Full connectivity in spatially confined random networks

Justin P. Coon[1], Carl P. Dettmann[2], and Orestis Georgiou[3]

[1] Toshiba Telecommunications Research Laboratory

[2] University of Bristol School of Mathematics

[3] Max-Planck-Institute for the Physics of Complex Systems

Introduction - Motivation

Many complex networks exhibit a percolation transition involving a macroscopic connected component, with universal features largely independent of the microscopic model and the macroscopic domain geometry.

In contrast, the transition to full network connectivity in spatially confined random networks is strongly influenced by the details of the boundary. This is particularly important for example in wireless *Multi-Hop* relay networks [1]. We are interested in understanding how the probability of achieving a fully connected network P_{fc} is affected by the shape of the network domain?

Examples of networks in 2 dimensions

In both Figs 1&2 a Gaussian pair-connectedness function $H(r)$ is used.

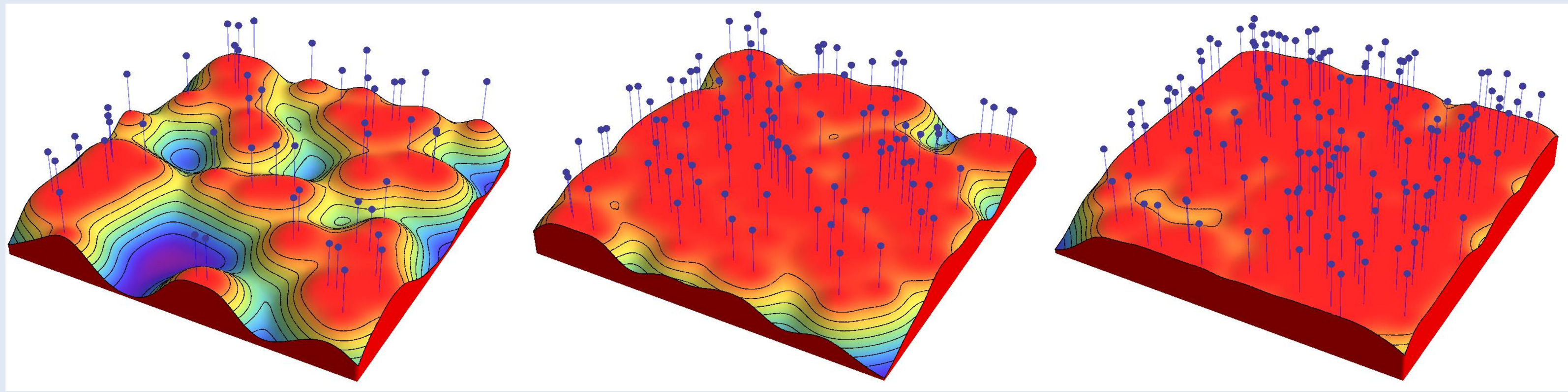


Fig. 1. Surface plots of the connectivity at node densities ρ = 0.5, 1, and 1.5.
The positions of the nodes are signified by pins, while the total probability of a new node introduced to the network connecting to any other node is indicated by the contours and shading (blue valleys indicate a low probability of connection while red peaks indicate a high probability). Notice that at high densities, low connection probabilities are concentrated near the corners.

A cluster expansion approach

The probability that two nodes i and j are connected H_{ij} or not $(1-H_{ij})$ gives a trivial identity:

$$1 \equiv H_{ij} + (1 - H_{ij}) \qquad (1)$$

Multiplying over all links between nodes expresses the probability of all possible combinations and therefore the sum of all possible networks $\mathcal{H}_g$:

$$1 \equiv \prod_{i<j} [H_{ij} + (1 - H_{ij})] = \sum_g \mathcal{H}_g \qquad (2)$$

The $N(N-1)/2$ terms in the sum can be grouped into smaller sums determined by the size of the largest connected component (cluster):

$$1 \equiv \sum_{g \in G_N} \mathcal{H}_g + \sum_{g \in G_{N-1}} \mathcal{H}_g + \ldots + \sum_{g \in G_1} \mathcal{H}_g \qquad (3)$$

such that the first term is the probability of a network consisting of a single cluster of size N (fully connected), and the last term is the probability of a network where all nodes are isolated (disconnected).

Rearranging and keeping only leading order terms we conclude that:

$$P_{fc} = 1 - \sum_{g \in G_{N-1}} \mathcal{H}_g - \ldots \qquad (4)$$

At high densities P_{fc} is the *complement* of the probability of an *isolated* node

Averaging (see Ref.[2]) and assuming that N is large, Eq.(4) can be expressed as:

$$P_{fc} \approx 1 - \frac{N}{V} \int_{\mathcal{V}} \left(1 - \frac{1}{V} \int_{\mathcal{V}} H(\mathbf{r}_{12}) \mathrm{d}\mathbf{r}_1 \right)^{N-1} \mathrm{d}\mathbf{r}_2$$
$$= 1 - \rho \int_{\mathcal{V}} e^{-\rho \int_{\mathcal{V}} H(\mathbf{r}_{12}) \mathrm{d}\mathbf{r}_1} \left(1 + \mathcal{O}(N^{-1})\right) \mathrm{d}\mathbf{r}_2 \qquad (5)$$

Notice that Eq.(5) is completely general and applies to any domain geometry $\mathcal{V}$, in any dimension d, and any pair connectedness function $H(r)$ where r is the distance between nodes i and j.

References

1. Impact of boundaries on fully connected random geometric networks, J. Coon, C. P. Dettmann and O. Georgiou, Phys. Rev E., 85, 011138, (2012).
2. Full Connectivity: Corners, edges and faces, J. Coon, C. P. Dettmann and O. Georgiou, to appear in J. Stat. Phys. (2012).

The authors thank the Directors of the Toshiba Telecommunications Research Laboratory for their support.

Examples of networks in 3 dimensions

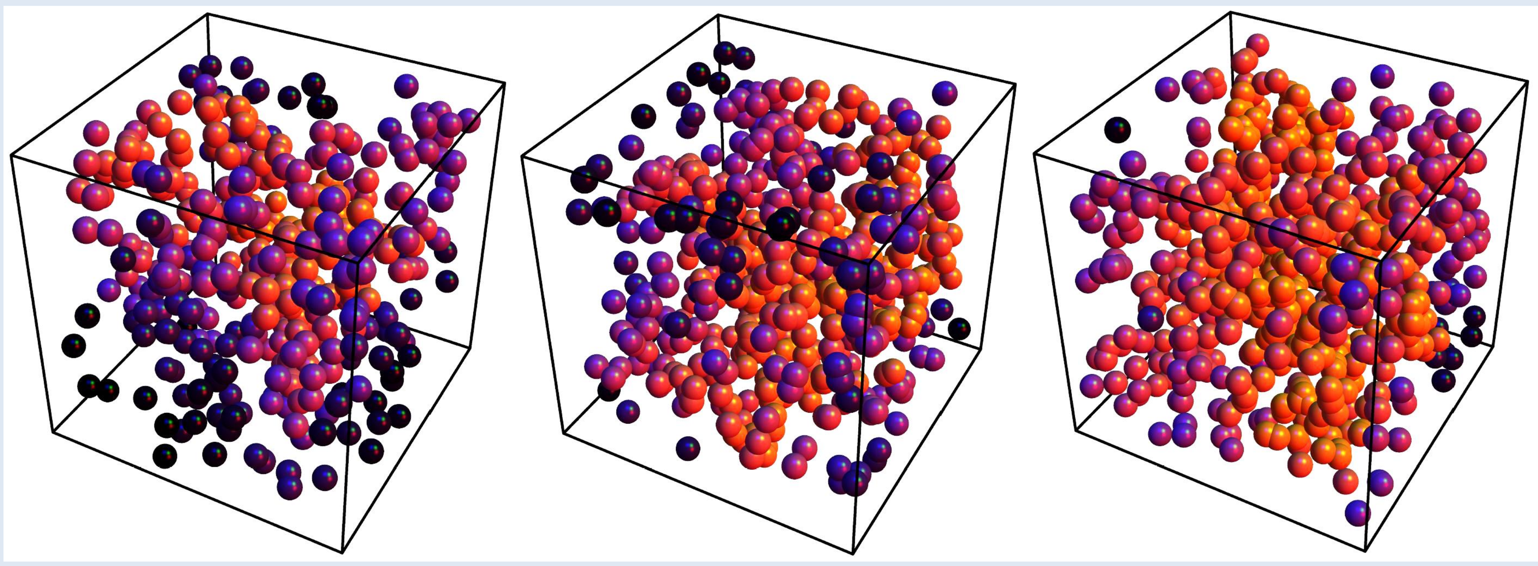


Fig 2. High density realizations for N = 450, 500 and 600 nodes (represented as balls) randomly placed inside a cubed domain of side. Lighter colors indicate a higher probability of being in the largest connected component. Notice that isolated nodes (darker colors) are concentrated at the edges and corners at higher densities.

General high-density Formula

Assuming that the pair connectedness function is decaying fast enough (e.g. exponentially), P_{fc} to leading order can be expressed as a sum of contributions due to objects of different codimension $i = 0, 1, \ldots, d$ (with $i = 0$ corresponding to the bulk/volume term)

$$P_{fc} \approx 1 - \sum_{i=0}^{d} \sum_{j_i} \rho^{1-i} \mathbf{G}_{j_i} \mathbf{V}_{j_i} e^{-\rho \omega_{j_i} \mathcal{H}} \qquad (6)$$

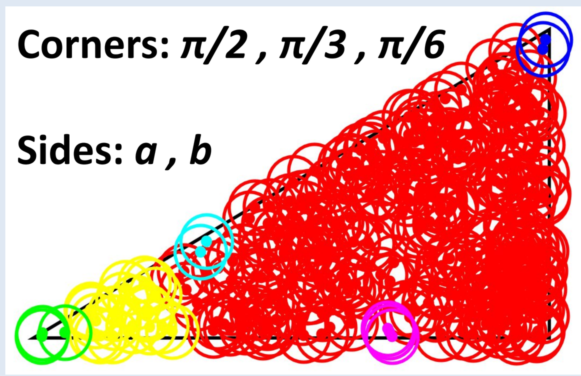
$$\mathcal{H} = \int_0^\infty r^{d-1} H(r) \mathrm{d}r$$

where ρ is the density of nodes, $\mathbf{G}$ is a geometrical factor which is $H(r)$ dependent, $\mathbf{V}$ and ω are the d-i dimensional volume and the available solid angle of object j respectively.

Below are two worked examples illustrating how Eq.(6) can be calculated and how boundary effects dominate P_{fc} at high densities:

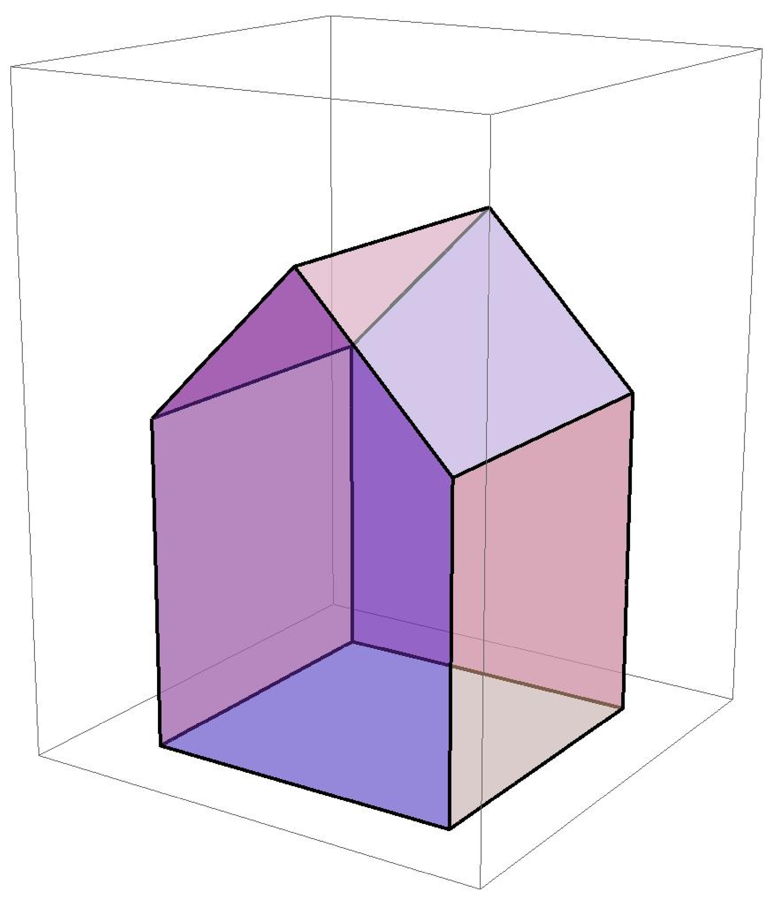
Unit-disk model in a triangular domain

$$P_{fc} \approx 1 - \rho \frac{ab}{2} e^{-\frac{\rho}{2} 2\pi} - \frac{1}{2}(a + b + \sqrt{a^2 + b^2}) e^{-\frac{\rho}{2}\pi}$$
$$- \frac{4}{\rho} \left(e^{-\frac{\rho}{2}\frac{\pi}{2}} + e^{-\frac{\rho}{2}\frac{\pi}{3}} + e^{-\frac{\rho}{2}\frac{\pi}{6}} \right) \qquad (7)$$

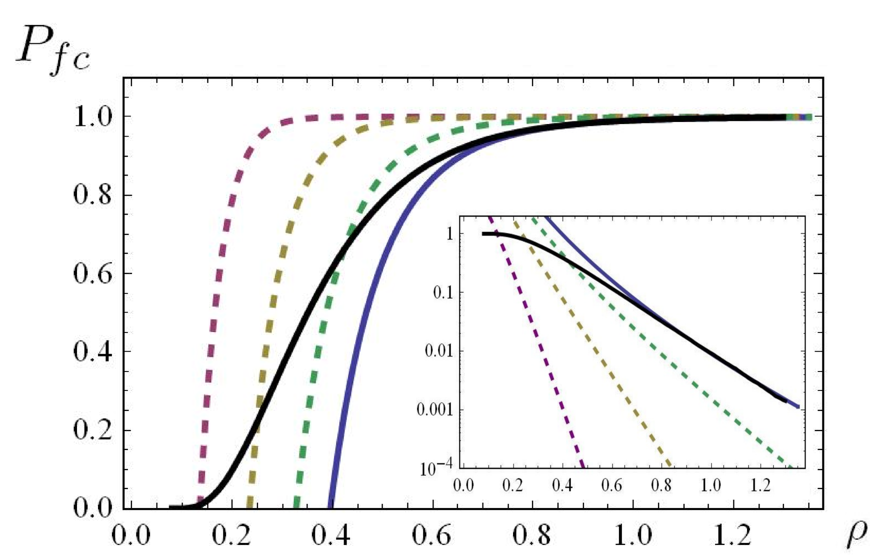


Full connectivity at high densities is dominated by the sharpest corner

Multiple input multiple output (MIMO) model in a "house"



$$H(r) = e^{-\beta r^2} \left(\beta^2 r^4 + 2 - e^{-\beta r^2} \right)$$



Analytical and numerical results for P_{fc} as a function of the density ρ in a typical "house". The two solid curves are the numerical (black) and full analytical results (blue). The dashed curves are the volume, surface and edge terms plotted independently from left to right. The inset shows ***1- P_{fc}***

$$P_{fc} \approx 1 - \rho(V + S + 9E_1 + 4E_2 + 2E_3 + 6C_1 + 4C_2) \qquad (8)$$

General Formula Parameter	Corners	Edges	Faces	Volume
Volume (V_{j_i})	1	$L, \frac{L}{\sqrt{2}}$	S	V
Solid Angle (ω_{j_i})	ϑ	2ϑ	2π	4π
Geometrical Factor (G_{j_i})	$\frac{256\beta^3 \csc\vartheta}{343\pi^3\vartheta}$	$\frac{16\beta^2 \csc\vartheta}{49\pi^2}$	$\frac{2\beta}{7\pi}$	1