Structure dependent dynamics

- Collective dynamics of coupled dynamical units are often intimately related to the structure of the coupling network.
- Statistical methods have revealed that structural properties on all scales from individual nodes to the entire network may play a role.

Paradigmatic example: Synchronization of phase-oscillators

- e.g., heterogeneous Kuramoto model
- Stability of phase locked states determined by spectrum of Jacobian matrix $J \in \mathbb{R}^{N \times N}$: $J_{ik} = \partial x_i / \partial x_k = 0$ if $x_i, x_k$ not coupled

Jacobi’s Signature Criterion

A symmetric matrix $J \in \mathbb{R}^{N \times N}$ is negative semi-definite if all its principal minors $D_{|S|} = \det(J_{ik}), \ i,k \in S, \ S \subset \{1, \ldots, N\}$ satisfy $\text{sign}(D_{|S|}) = (-1)^{|S|}$

Topological Interpretation

Interprete $J$ as adjacency matrix of a graph $G$

- Entry $J_{ij}$: Weight of a link $ij$
- Term $J_{ij} J_{jk}$: Subgraph of $G$ spanned by the links $ij$ and $jk$
- Minor $D_{|S|}$: Sum over subgraphs with $|S|$ links

Graphical notation and calculus

- Basis of symbols denoting cycles of lengths $m = 1, 2, 3, 4, 5, \ldots$:
  - $\bigcirc$, $\triangle$, $\square$, $\lozenge$, $\ldots$
- Summation convention: symbol in $D_{|S|}$ denotes the sum over all non-equivalent possibilities to build the depicted subgraph with the nodes in $S$
  - $S = \{i,j,k,l\}$

Zero row sum

In many systems, including the Kuramoto model, $J$ has zero row sums (force balance along links). In these systems

$D_{|S|} = (-1)^{|S|} \sum$ all acyclic subgraphs of $G$ with $|S|$ links and no two nodes \( \notin S \) in the same component

Necessary topological stability criteria

Stability requires that $\Phi_S > 0$ for all $S$. We can show that this restricts the number, position and strength of potential links with negative weight: Every component of $G$ has to have a spanning tree of links with positive weight. Moreover, negative link weights are subject to a topology dependent lower bound.

Summary

- The topological interpretation of Jacobi’s Signature Criterion allows to study the interplay of structure and dynamics in complex networks.
- The derived topological stability criteria pertain to structures on all scales from single nodes to the entire network. They apply to all systems with symmetric or Hermitian Jacobian, i.e., in particular to networks of symmetrically coupled units.