarrow \emptyset$</td>
<td>diffusion</td>
<td>$\infty$</td>
</tr>
<tr>
<td>BCPL</td>
<td>$A \rightarrow 2A$ $A \rightarrow \emptyset$</td>
<td>Lévy flight</td>
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<td>FA</td>
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<td>contact process</td>
<td>$A \rightarrow 2A$ $A \rightarrow \emptyset$</td>
<td>diffusion</td>
<td>1</td>
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<td>NEKIM</td>
<td>$2A \rightarrow \emptyset$ $A \leftrightarrow 3A$</td>
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<td>1</td>
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<tr>
<td>model</td>
<td>$d$</td>
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<td>$\Xi_\infty$</td>
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<td>BCPD</td>
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<td>$b = a$</td>
<td>1/2</td>
</tr>
<tr>
<td>FA</td>
<td>1</td>
<td>$b = a$</td>
<td>$-3\pi/(6\pi - 16)$</td>
</tr>
<tr>
<td>contact process</td>
<td>1</td>
<td>$b = 1 + a$</td>
<td>1.15(5)</td>
</tr>
<tr>
<td>(directed perco.)</td>
<td>$4 - \varepsilon$</td>
<td>$b = 1 + a$</td>
<td>$2 - \varepsilon\left(\frac{119}{240} - \frac{\pi^2}{60}\right)$</td>
</tr>
<tr>
<td>NEKIM</td>
<td>1</td>
<td>$b = 1 + a$</td>
<td>$\approx 0.1$</td>
</tr>
<tr>
<td>coag.-diff.</td>
<td>1</td>
<td>$b = 3/2 + a$</td>
<td>$3\pi/(6\pi - 16)$</td>
</tr>
</tbody>
</table>
**Models with $a = b$**

- contains critical systems with detailed balance (FA model)
- in systems without detailed balance, we use

$$X(t, s) = \left( \frac{R(s, s)}{\partial_s C(s, s)} \right)^{-1} \frac{R(t, s)}{\partial_s C(t, s)} = X\left(\frac{t}{s}\right) \rightarrow X_\infty$$

since there is no longer reference to an equilibrium temperature

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**Models with $1 + a = b$**

- Limit value $\Xi_\infty$ is directly universal since the dependence on the scale $\vartheta$ drops out
- In the stationary state, one has $\Xi_{stat}^{-1} = 0$
- In the directed percolation class, the universality of $\Xi_\infty$ has been proven to one-loop order.  

F. Baumann and *al.* 2007

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