Dynamics of Voter Models on Complex & Simple Networks

Sid Redner (physics.bu.edu/~redner)

Mathematical Physics of Complex Networks (MAPCON), MPI Dresden, May 14-18, 2012

T. Antal (BU→Edinburgh), V. Sood (BU→Lausanne), D. Volovik (BU)

NSF DMR0906504

Dynamics of Voter Models on Complex & Simple Networks

Sid Redner (physics.bu.edu/~redner)

Mathematical Physics of Complex Networks (MAPCON), MPI Dresden, May 14-18, 2012

T. Antal (BU→Edinburgh), V. Sood (BU→Lausanne), D. Volovik (BU)

NSF DMR0906504

The classic voter model

3 basic results

Dynamics of Voter Models on Complex & Simple Networks

Sid Redner (physics.bu.edu/~redner)

Mathematical Physics of Complex Networks (MAPCON), MPI Dresden, May 14-18, 2012

T. Antal (BU→Edinburgh), V. Sood (BU→Lausanne), D. Volovik (BU)

NSF DMR0906504

The classic voter model

3 basic results

Voting on complex networks

T. Antal, V. Sood

new conservation law
two time-scale route to consensus
short consensus time

Dynamics of Voter Models on Complex & Simple Networks

Sid Redner (physics.bu.edu/~redner)

Mathematical Physics of Complex Networks (MAPCON), MPI Dresden, May 14-18, 2012

T. Antal (BU→Edinburgh), V. Sood (BU→Lausanne), D. Volovik (BU)

NSF DMR0906504

The classic voter model

3 basic results

Voting on complex networks

T. Antal, V. Sood

new conservation law
two time-scale route to consensus
short consensus time

Confident/Reinforced voting

D. Volovik

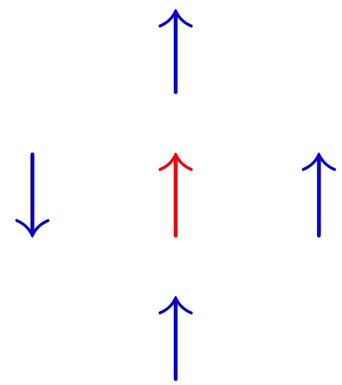
two time-scale dynamics
symmetry breaking clustering

Classic Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)

Classic Voter Model

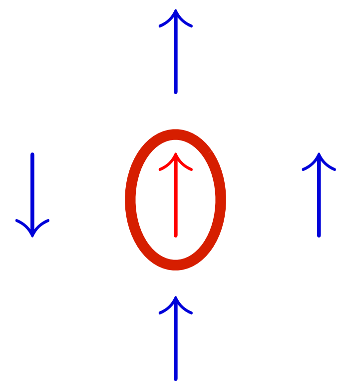
Clifford & Sudbury (1973)
Holley & Liggett (1975)



0. Binary voter variable at each site i

Classic Voter Model

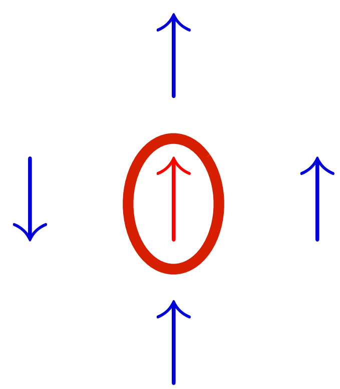
Clifford & Sudbury (1973)
Holley & Liggett (1975)



0. Binary voter variable at each site i
1. Pick a random voter

Classic Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)



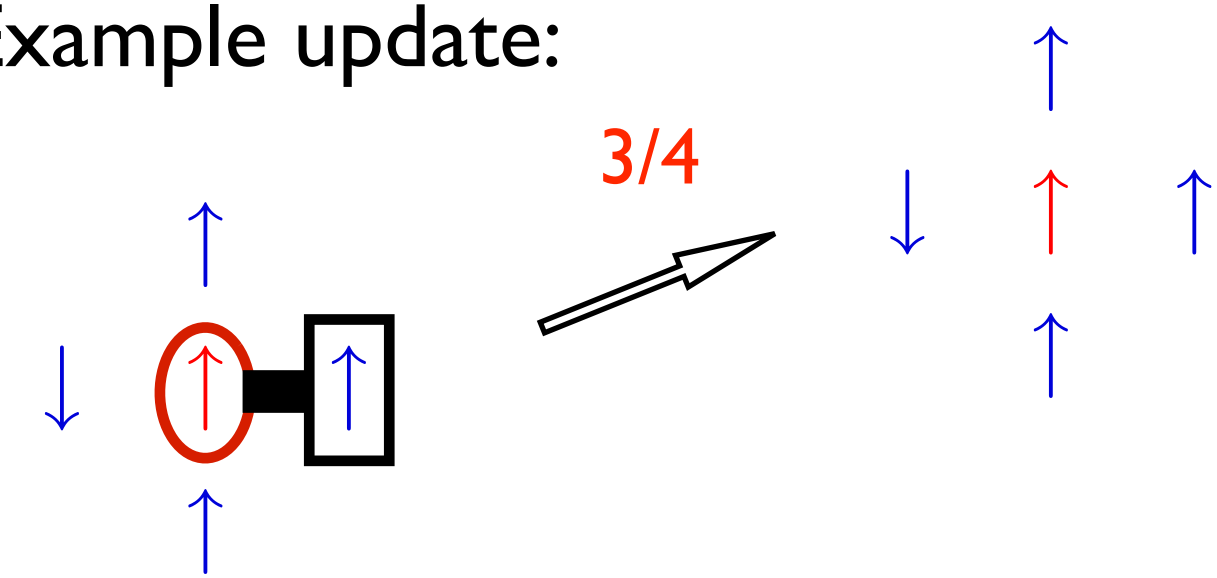
0. Binary voter variable at each site i
1. Pick a random voter
2. Assume state of randomly-selected neighbor

   individual has no self-confidence & adopts neighbor's state

Classic Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)

Example update:

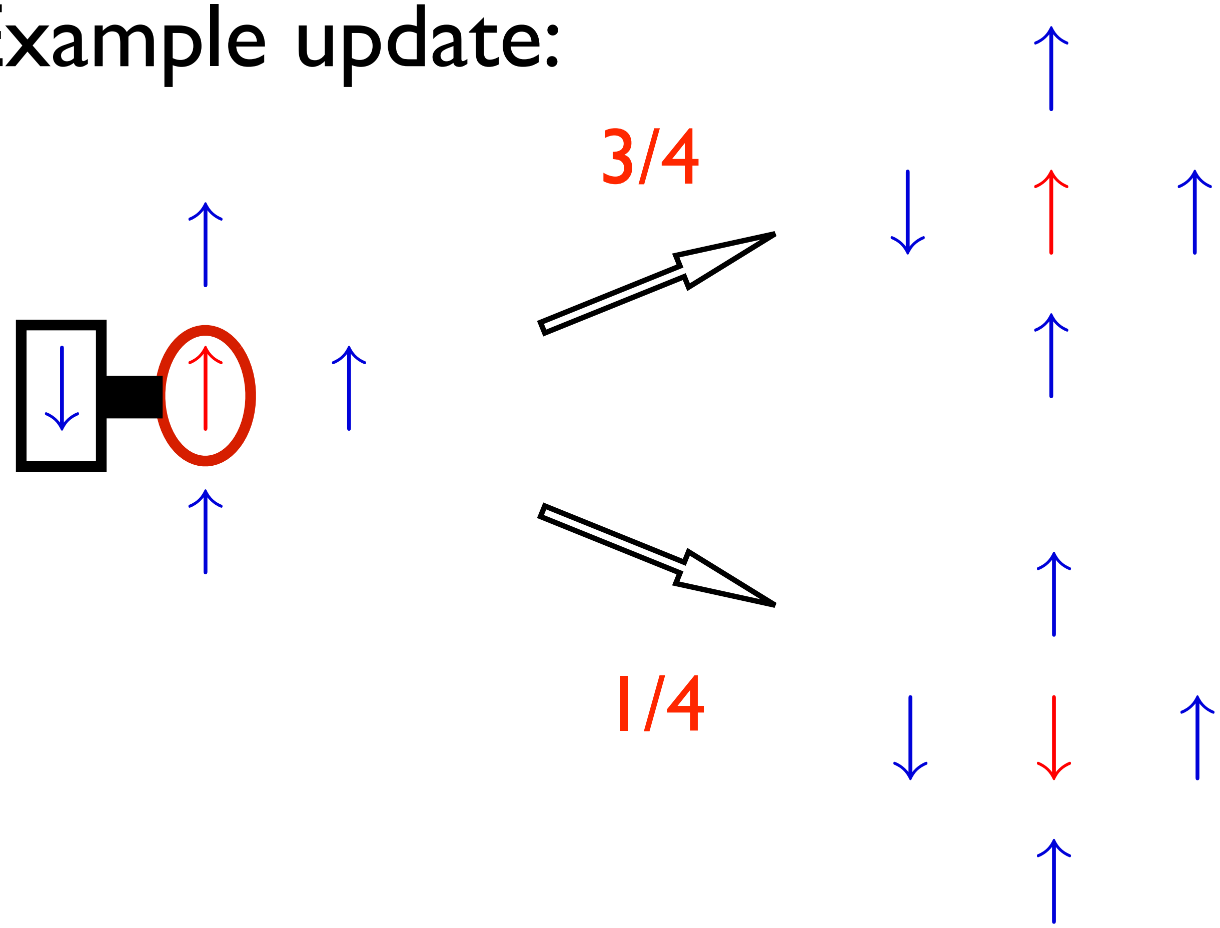


0. Binary voter variable at each site i
1. Pick a random voter
2. Assume state of randomly-selected neighbor
   individual has no self-confidence & adopts neighbor's state

Classic Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)

Example update:



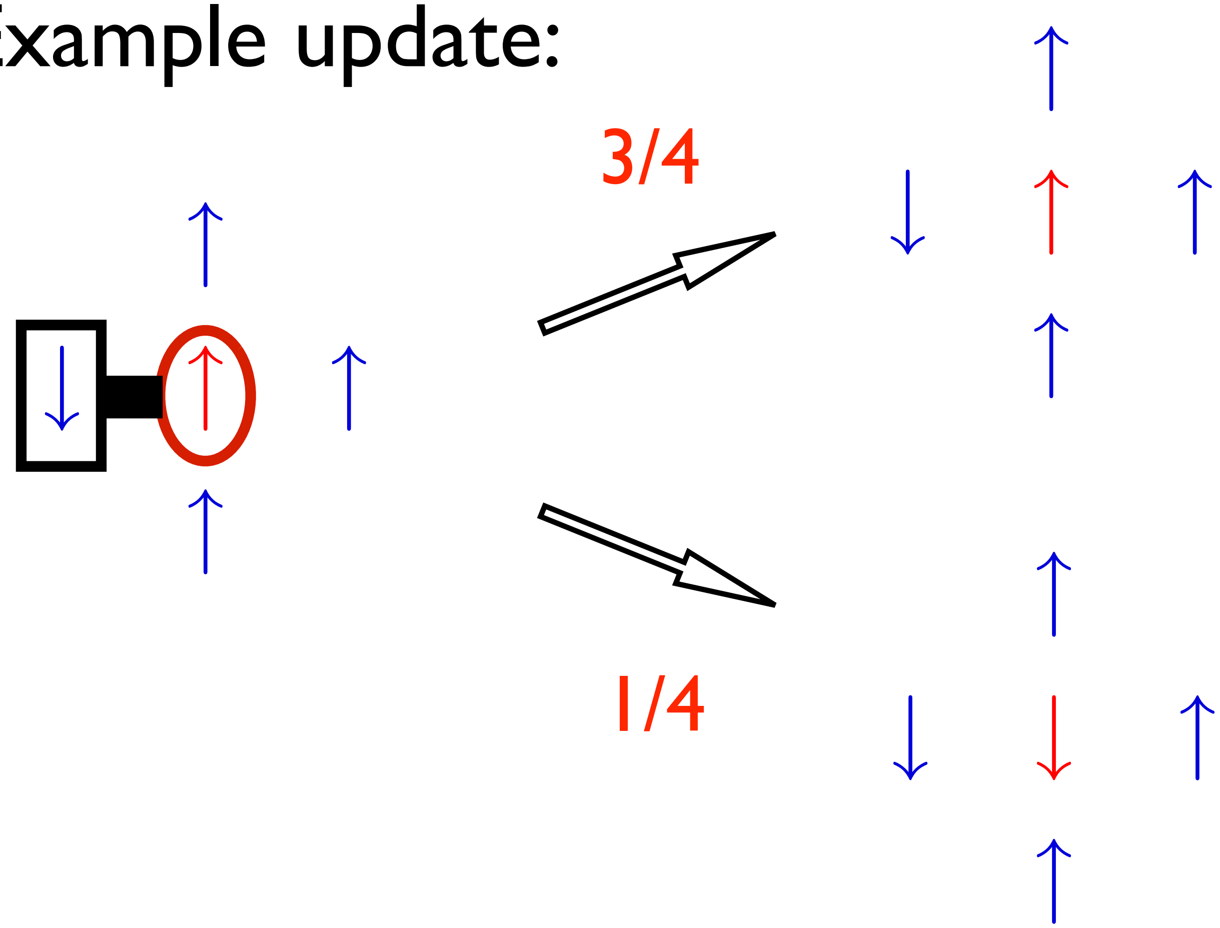
0. Binary voter variable at each site i
1. Pick a random voter
2. Assume state of randomly-selected neighbor

   individual has no self-confidence & adopts neighbor's state

Classic Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)

Example update:



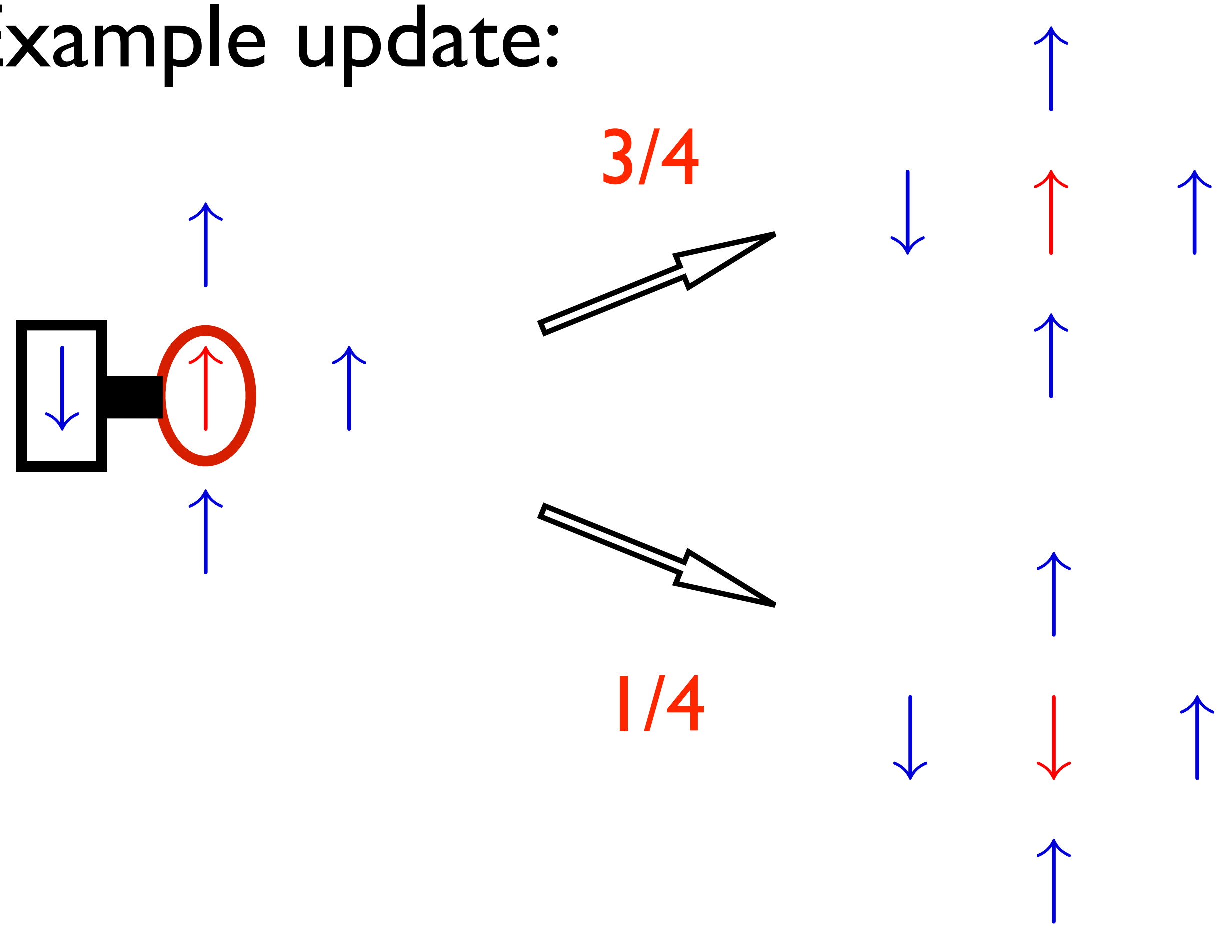
proportional rule

0. Binary voter variable at each site i
1. Pick a random voter
2. Assume state of randomly-selected neighbor
   individual has no self-confidence & adopts neighbor's state

Classic Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)

Example update:



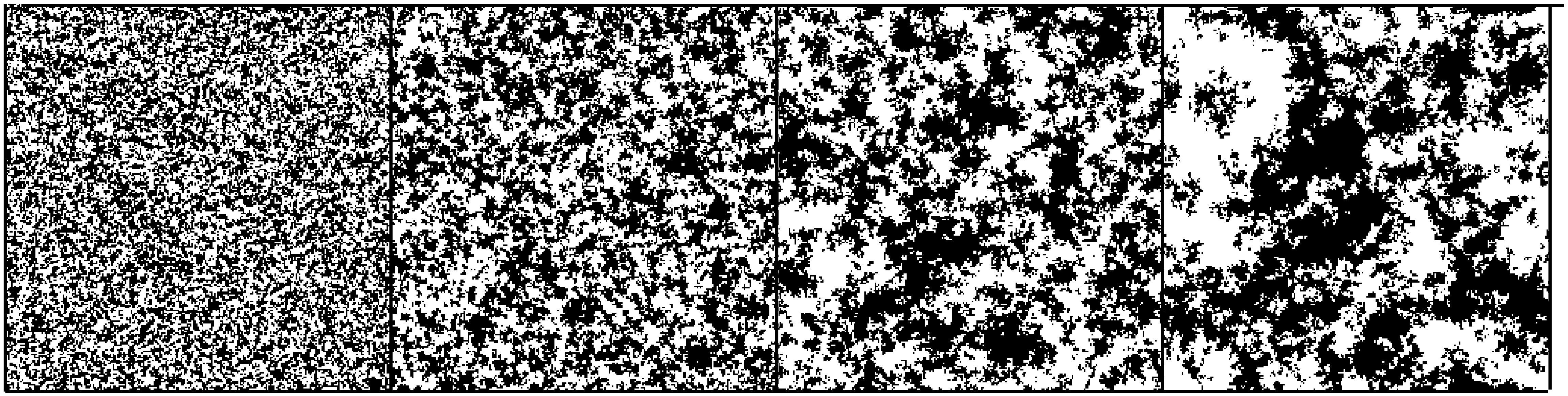
proportional rule

0. Binary voter variable at each site i
1. Pick a random voter
2. Assume state of randomly-selected neighbor
   individual has no self-confidence & adopts neighbor's state
3. Repeat 1 & 2 until consensus *necessarily* occurs in a finite system

Voter Model Evolution

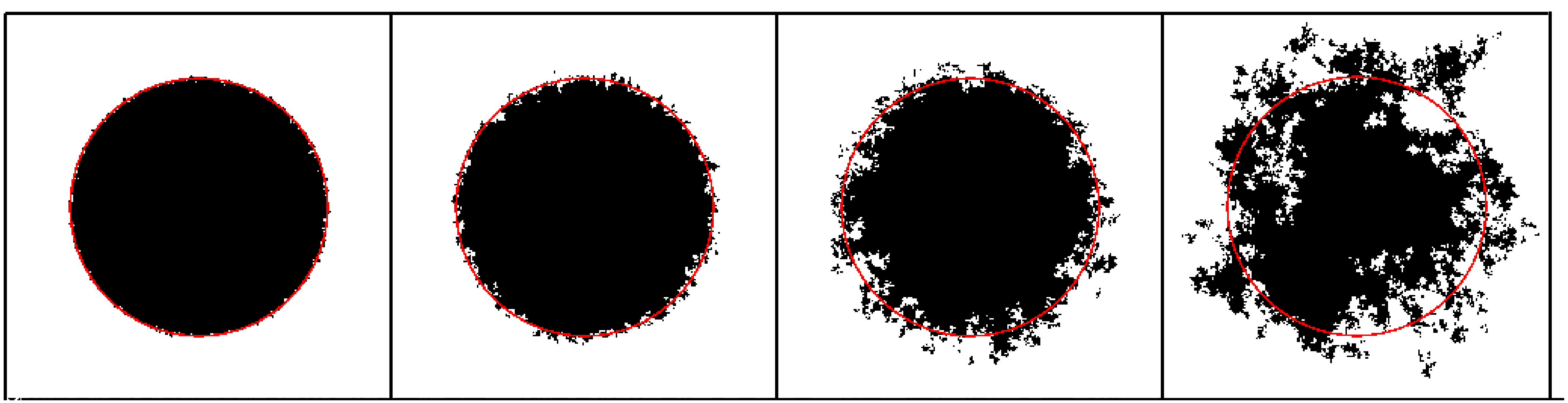
Dornic et al. (2001)

random initial condition:

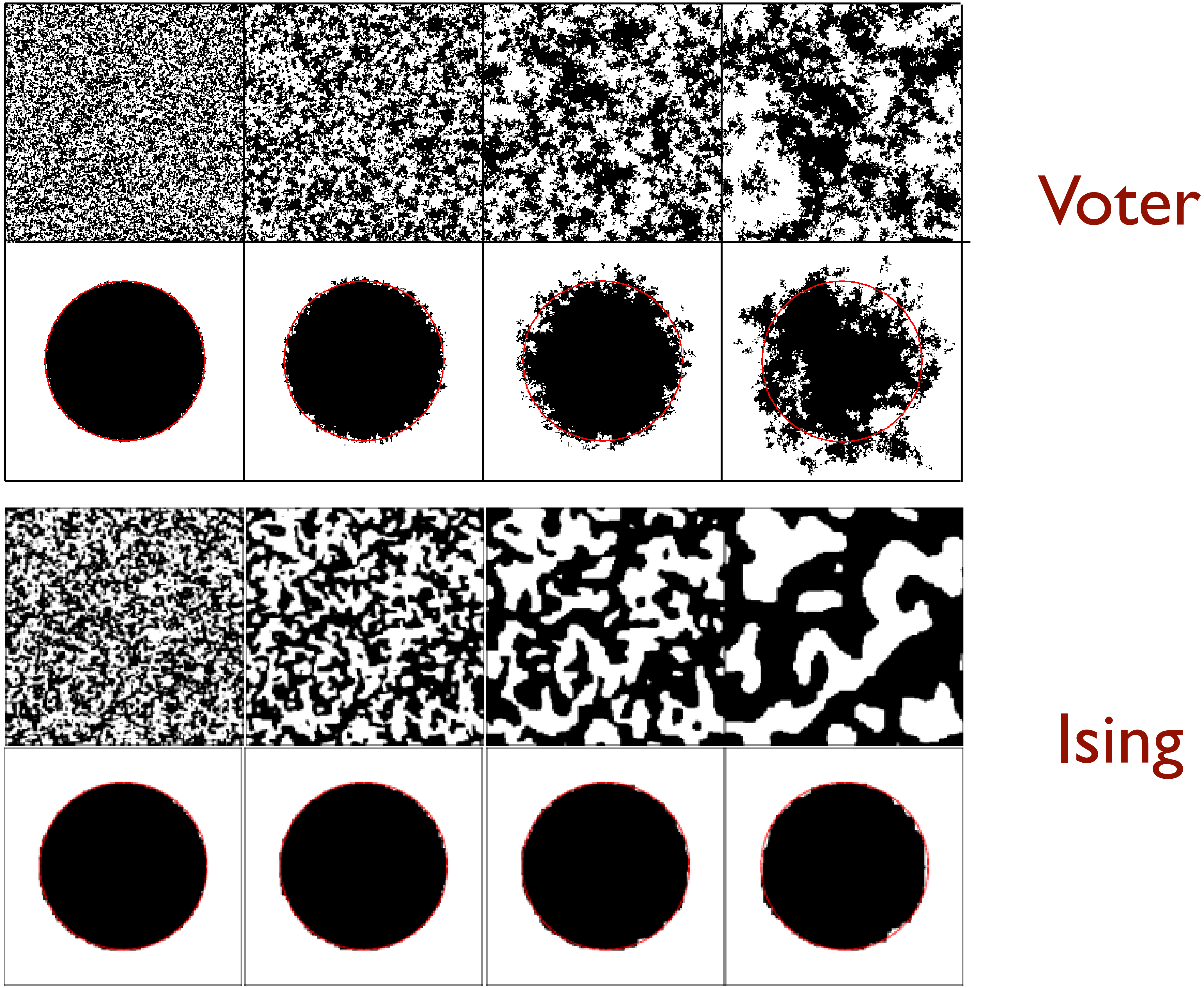


t=4 t=16 t=64 t=256

droplet initial condition:



Voter versus Ising Evolution



Voter Model & Cousins

Voter Model & Cousins

lemming

Voter Model: Tell me how to vote

Voter Model & Cousins

lemming

Voter Model: Tell me how to vote

Invasion Process: I tell you how to vote

Voter Model & Cousins

lemming

Voter Model: Tell me how to vote

Invasion Process: I tell you how to vote

Link Dynamics: Pick two disagreeing agents and change one at random

Voter Model & Cousins

lemming

Voter Model: Tell me how to vote

Invasion Process: I tell you how to vote

Link Dynamics: Pick two disagreeing agents and change one at random

identical on lattices, *distinct* on degree-heterogenous graphs

Suchecki, Eguiluz & San Miguel (2005), Castellano (2005), Sood & SR (2005)

Lattice Voter Model: 3 Basic Properties

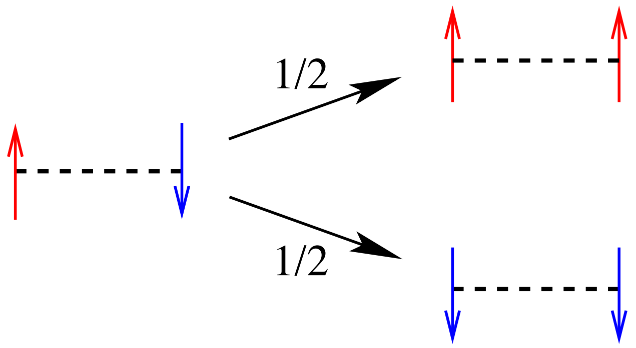
Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability $\mathcal{E}(\rho_0)$

Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability $\mathcal{E}(\rho_0)$

Evolution of a single active link:

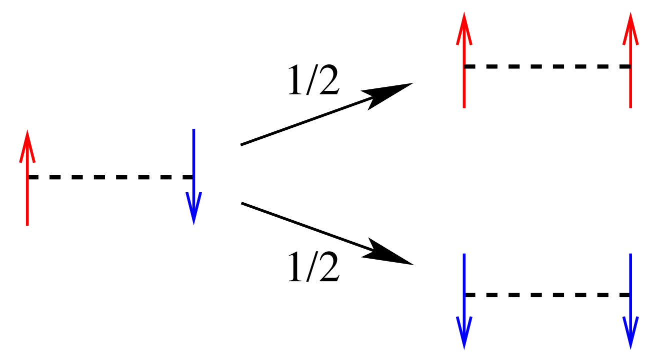


average magnetization conserved

Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability $\mathcal{E}(\rho_0) = \rho_0$

Evolution of a single active link:

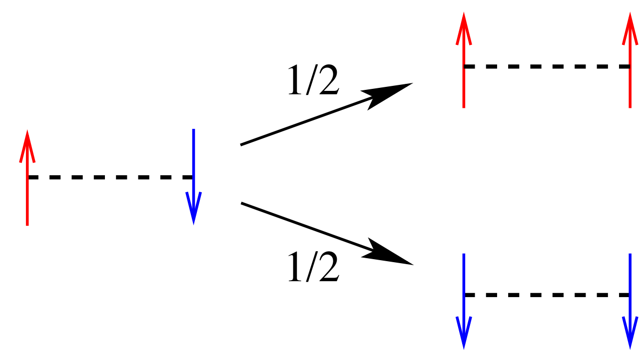


average magnetization conserved

Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability $\mathcal{E}(\rho_0) = \rho_0$

Evolution of a single active link:



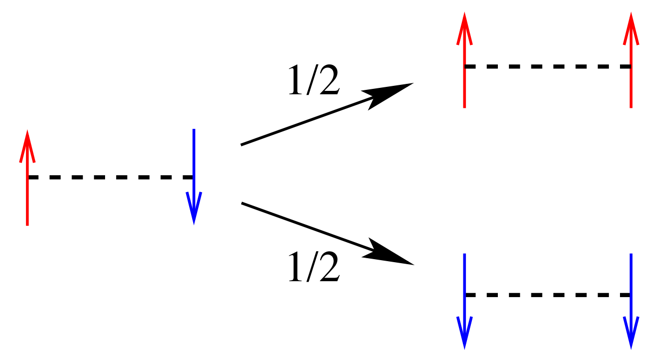
average magnetization conserved

2. Two-Spin Correlations

Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability $\mathcal{E}(\rho_0) = \rho_0$

Evolution of a single active link:



average magnetization conserved

2. Two-Spin Correlations

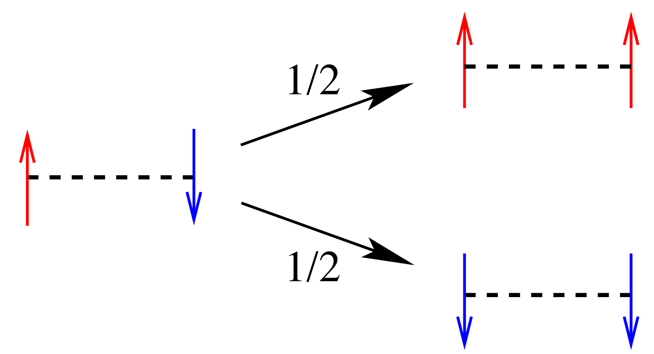
$$\frac{\partial c_2(\mathbf{r},t)}{\partial t} = \nabla^2 c_2(\mathbf{r},t)$$

$$c_2(r=0,t) = 1$$
$$c_2(r>0,t=0) = 0$$

Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability $\mathcal{E}(\rho_0) = \rho_0$

Evolution of a single active link:

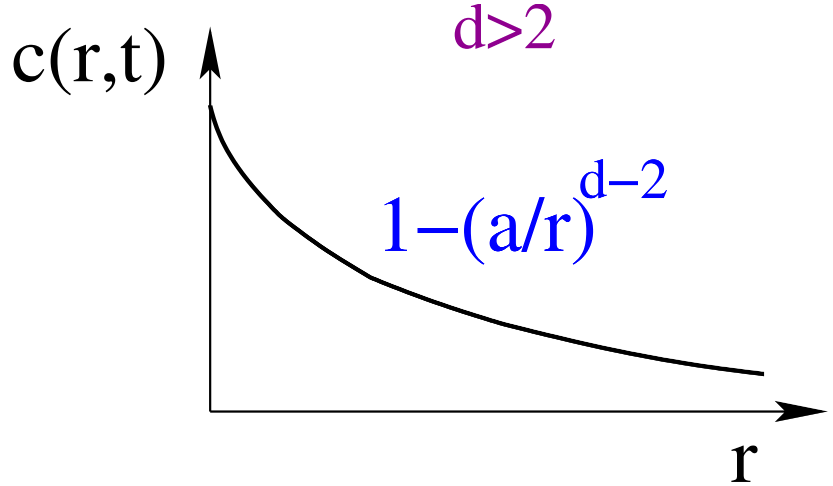


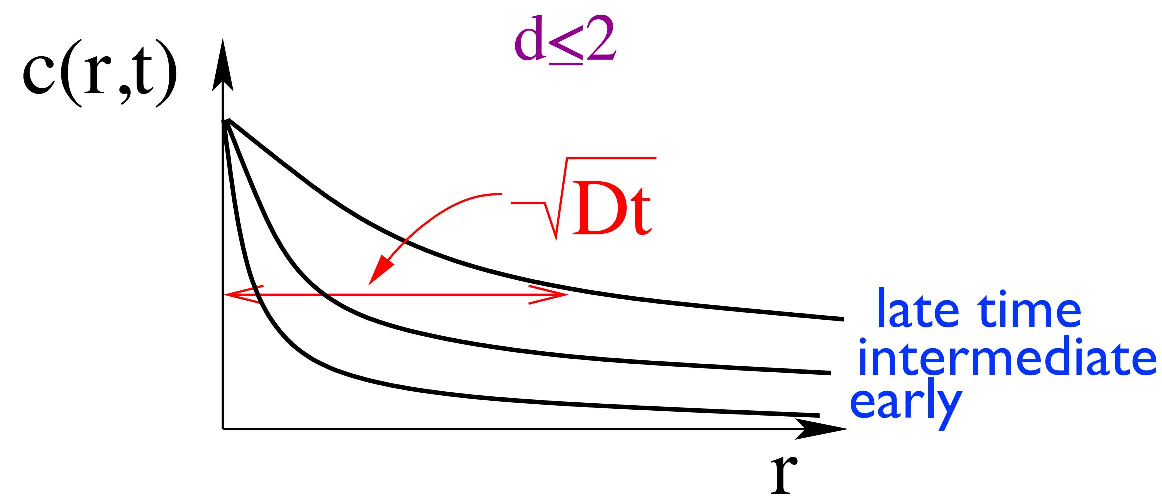
average magnetization conserved

2. Two-Spin Correlations

$$\frac{\partial c_2(\mathbf{r},t)}{\partial t} = \nabla^2 c_2(\mathbf{r},t)$$

$$c_2(r=0,t) = 1$$
$$c_2(r>0,t=0) = 0$$

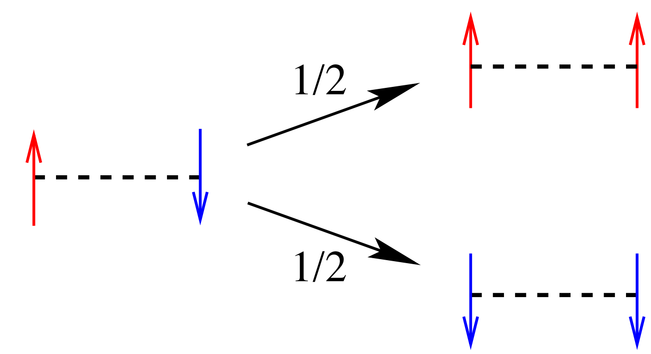




Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability $\mathcal{E}(\rho_0) = \rho_0$

Evolution of a single active link:



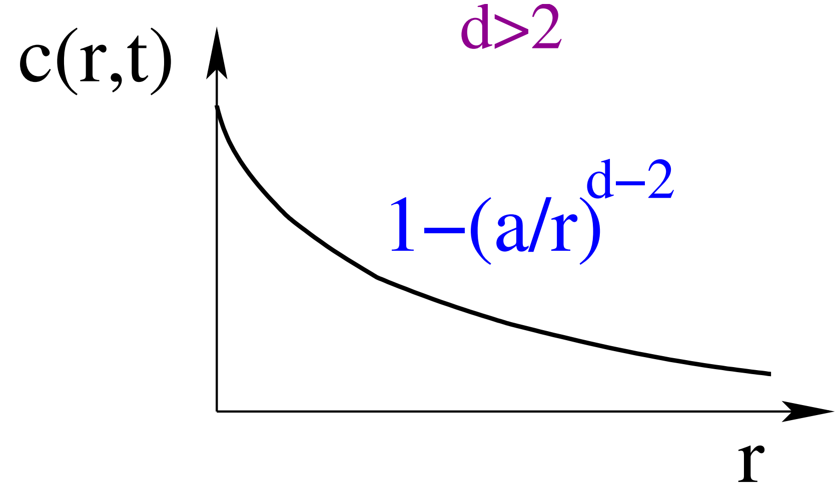
average magnetization conserved

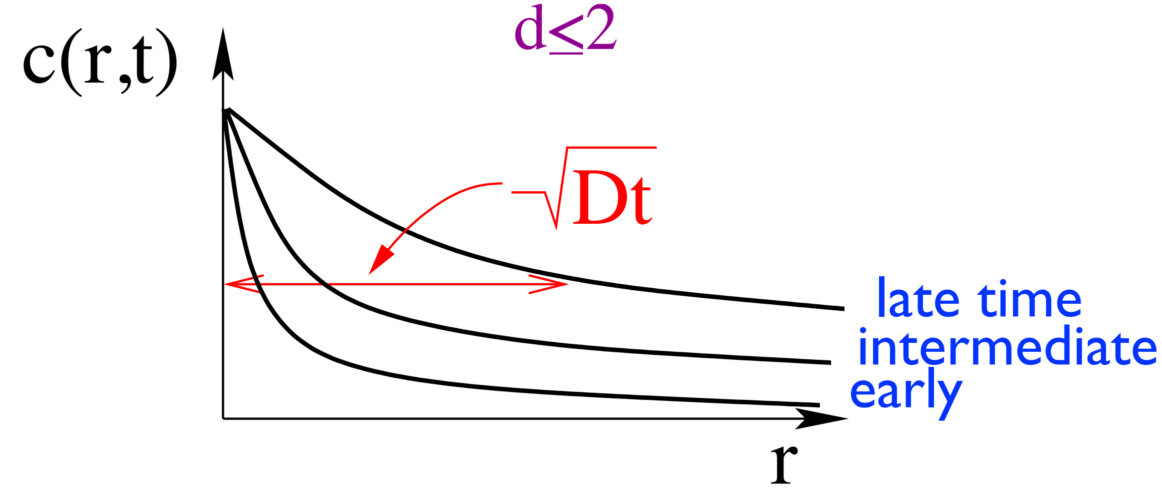
2. Two-Spin Correlations

$$\frac{\partial c_2(\mathbf{r},t)}{\partial t} = \nabla^2 c_2(\mathbf{r},t)$$

$c_2(r=0,t) = 1$

$c_2(r>0,t=0) = 0$



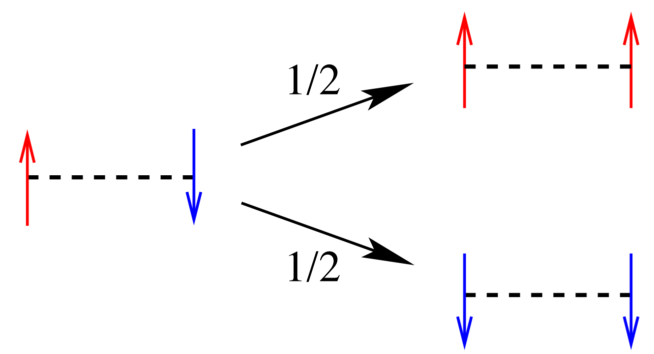


3. Consensus Time

Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability $\mathcal{E}(\rho_0) = \rho_0$

Evolution of a single active link:

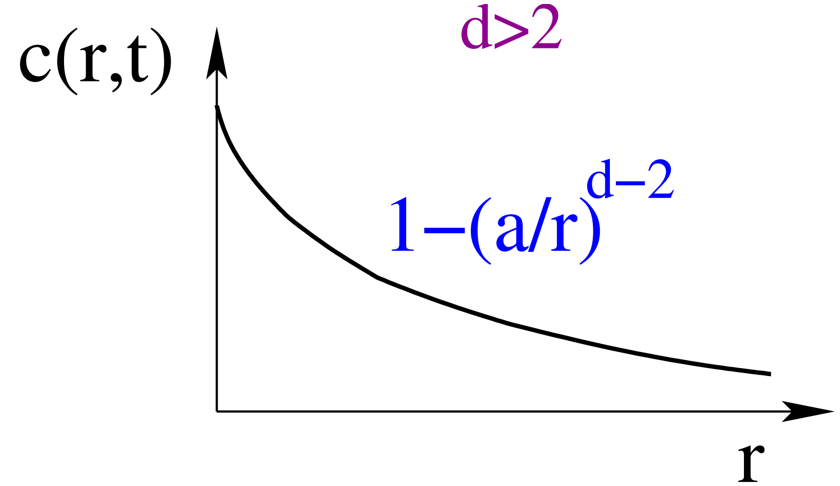


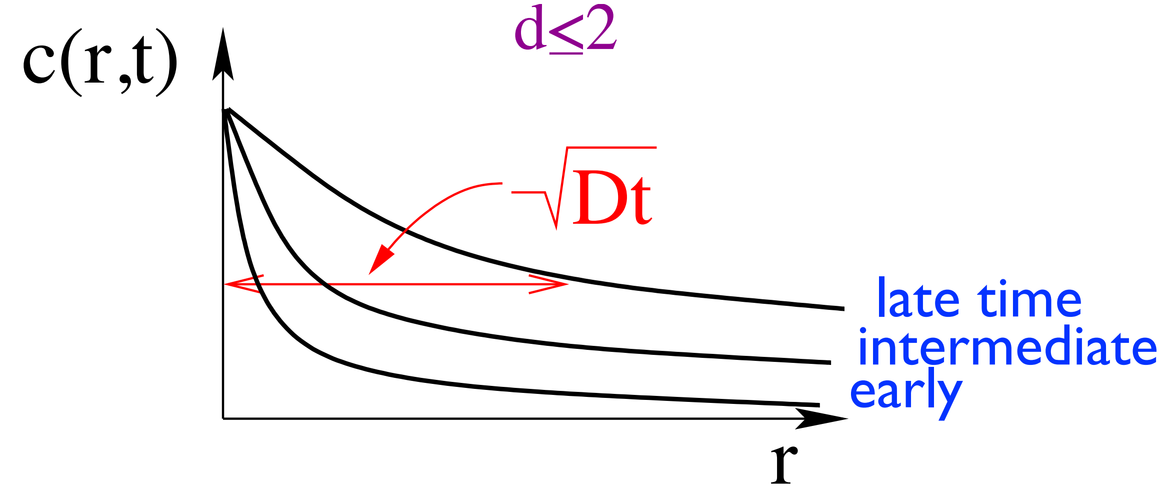
average magnetization conserved

2. Two-Spin Correlations

$$\frac{\partial c_2(\mathbf{r},t)}{\partial t} = \nabla^2 c_2(\mathbf{r},t)$$

$$c_2(r=0,t) = 1$$
$$c_2(r>0,t=0) = 0$$





3. Consensus Time

dimension	consensus time
1	N^2
2	N ln N
>2	N

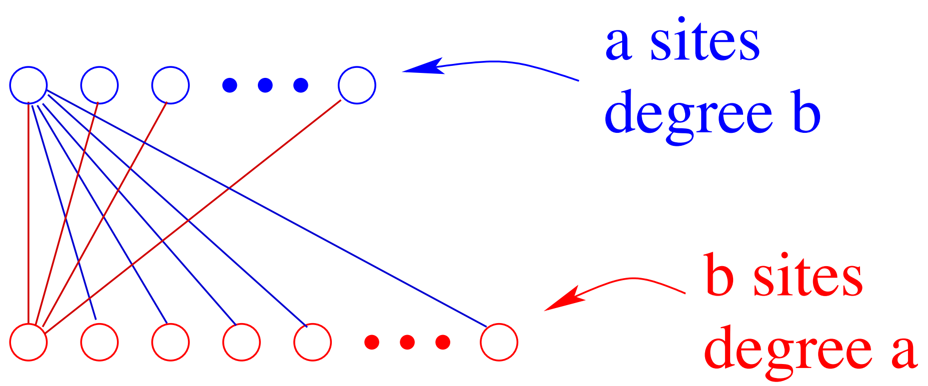
Voter Model on Complex Networks

C. Castellano, D.Vilon, A.Vespignani, EPL **63**, 153 (2003)
K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL **69**, 228 (2005)
V. Sood, SR, PRL **94**, 178701 (2005); T. Antal, V. Sood, SR, PRE **77**, 041121 (2008)

Voter Model on Complex Networks

C. Castellano, D. Vilon, A. Vespignani, EPL **63**, 153 (2003)
K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL **69**, 228 (2005)
V. Sood, SR, PRL **94**, 178701 (2005); T. Antal, V. Sood, SR, PRE **77**, 041121 (2008)

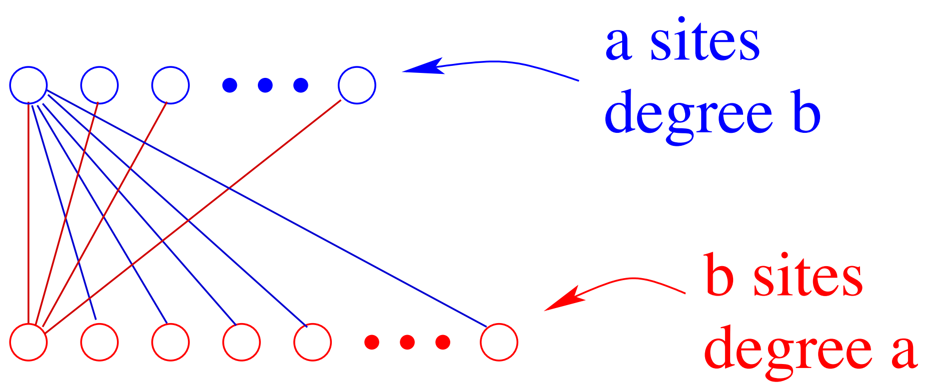
illustrative example:
complete bipartite graph



Voter Model on Complex Networks

C. Castellano, D. Vilon, A. Vespignani, EPL **63**, 153 (2003)
K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL **69**, 228 (2005)
V. Sood, SR, PRL **94**, 178701 (2005); T. Antal, V. Sood, SR, PRE **77**, 041121 (2008)

illustrative example:
complete bipartite graph



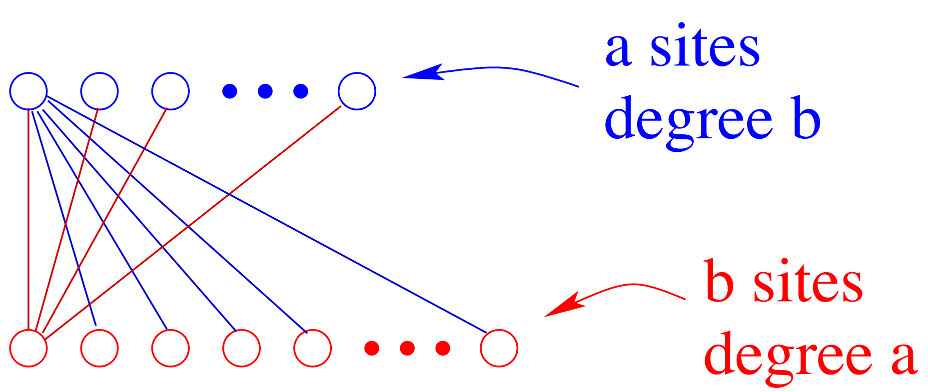
$$dN_a = \frac{a}{a+b}\left[\frac{a-N_a}{a}\frac{N_b}{b} - \frac{N_a}{a}\frac{b-N_b}{b}\right]$$

$$dN_b = \frac{b}{a+b}\left[\frac{b-N_b}{b}\frac{N_a}{a} - \frac{N_b}{b}\frac{a-N_a}{a}\right]$$

Voter Model on Complex Networks

C. Castellano, D. Vilon, A. Vespignani, EPL **63**, 153 (2003)
K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL **69**, 228 (2005)
V. Sood, SR, PRL **94**, 178701 (2005); T. Antal, V. Sood, SR, PRE **77**, 041121 (2008)

illustrative example:
complete bipartite graph



pick site on
a sublattice

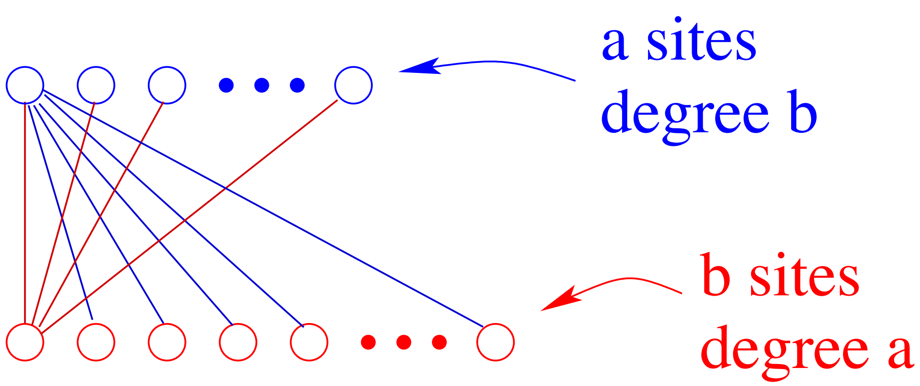
$$dN_a = \frac{a}{a+b}\left[\frac{a-N_a}{a}\,\frac{N_b}{b} - \frac{N_a}{a}\,\frac{b-N_b}{b}\right]$$

$$dN_b = \frac{b}{a+b}\left[\frac{b-N_b}{b}\,\frac{N_a}{a} - \frac{N_b}{b}\,\frac{a-N_a}{a}\right]$$

Voter Model on Complex Networks

C. Castellano, D. Vilon, A. Vespignani, EPL **63**, 153 (2003)
K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL **69**, 228 (2005)
V. Sood, SR, PRL **94**, 178701 (2005); T. Antal, V. Sood, SR, PRE **77**, 041121 (2008)

illustrative example:
complete bipartite graph



pick site on **a** sublattice

pick ↓ on **a**

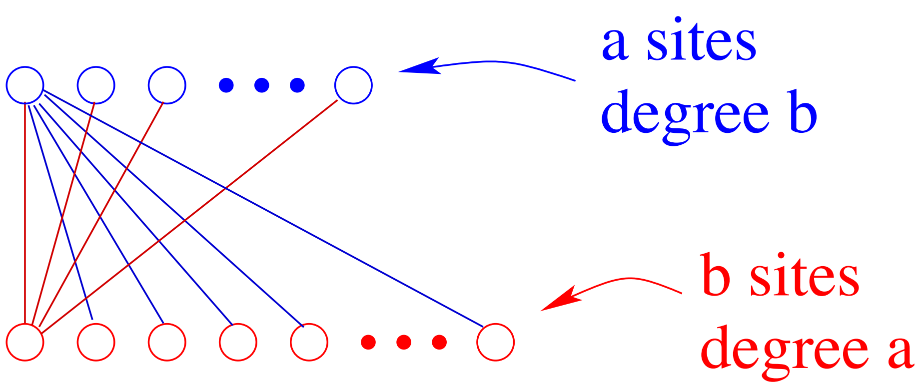
$$dN_a = \frac{a}{a+b}\left[\frac{a-N_a}{a}\,\frac{N_b}{b} - \frac{N_a}{a}\,\frac{b-N_b}{b}\right]$$

$$dN_b = \frac{b}{a+b}\left[\frac{b-N_b}{b}\,\frac{N_a}{a} - \frac{N_b}{b}\,\frac{a-N_a}{a}\right]$$

Voter Model on Complex Networks

C. Castellano, D. Vilon, A. Vespignani, EPL **63**, 153 (2003)
K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL **69**, 228 (2005)
V. Sood, SR, PRL **94**, 178701 (2005); T. Antal, V. Sood, SR, PRE **77**, 041121 (2008)

illustrative example:
complete bipartite graph



pick site on **a** sublattice

pick ↓ on **a**

pick ↑ on ***b*** sublattice

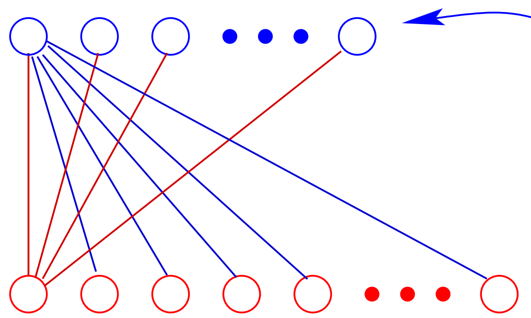
$$dN_a = \frac{a}{a+b}\left[\frac{a-N_a}{a}\,\frac{N_b}{b} - \frac{N_a}{a}\,\frac{b-N_b}{b}\right]$$

$$dN_b = \frac{b}{a+b}\left[\frac{b-N_b}{b}\,\frac{N_a}{a} - \frac{N_b}{b}\,\frac{a-N_a}{a}\right]$$

Voter Model on Complex Networks

C. Castellano, D. Vilon, A. Vespignani, EPL **63**, 153 (2003)
K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL **69**, 228 (2005)
V. Sood, SR, PRL **94**, 178701 (2005); T. Antal, V. Sood, SR, PRE **77**, 041121 (2008)

illustrative example:
complete bipartite graph



pick site on a sublattice

pick ↓ on **a**

pick ↑ on ***b*** sublattice

$$dN_a = \frac{a}{a+b}\left[\frac{a-N_a}{a}\frac{N_b}{b} - \frac{N_a}{a}\frac{b-N_b}{b}\right]$$

$$dN_b = \frac{b}{a+b}\left[\frac{b-N_b}{b}\frac{N_a}{a} - \frac{N_b}{b}\frac{a-N_a}{a}\right]$$

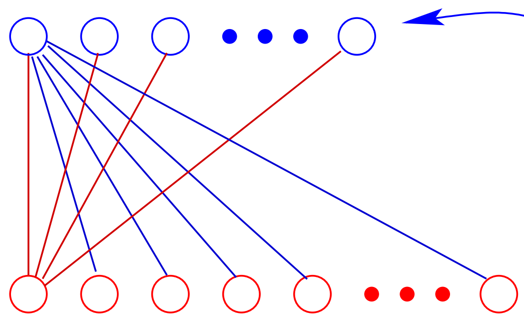
Subgraph densities: $\rho_a = N_a/a,\ \rho_b = N_b/b \quad dt = 1/(a+b)$

$$\begin{aligned}\rho_{a,b}(t) &= \frac{1}{2}[\rho_{a,b}(0) - \rho_{b,a}(0)]\, e^{-2t} + \frac{1}{2}[\rho_a(0) + \rho_b(0)] \\ &\to \frac{1}{2}[\rho_a(0) + \rho_b(0)]\end{aligned}$$

Voter Model on Complex Networks

C. Castellano, D. Vilon, A. Vespignani, EPL **63**, 153 (2003)
K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL **69**, 228 (2005)
V. Sood, SR, PRL **94**, 178701 (2005); T. Antal, V. Sood, SR, PRE **77**, 041121 (2008)

illustrative example:
complete bipartite graph



pick site on a sublattice

pick $\downarrow$ on **a**

pick $\uparrow$ on ***b*** sublattice

$$dN_a = \frac{a}{a+b}\left[\frac{a-N_a}{a}\frac{N_b}{b} - \frac{N_a}{a}\frac{b-N_b}{b}\right]$$

$$dN_b = \frac{b}{a+b}\left[\frac{b-N_b}{b}\frac{N_a}{a} - \frac{N_b}{b}\frac{a-N_a}{a}\right]$$

Subgraph densities: $\rho_a = N_a/a,\ \rho_b = N_b/b \quad dt = 1/(a+b)$

$$\begin{aligned}\rho_{a,b}(t) &= \frac{1}{2}[\rho_{a,b}(0) - \rho_{b,a}(0)]\,e^{-2t} + \frac{1}{2}[\rho_a(0)+\rho_b(0)] \\ &\to \frac{1}{2}[\rho_a(0)+\rho_b(0)]\end{aligned}$$

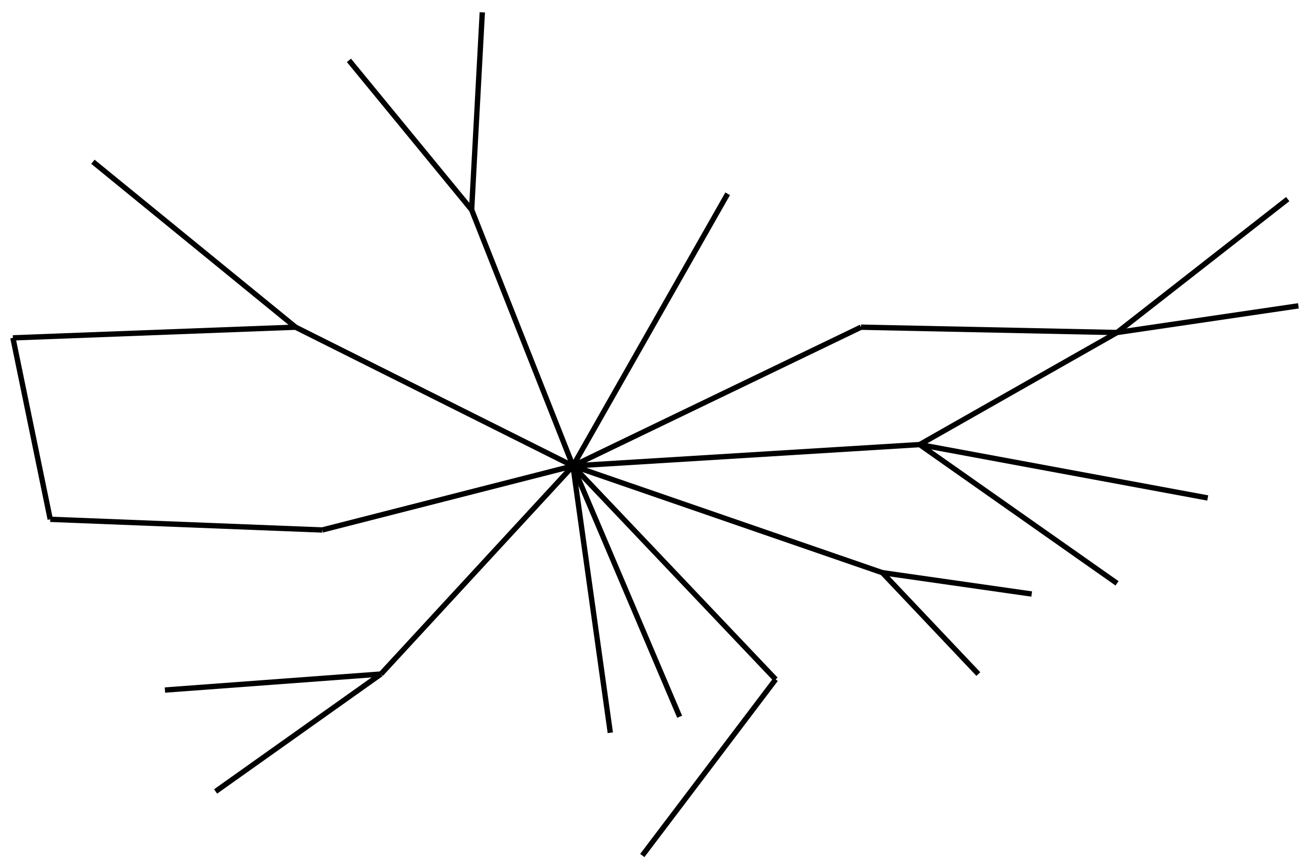
magnetization ***not*** conserved

Voter Model on Complex Networks

mechanism for non-conservation

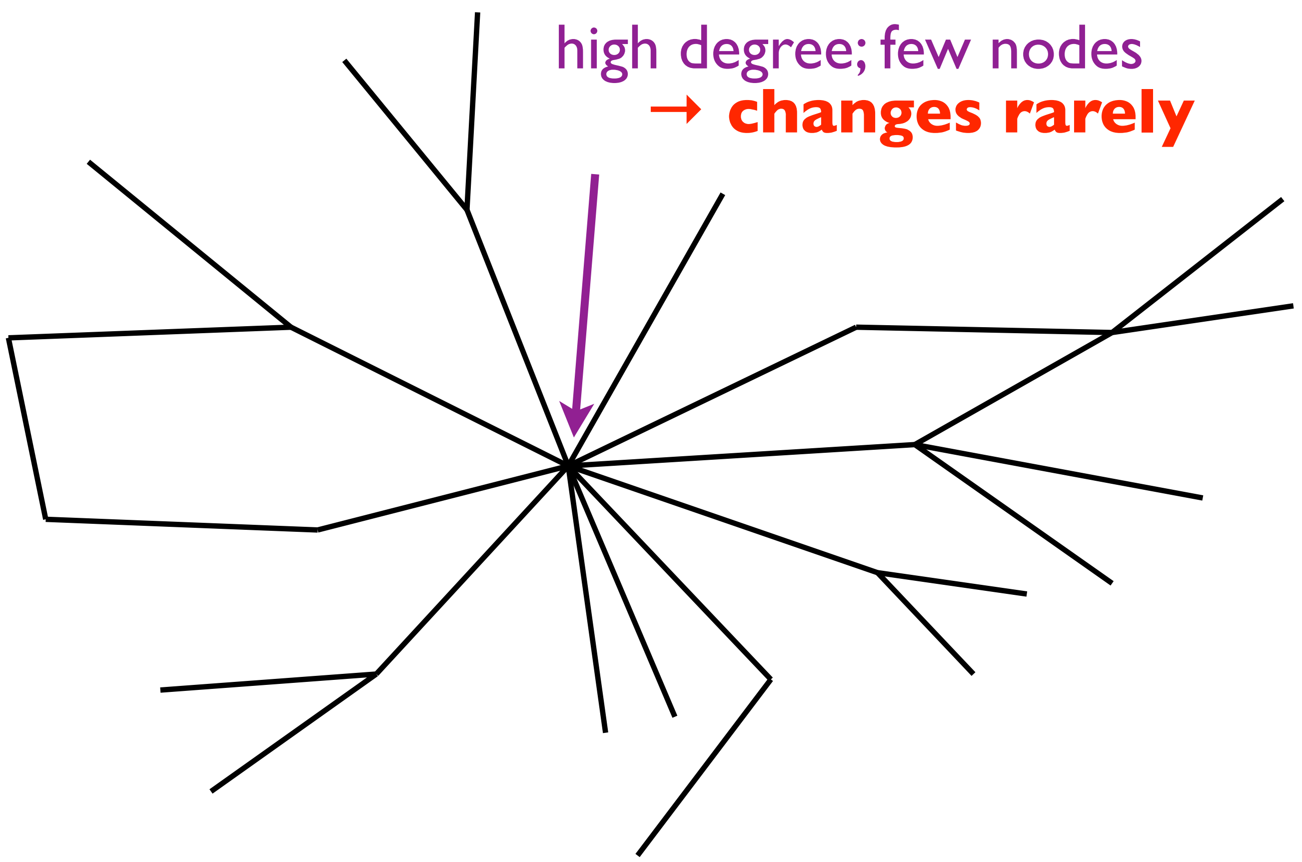
Voter Model on Complex Networks

mechanism for non-conservation



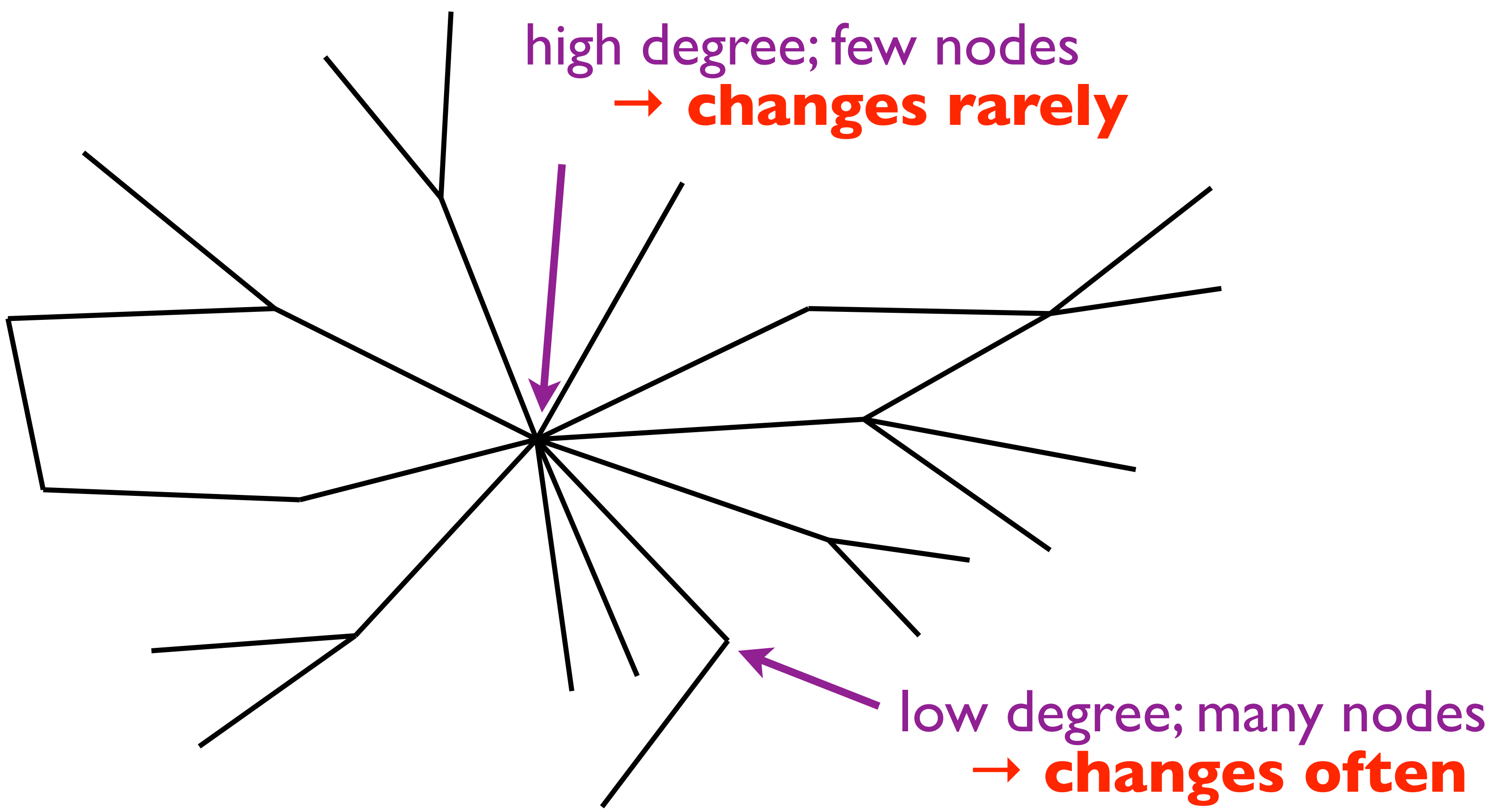
Voter Model on Complex Networks

mechanism for non-conservation



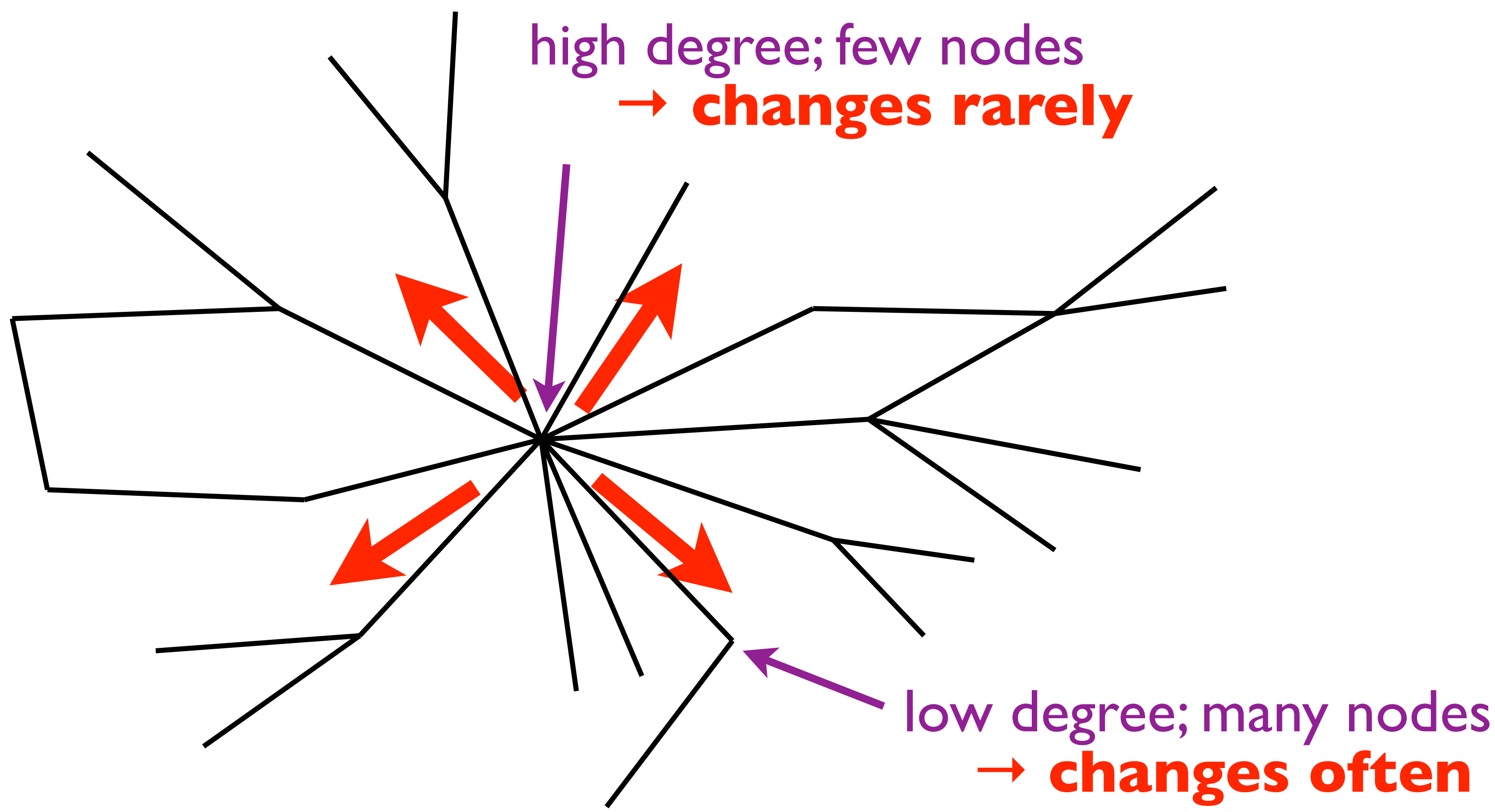
Voter Model on Complex Networks

mechanism for non-conservation



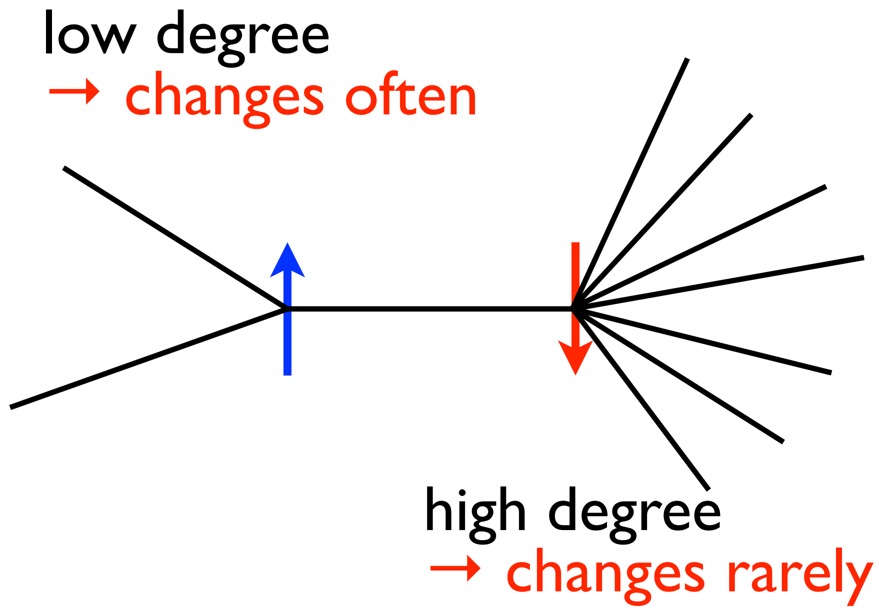
Voter Model on Complex Networks

mechanism for non-conservation

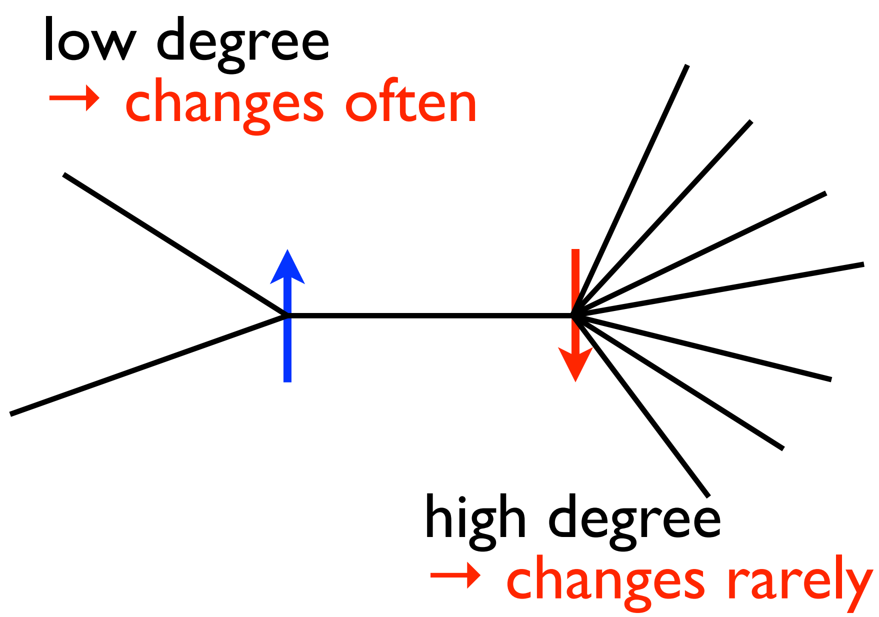


*"flow" from **high** degree to **low** degree*

New Conservation Law



New Conservation Law



to compensate the different rates:

degree-weighted 1st moment:

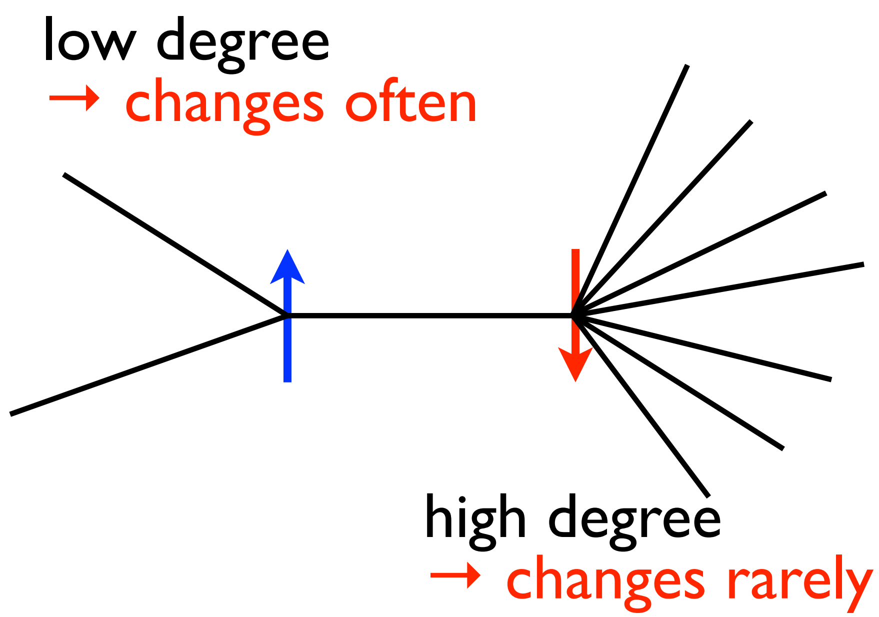
$$\omega = \frac{1}{\mu_1} \sum_k k\, n_k \rho_k$$

$$\mu_1 = \text{av. degree}$$

$$n_k = \text{frac. nodes of degree } k$$

$$\rho_k = \text{frac.} \uparrow \text{ on nodes of degree } k$$

New Conservation Law



to compensate the different rates:

degree-weighted 1st moment: $$\omega = \frac{1}{\mu_1} \sum_k k\, n_k \rho_k$$ *conserved!*

$\mu_1 = \text{av. degree}$

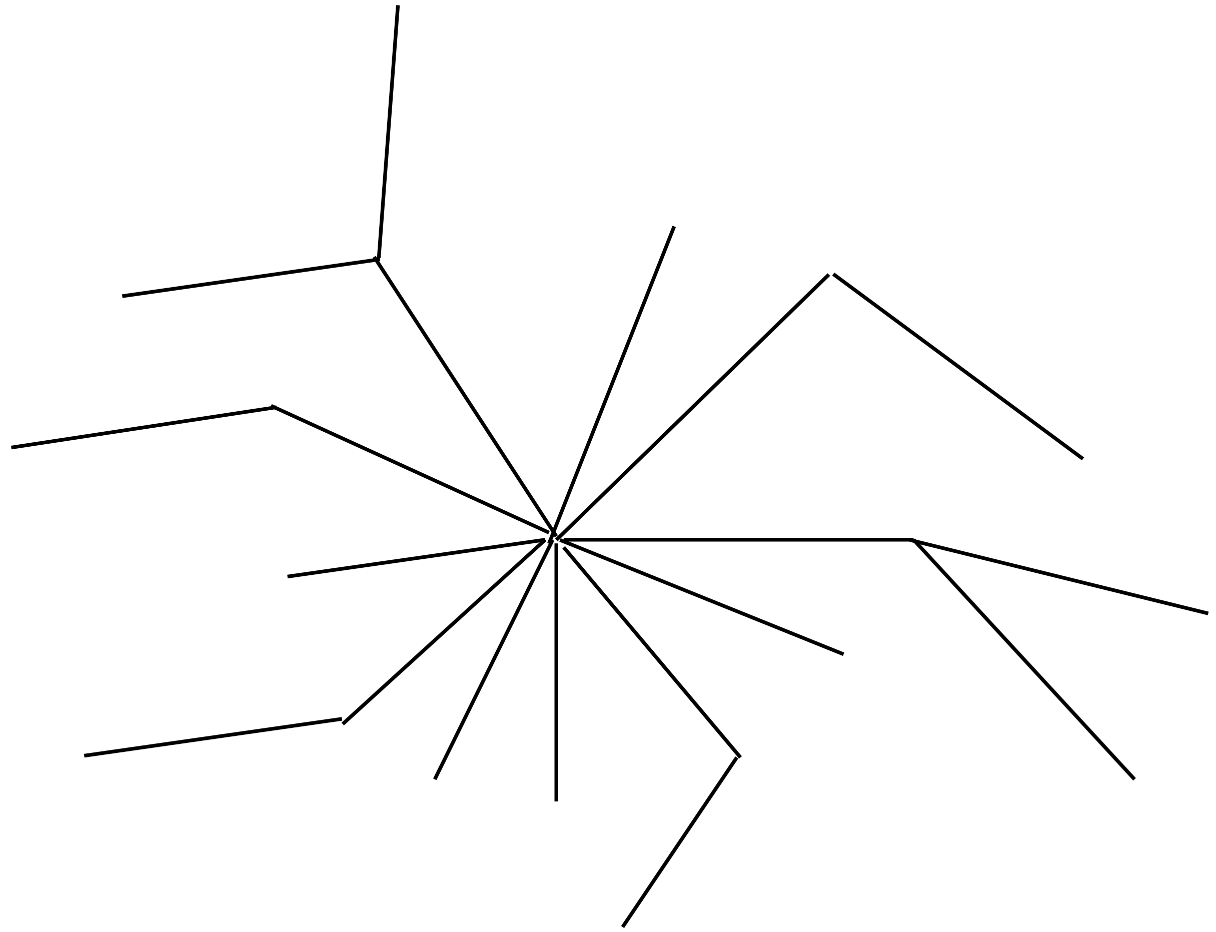
$n_k = \text{frac. nodes of degree } k$

$\rho_k = \text{frac. } \uparrow \text{ on nodes of degree } k$

Invasion Process on Complex Networks

Castellano (2005)

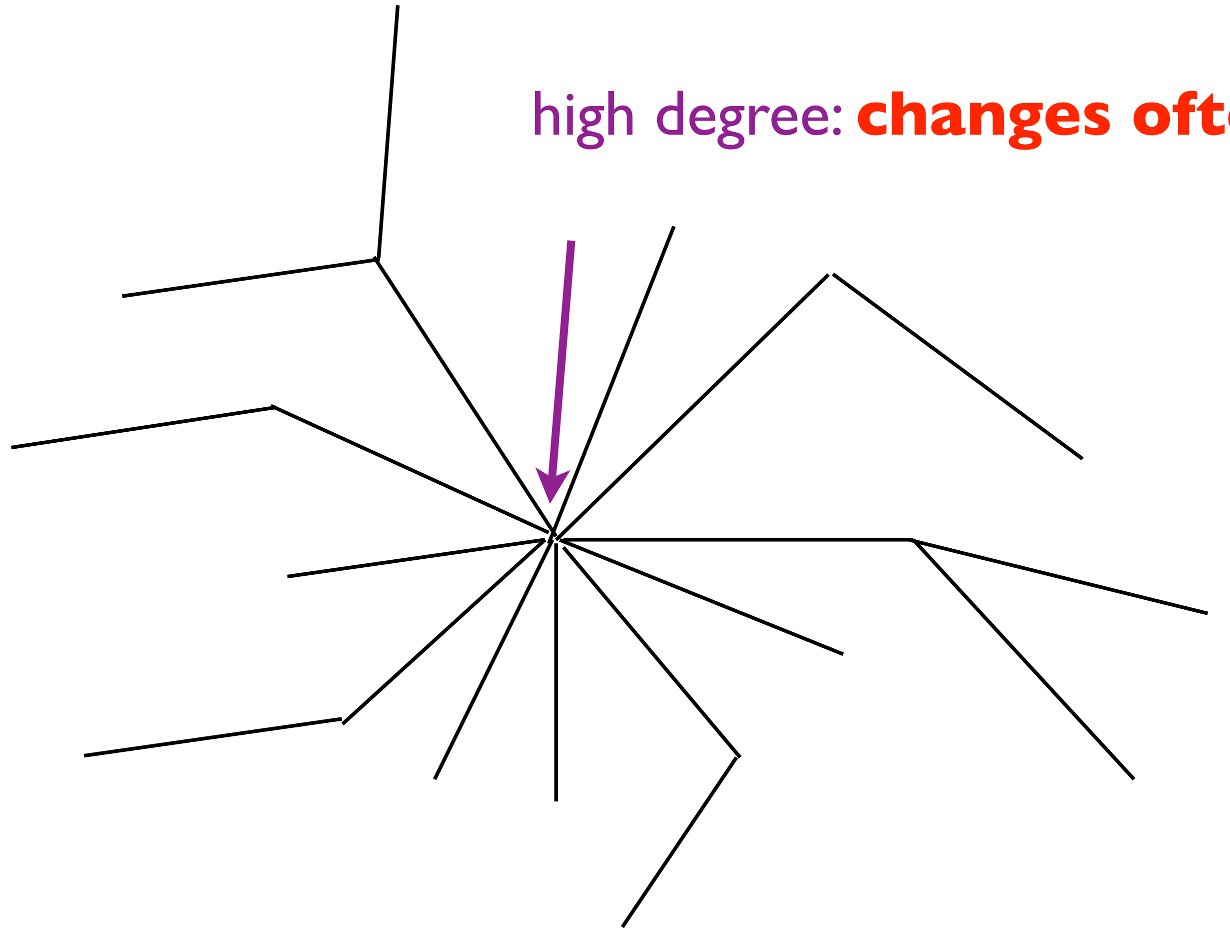
Antal, Sood, SR (2005, 06, 08)



Invasion Process on Complex Networks

Castellano (2005)

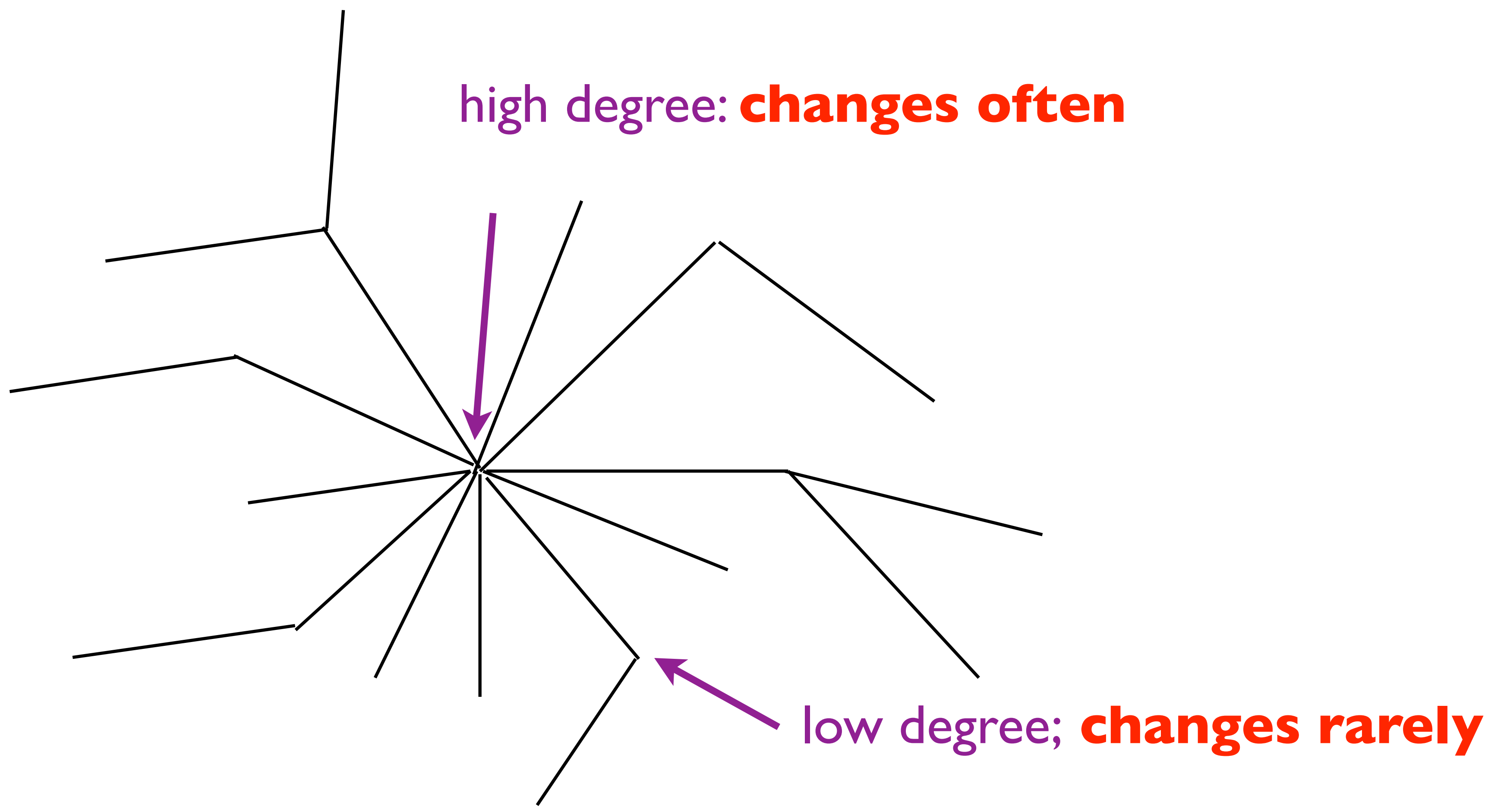
Antal, Sood, SR (2005, 06, 08)



Invasion Process on Complex Networks

Castellano (2005)

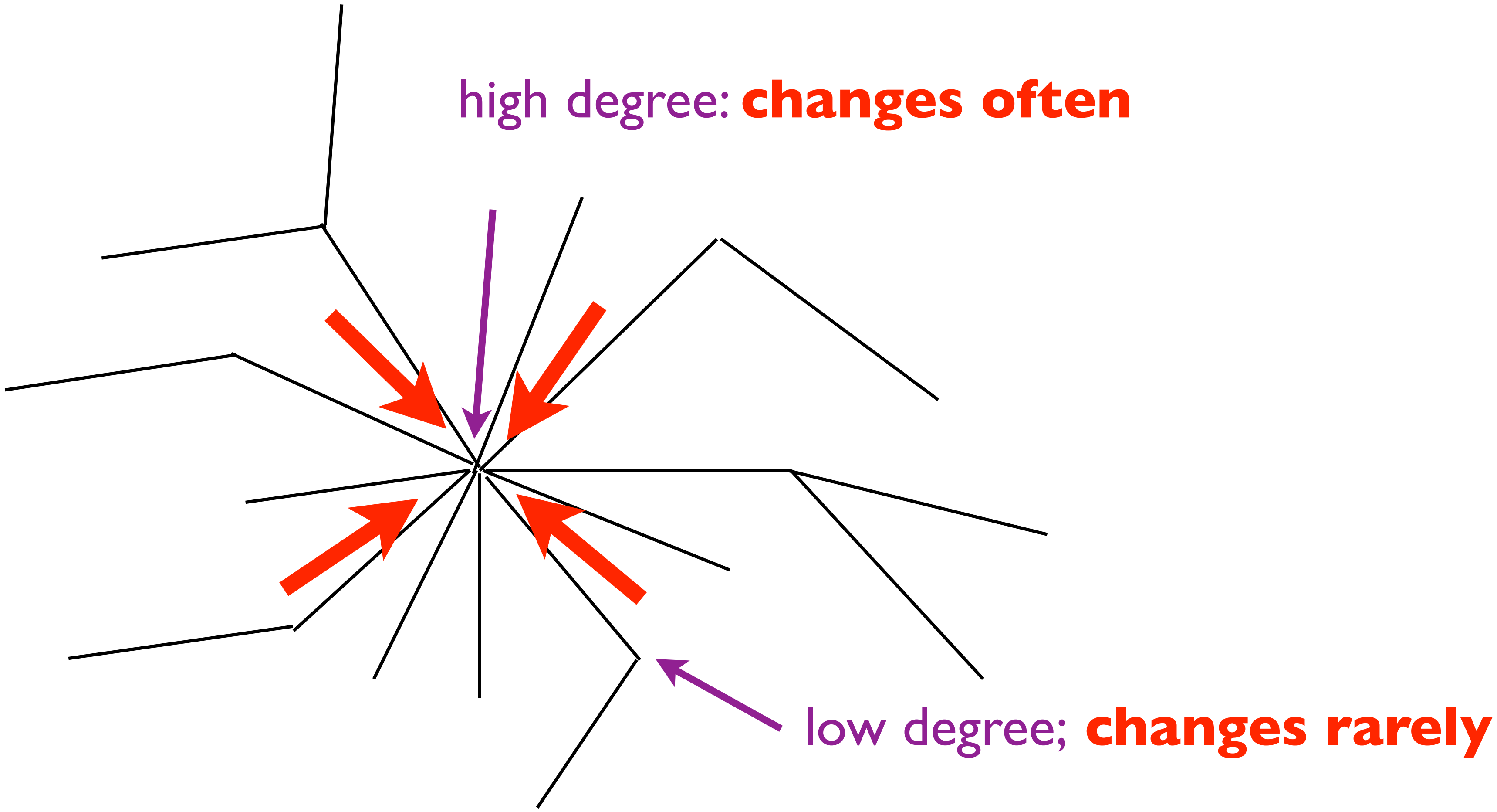
Antal, Sood, SR (2005, 06, 08)



Invasion Process on Complex Networks

Castellano (2005)

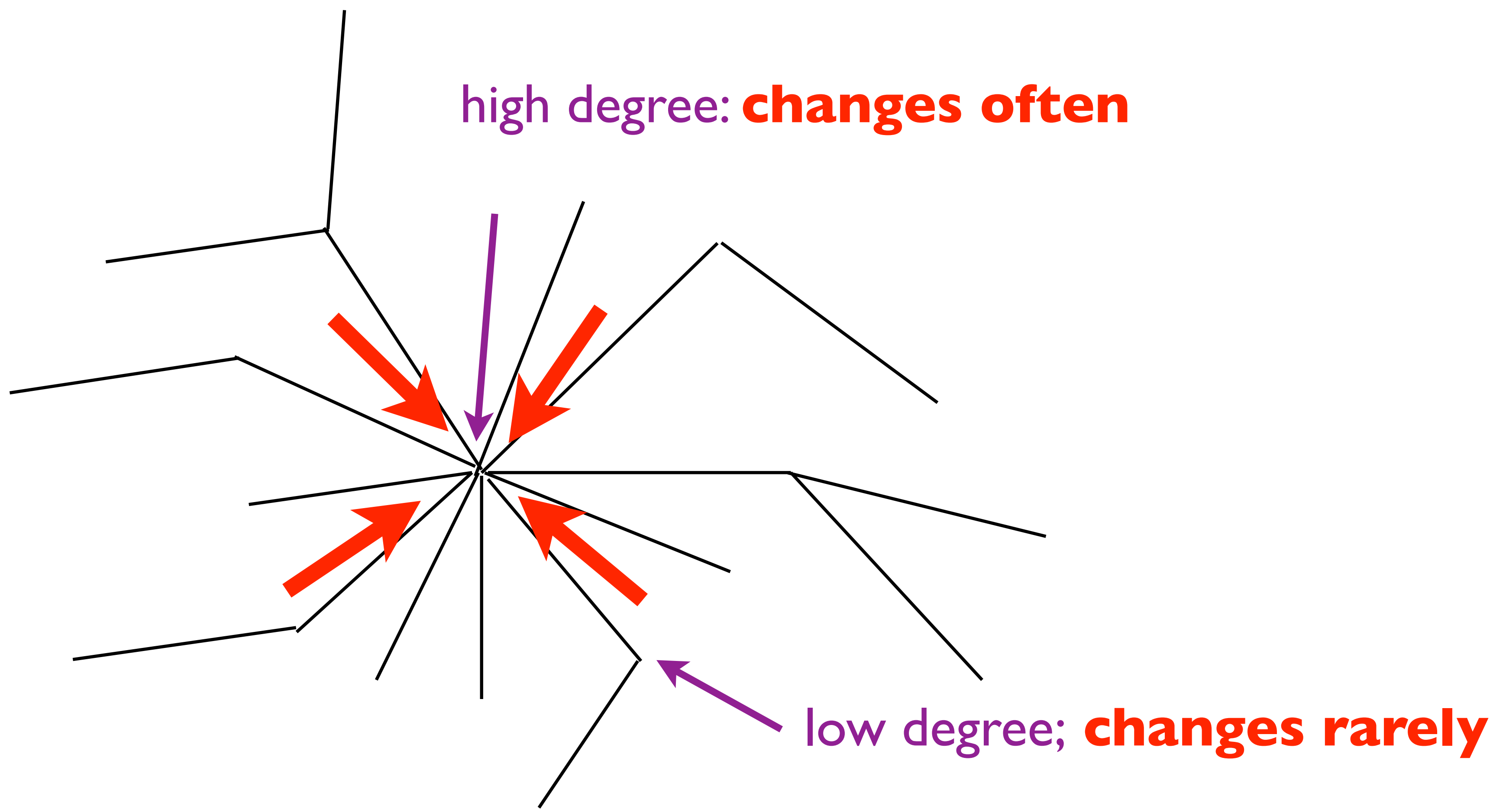
Antal, Sood, SR (2005, 06, 08)



*"flow" from **low** degree to **high** degree*

Invasion Process on Complex Networks

Castellano (2005)
Antal, Sood, SR (2005, 06, 08)



*"flow" from **low** degree to **high** degree*

degree-weighted inverse moment $$\omega_{-1} = \frac{1}{\mu_1}\sum_k k^{-1} n_k\, \rho_k$$ *conserved!*

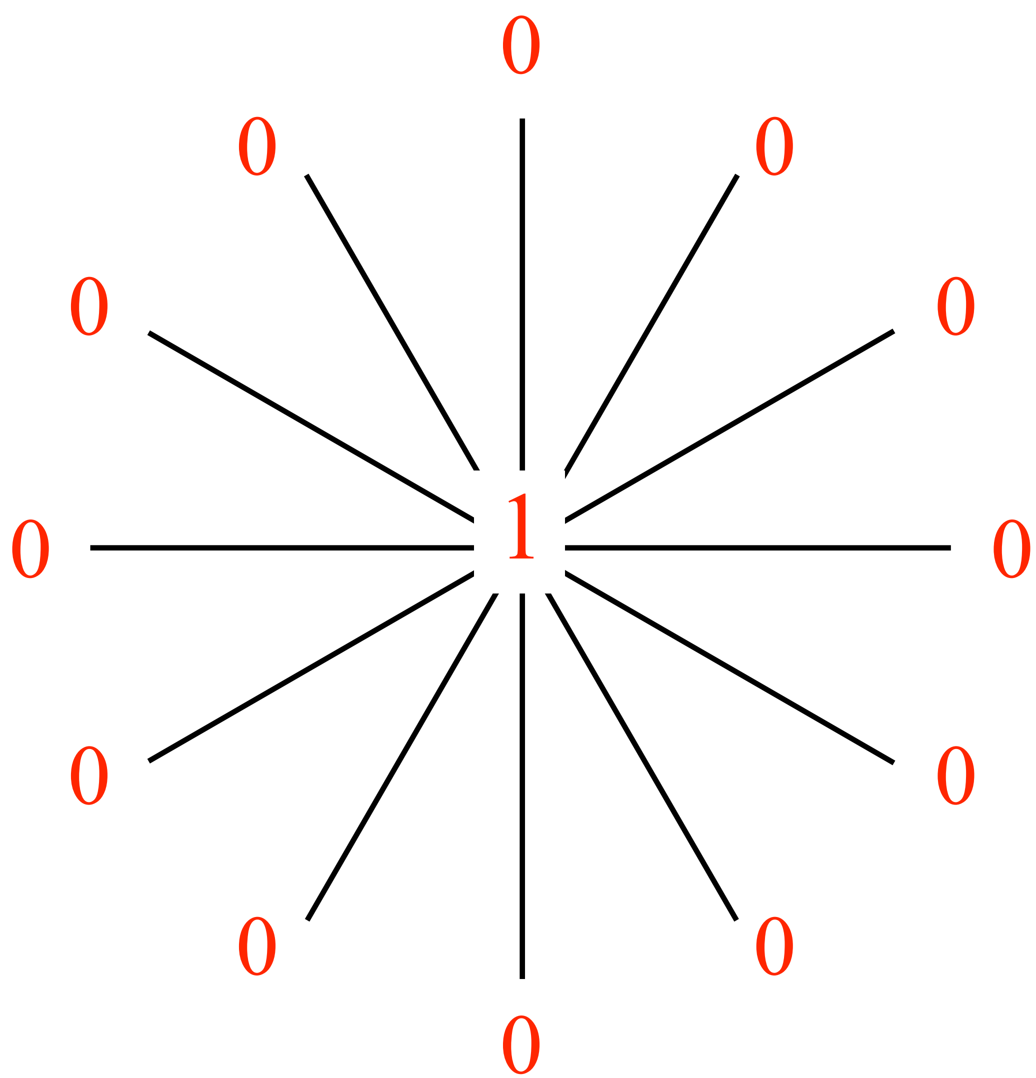
Exit Probability on Complex Graphs

$$\mathcal{E}(\omega) = \omega$$

Exit Probability on Complex Graphs

$$\mathcal{E}(\omega) = \omega$$

Extreme case: star graph



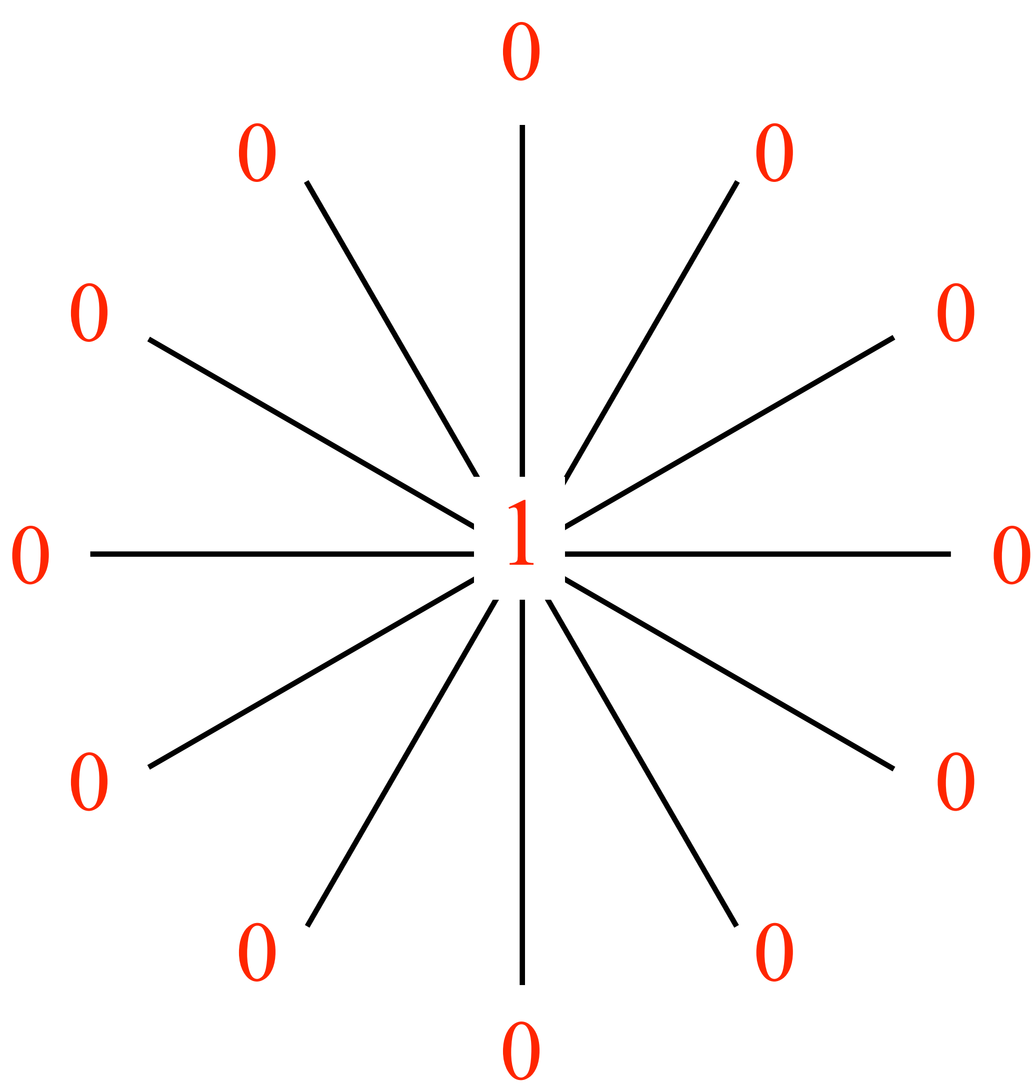
N nodes: degree 1
1 node: degree N

Exit Probability on Complex Graphs

$$\mathcal{E}(\omega) = \omega$$

Extreme case: star graph

N nodes: degree 1
1 node: degree N

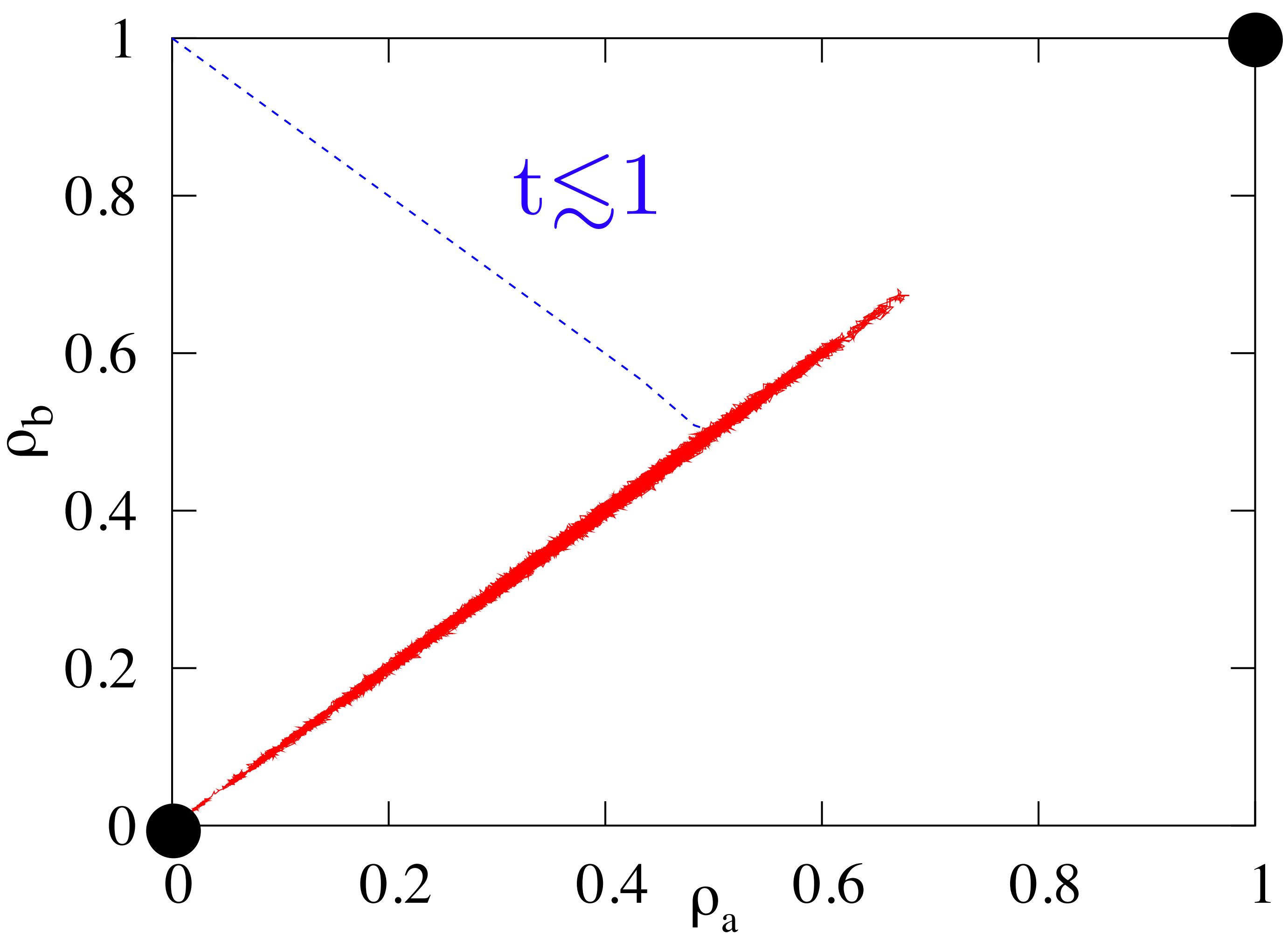


$$\omega = \frac{1}{\mu_1}\sum_k k\, n_k \rho_k = \frac{1}{2}$$

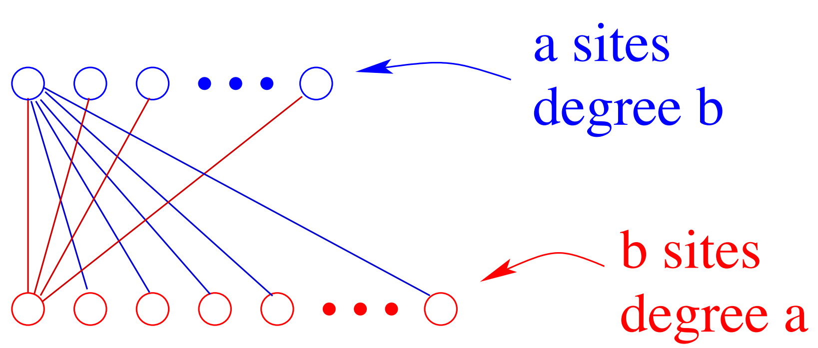
Final state: all 1 with prob. 1/2!

Route to Consensus on Complex Graphs

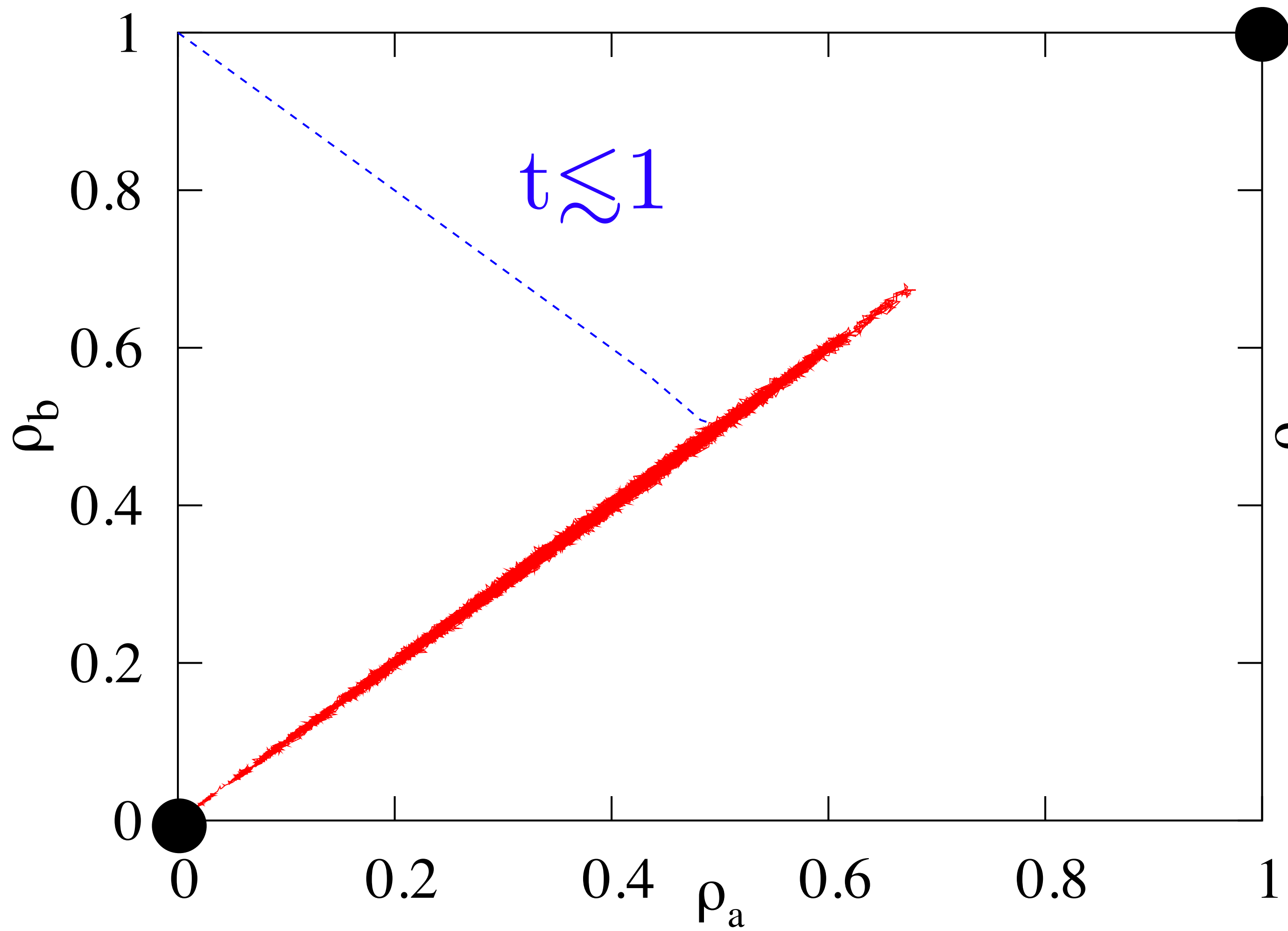
Route to Consensus on Complex Graphs

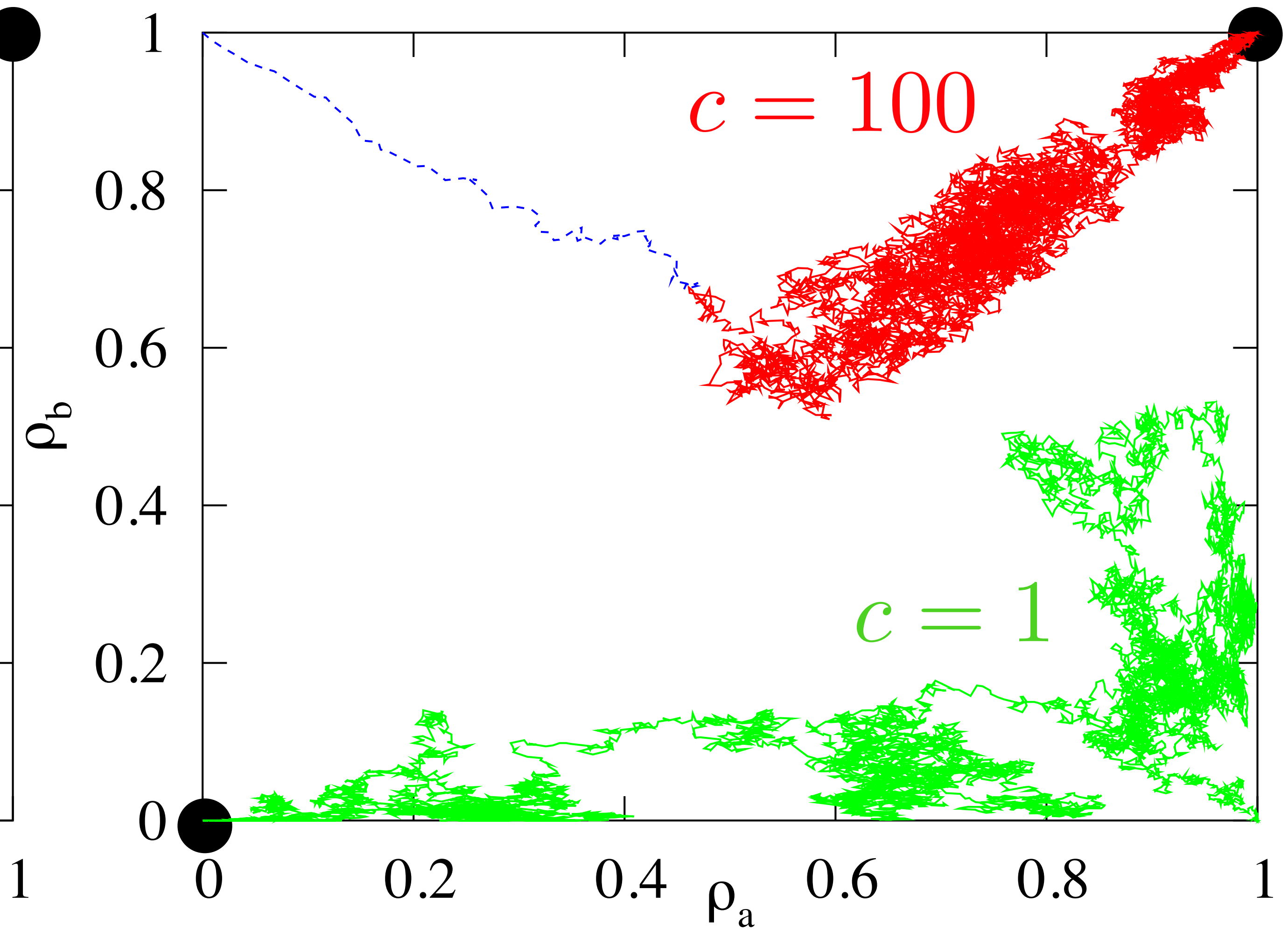


complete bipartite graph

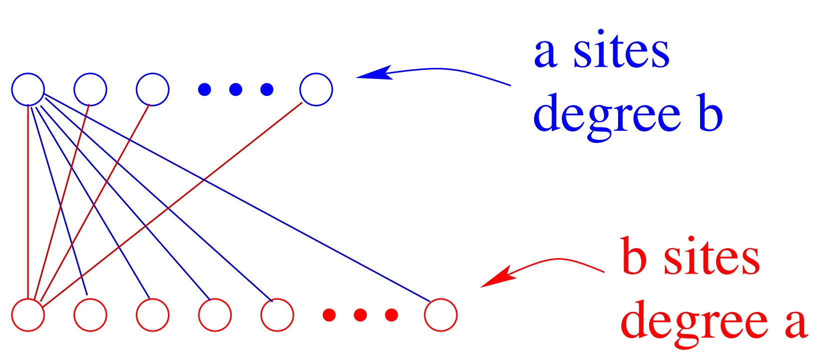


Route to Consensus on Complex Graphs

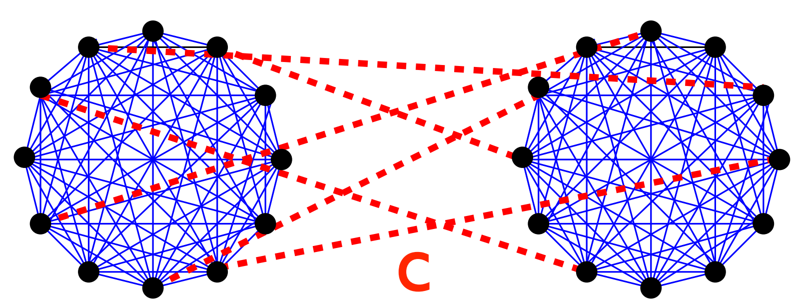




complete bipartite graph



two-clique graph



N=10000, C links/node

Consensus Time Evolution Equation

Consensus Time Evolution Equation

warmup: complete graph

$T(\rho) \equiv$ av. consensus time starting with density ρ

Consensus Time Evolution Equation

warmup: complete graph

$T(\rho) \equiv$ av. consensus time starting with density ρ

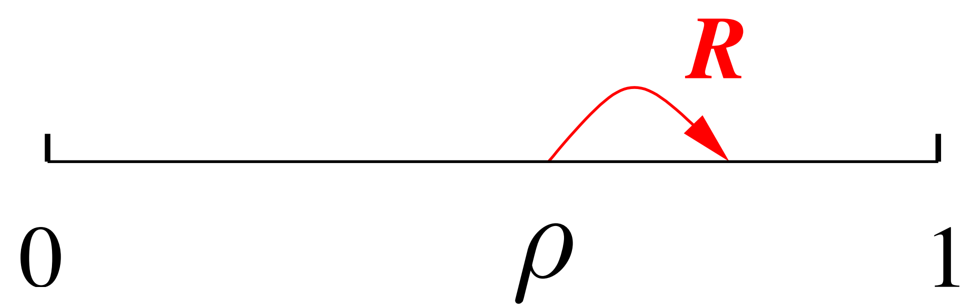
$$\begin{aligned} T(\rho) &= \mathcal{R}(\rho)[T(\rho+d\rho)+dt] \\ &\quad +\mathcal{L}(\rho)[T(\rho-d\rho)+dt] \\ &\quad +[1-\mathcal{R}(\rho)-\mathcal{L}(\rho)][T(\rho)+dt] \end{aligned}$$

Consensus Time Evolution Equation

warmup: complete graph

$T(\rho) \equiv$ av. consensus time starting with density ρ

$$
\begin{aligned}
T(\rho) \quad &= \quad \mathcal{R}(\rho)[T(\rho + d\rho) + dt] \\
&\quad\quad +\mathcal{L}(\rho)[T(\rho - d\rho) + dt] \\
&\quad\quad +[1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)][T(\rho) + dt]
\end{aligned}
$$



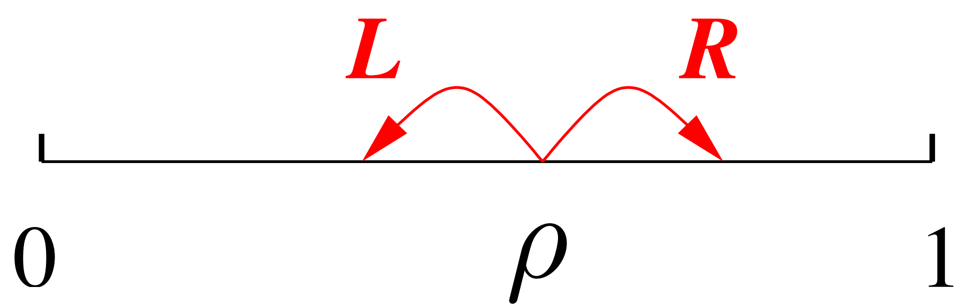
$$
\begin{aligned}
\mathcal{R}(\rho) &\equiv \mathrm{prob}(\downarrow\uparrow\rightarrow\uparrow\uparrow) \\
&= \rho(1-\rho)
\end{aligned}
$$

Consensus Time Evolution Equation

warmup: complete graph

$T(\rho) \equiv$ av. consensus time starting with density ρ

$$\begin{aligned} T(\rho) &= \mathcal{R}(\rho)[T(\rho+d\rho)+dt] \\ &\quad +\mathcal{L}(\rho)[T(\rho-d\rho)+dt] \\ &\quad +[1-\mathcal{R}(\rho)-\mathcal{L}(\rho)][T(\rho)+dt] \end{aligned}$$



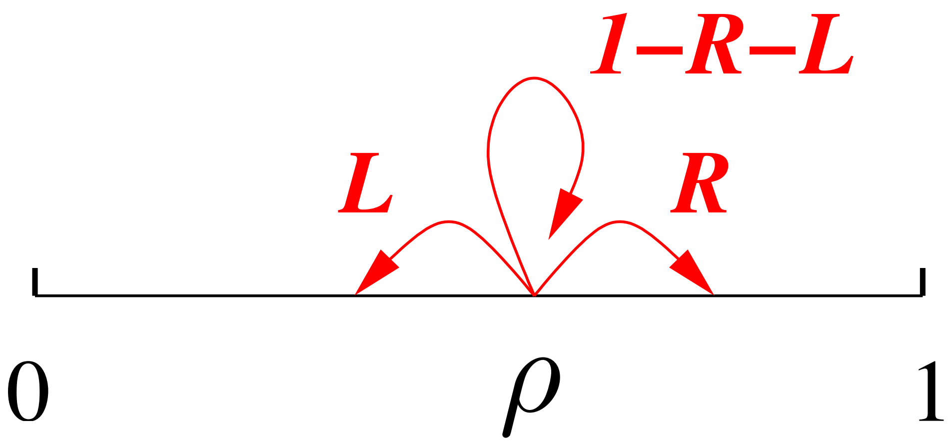
$$\begin{aligned} \mathcal{R}(\rho) &\equiv \mathrm{prob}(\downarrow\uparrow\rightarrow\uparrow\uparrow) \\ \mathcal{L}(\rho) &\equiv \mathrm{prob}(\uparrow\downarrow\rightarrow\downarrow\downarrow) \\ &= \rho(1-\rho) \end{aligned}$$

Consensus Time Evolution Equation

warmup: complete graph

$T(\rho) \equiv$ av. consensus time starting with density ρ

$$\begin{aligned} T(\rho) &= \mathcal{R}(\rho)[T(\rho+d\rho)+dt] \\ &\quad +\mathcal{L}(\rho)[T(\rho-d\rho)+dt] \\ &\quad +[1-\mathcal{R}(\rho)-\mathcal{L}(\rho)][T(\rho)+dt] \end{aligned}$$



$$\begin{aligned} \mathcal{R}(\rho) &\equiv \mathrm{prob}(\downarrow\uparrow\rightarrow\uparrow\uparrow) \\ \mathcal{L}(\rho) &\equiv \mathrm{prob}(\uparrow\downarrow\rightarrow\downarrow\downarrow) \\ &= \rho(1-\rho) \end{aligned}$$

Consensus Time on Complete Graph

$$
\begin{aligned}
T(\rho) \quad = \quad & \mathcal{R}(\rho)[T(\rho + d\rho) + dt] \\
& + \mathcal{L}(\rho)[T(\rho - d\rho) + dt] \\
& + [1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)][T(\rho) + dt]
\end{aligned}
$$

continuum limit: $$T'' = -\frac{N}{\rho(1-\rho)}$$

Consensus Time on Complete Graph

$$\begin{aligned} T(\rho) &= \mathcal{R}(\rho)[T(\rho+d\rho)+dt] \\ &\quad +\mathcal{L}(\rho)[T(\rho-d\rho)+dt] \\ &\quad +[1-\mathcal{R}(\rho)-\mathcal{L}(\rho)][T(\rho)+dt] \end{aligned}$$

continuum limit:

$$T'' = -\frac{N}{\rho(1-\rho)}$$

solution:

$$T(\rho) = -N\left[\rho\ln\rho + (1-\rho)\ln(1-\rho)\right]$$

Consensus Time on Heterogeneous Networks

$$T(\{\rho_k\}) \equiv \text{av. consensus time starting with density } \rho_k \text{ on nodes of degree } k$$

$$\begin{aligned}
T(\{\rho_k\}) = & \sum_k \mathcal{R}_k(\{\rho_k\})[T(\{\rho_k^+\}) + dt] \\
& + \sum_k \mathcal{L}_k(\{\rho_k\})[T(\{\rho_k^-\}) + dt] \\
& + \left[1 - \sum_k [\mathcal{R}_k(\{\rho_k\}) + \mathcal{L}_k(\{\rho_k\})]\right][T(\{\rho_k\}) + dt]
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_k(\{\rho_k\}) &= \text{prob}(\rho_k \to \rho_k^+) \\
&= \frac{1}{N} \sum_x{}' \frac{1}{k_x} \sum_y P(\downarrow, \text{---}, \uparrow) \\
&= n_k \omega (1 - \rho_k)
\end{aligned}$$

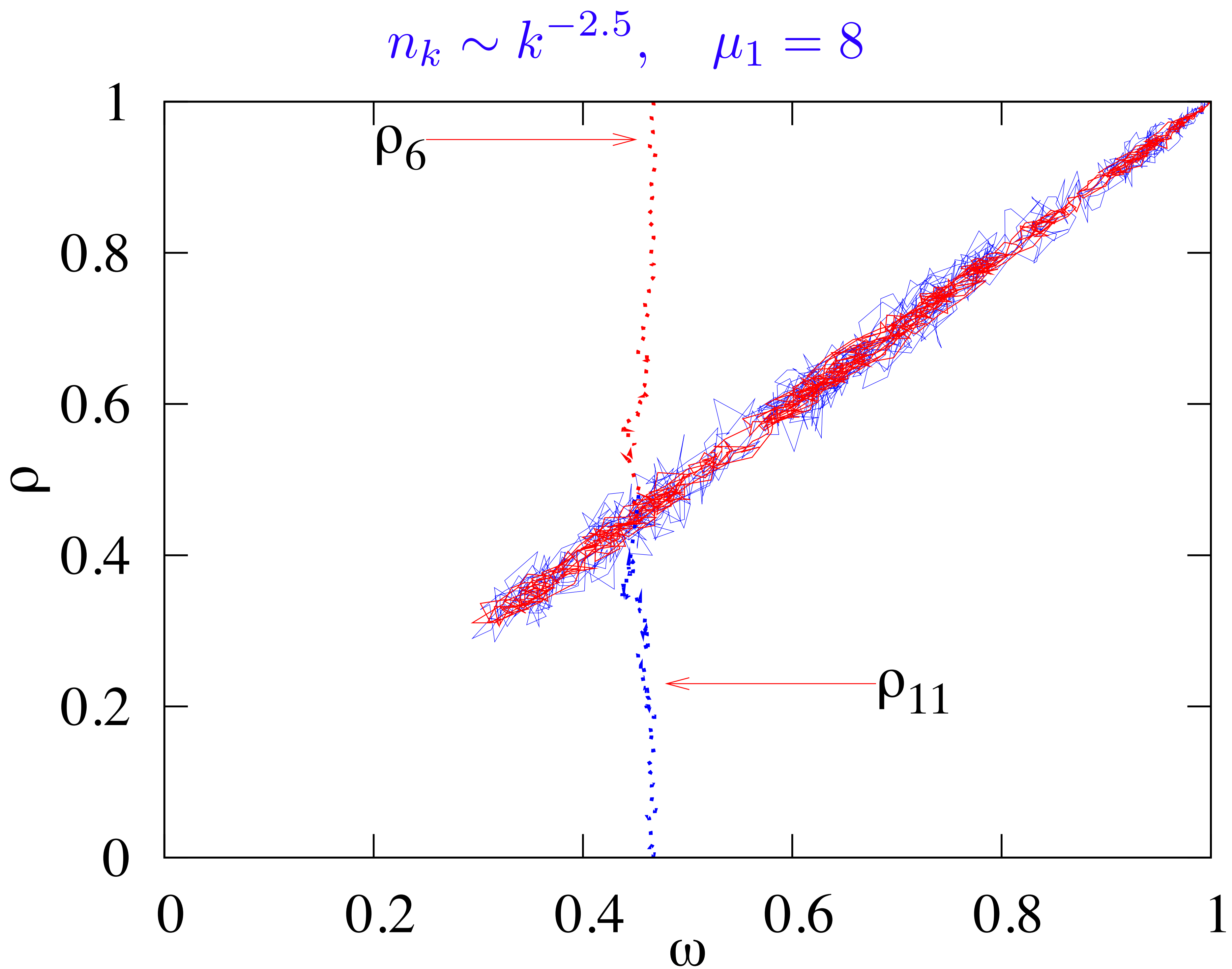
$$\mathcal{L}_k(\{\rho_k\}) = n_k \rho_k (1 - \omega)$$

Consensus Time on Heterogeneous Networks

continuum limit:

$$\sum_k \left[(\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega\rho_k}{2Nn_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1$$

Molloy-Reed Configuration Model



Consensus Time on Heterogeneous Networks

continuum limit:

$$\sum_k \left[(\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega\rho_k}{2Nn_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1$$

now use $\rho_k \longrightarrow \omega \quad \forall k$

and $\frac{\partial}{\partial \rho_k} = \frac{\partial \omega}{\partial \rho_k} \frac{\partial}{\partial \omega} = \frac{k n_k}{\mu_1} \frac{\partial}{\partial \omega}$

Consensus Time on Heterogeneous Networks

continuum limit:

$$\sum_k \left[(\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega\rho_k}{2Nn_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1$$

now use $\rho_k \to \omega \quad \forall k$

and $\frac{\partial}{\partial \rho_k} = \frac{\partial \omega}{\partial \rho_k} \frac{\partial}{\partial \omega} = \frac{kn_k}{\mu_1} \frac{\partial}{\partial \omega}$

to give $\frac{\partial^2 T}{\partial \omega^2} = -\frac{N\mu_1^2/\mu_2}{\omega(1-\omega)}$

Consensus Time on Heterogeneous Networks

continuum limit:

$$\sum_k \left[(\omega - \rho_k)\frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega\rho_k}{2Nn_k}\frac{\partial^2 T}{\partial \rho_k^2}\right] = -1$$

now use $\rho_k \longrightarrow \omega \quad \forall k$

and $\frac{\partial}{\partial \rho_k} = \frac{\partial \omega}{\partial \rho_k}\frac{\partial}{\partial \omega} = \frac{kn_k}{\mu_1}\frac{\partial}{\partial \omega}$

to give $\frac{\partial^2 T}{\partial \omega^2} = -\frac{N\mu_1^2/\mu_2}{\omega(1-\omega)}$ same as $T'' = -\frac{N}{\rho(1-\rho)}$

with *effective* size $N_{\text{eff}} = N\,\mu_1^2/\mu_2$

Consensus Time for Power-Law Degree Distribution $n_k \sim k^{-\nu}$

Voter model:

$$T_N \propto N_{\rm eff} = N\frac{\mu_1^2}{\mu_2} \sim \begin{cases} N & \nu > 3 \\ N/\ln N & \nu = 3 \\ N^{2(\nu-2)/(\nu-1)} & 2 < \nu < 3 \\ (\ln N)^2 & \nu = 2 \\ \mathcal{O}(1) & \nu < 2 \end{cases}$$

Consensus Time for Power-Law Degree Distribution $n_k \sim k^{-\nu}$

Voter model:

$$T_N \propto N_{\text{eff}} = N\frac{\mu_1^2}{\mu_2} \sim \begin{cases} N & \nu > 3 \\ N/\ln N & \nu = 3 \\ N^{2(\nu-2)/(\nu-1)} & 2 < \nu < 3 \\ (\ln N)^2 & \nu = 2 \\ \mathcal{O}(1) & \nu < 2 \end{cases}$$

fast (<N) consensus

Consensus Time for Power-Law Degree Distribution $n_k \sim k^{-\nu}$

Voter model:

$$T_N \propto N_{\rm eff} = N\frac{\mu_1^2}{\mu_2} \sim \begin{cases} N & \nu > 3 \\ N/\ln N & \nu = 3 \\ N^{2(\nu-2)/(\nu-1)} & 2 < \nu < 3 \\ (\ln N)^2 & \nu = 2 \\ \mathcal{O}(1) & \nu < 2 \end{cases}$$

fast (<N) consensus

Invasion process:

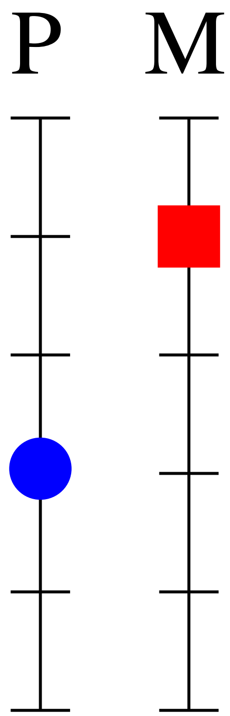
$$T_N \sim \begin{cases} N & \nu > 2, \\ N \ln N & \nu = 2, \\ N^{2-\nu} & \nu < 2. \end{cases}$$

Confident/Reinforced Voter Model

motivation: Centola (2010)
related work: Dall'Asta & Castellano (2007)

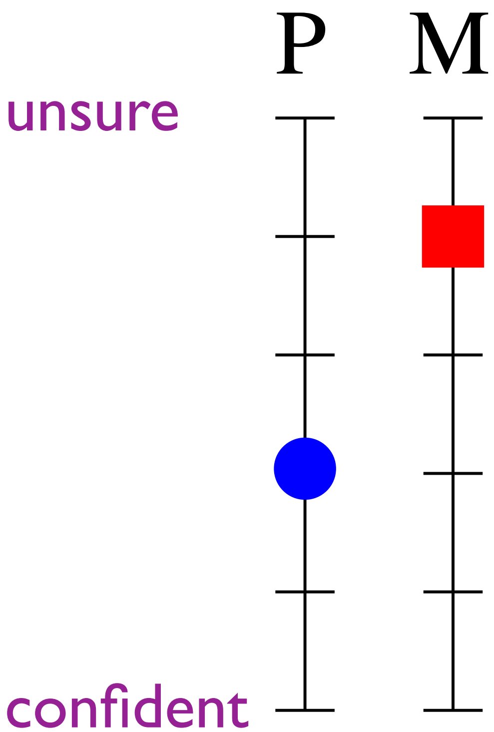
Confident/Reinforced Voter Model

motivation: Centola (2010)
related work: Dall'Asta & Castellano (2007)



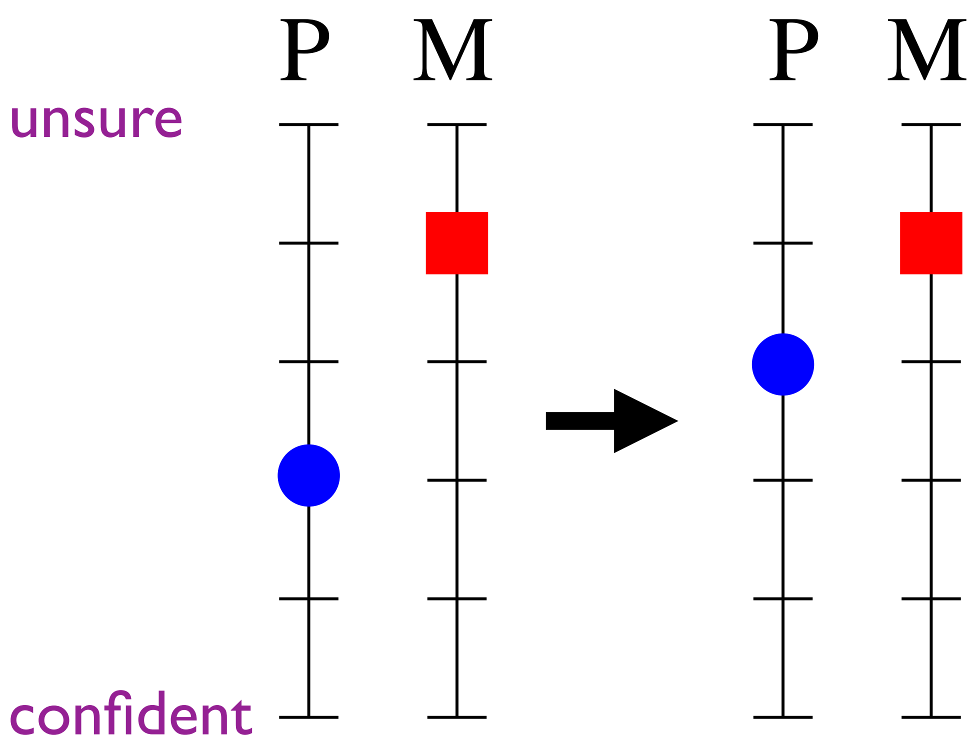
Confident/Reinforced Voter Model

motivation: Centola (2010)
related work: Dall'Asta & Castellano (2007)



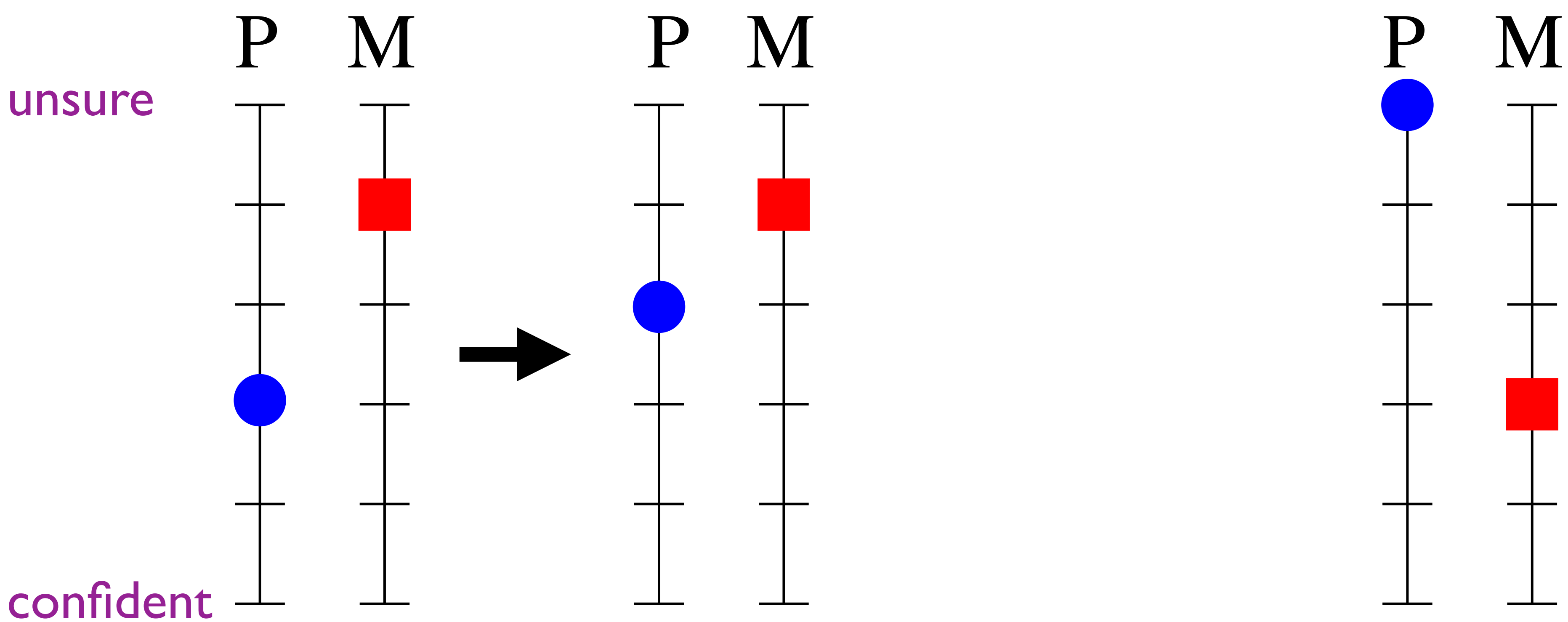
Confident/Reinforced Voter Model

motivation: Centola (2010)
related work: Dall'Asta & Castellano (2007)



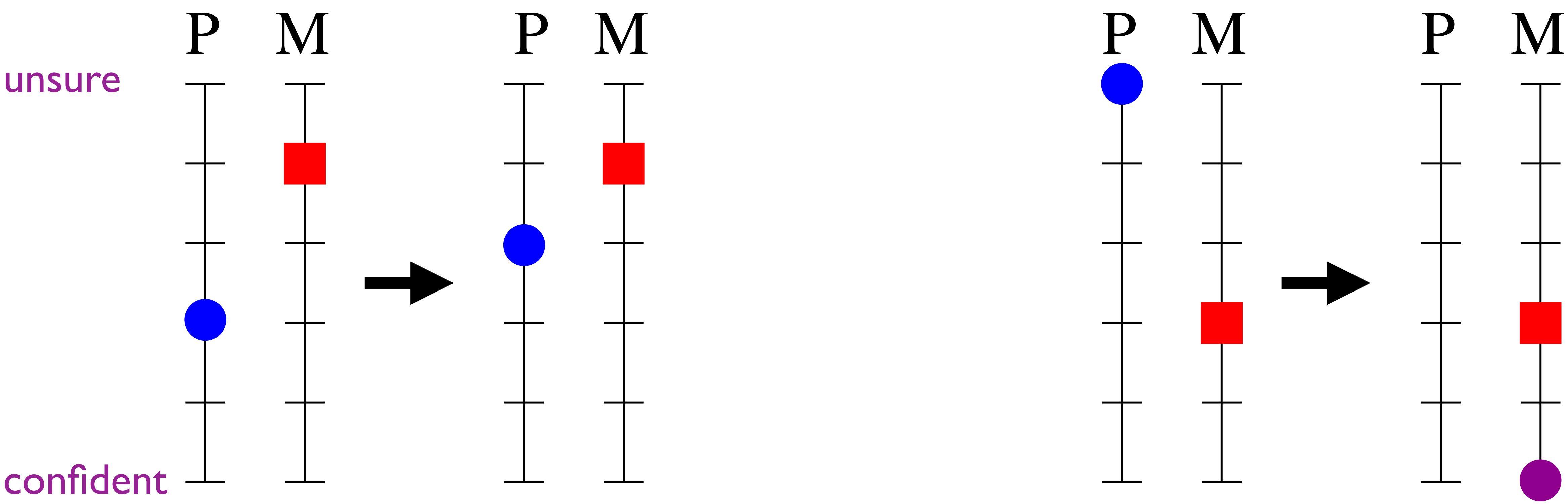
Confident/Reinforced Voter Model

motivation: Centola (2010)
related work: Dall'Asta & Castellano (2007)



Confident/Reinforced Voter Model

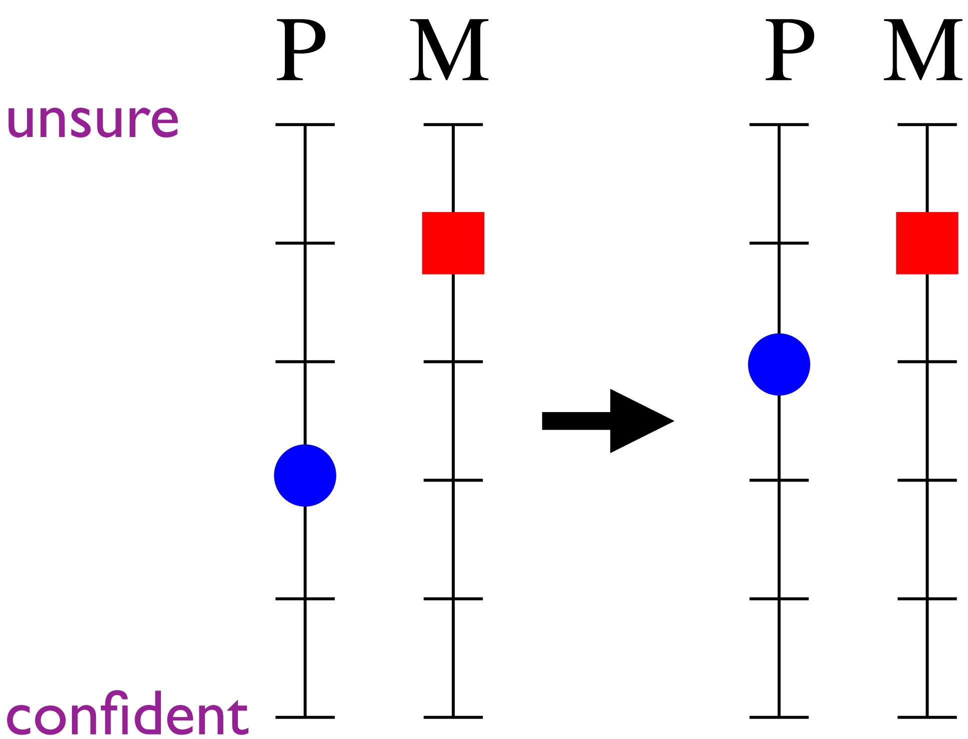
motivation: Centola (2010)
related work: Dall'Asta & Castellano (2007)

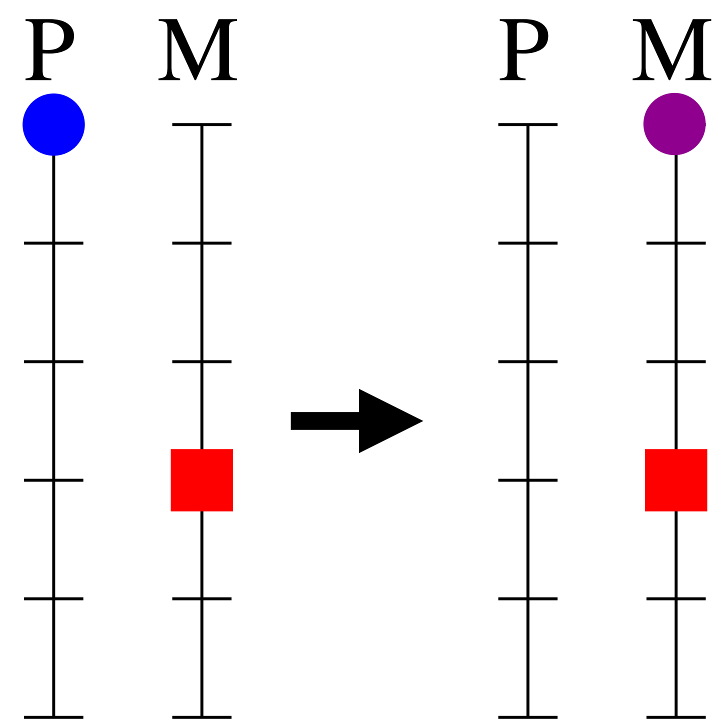


extremal

Confident/Reinforced Voter Model

motivation: Centola (2010)
related work: Dall'Asta & Castellano (2007)



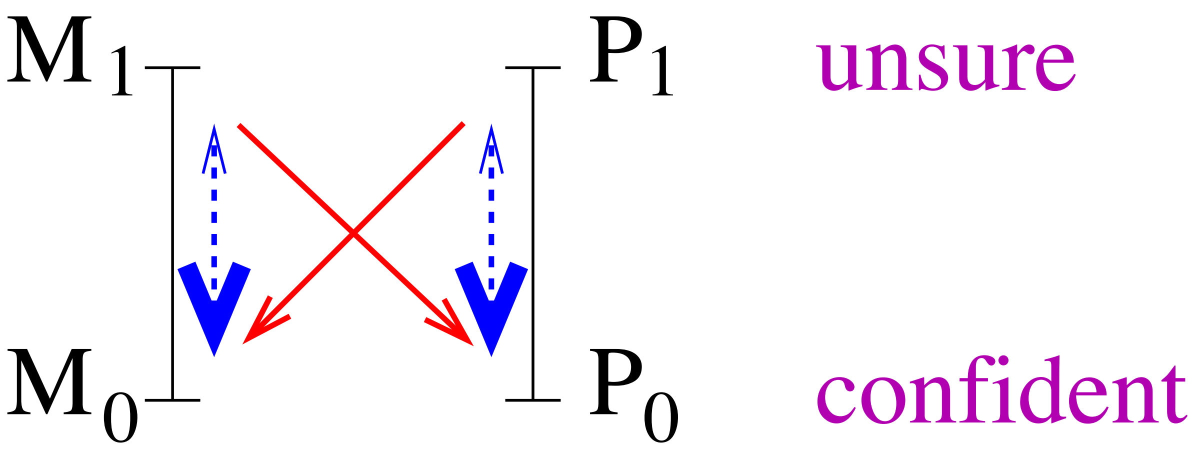


marginal

Simplest case: 2 internal states

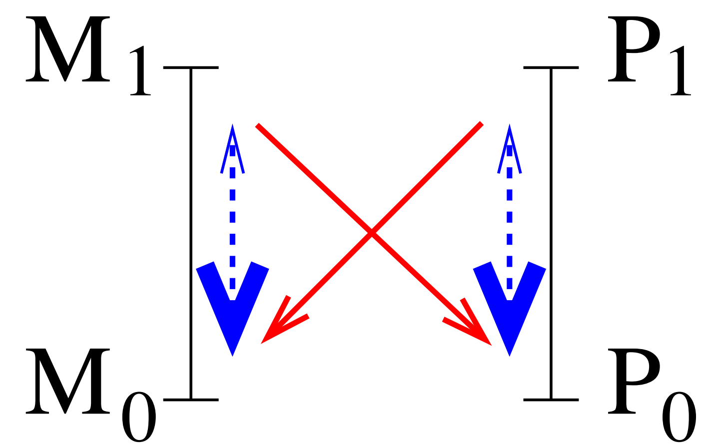
densities P_0, P_1, M_0, M_1,

with $P_0+P_1+M_0+M_1=1$



Simplest case: 2 internal states

densities P_0, P_1, M_0, M_1,
with $P_0+P_1+M_0+M_1=1$



unsure

confident

basic processes:

$$M_1\,P_1 \rightarrow P_0 P_1 \text{ or } M_0 M_1 \qquad M_0\,P_0 \rightarrow M_0 P_1 \text{ or } M_1 P_0$$

$$P_0\,P_1 \rightarrow P_0 P_1 \text{ or } P_0 P_0 \qquad M_0 M_1 \rightarrow M_0 M_1 \text{ or } M_0 M_0$$

$$M_1\,P_0 \rightarrow M_1 P_1 \text{ or } P_0 P_0 \qquad M_0\,P_1 \rightarrow M_1 P_1 \text{ or } M_0 M_0$$

rate equations/mean-field limit:

$$\dot{P}_0 = -M_0 P_0 + M_1 P_1 + P_0 P_1$$

$$\dot{P}_1 = \quad M_0 P_0 - M_1 P_1 - P_0 P_1 + (M_1 P_0 - M_0 P_1)$$

similarly for M_0, M_1

special soluble case: symmetric limit

$$P_0 + P_1 = M_0 + M_1 = \frac{1}{2}$$

$$\dot{P}_0 = -M_0 P_0 + M_1 P_1 + P_0 P_1$$

$$\dot{P}_1 = \quad M_0 P_0 - M_1 P_1 - P_0 P_1 + (M_1 P_0 - M_0 P_1)$$

special soluble case: symmetric limit

$$P_0 + P_1 = M_0 + M_1 = \tfrac{1}{2}$$

$$\dot{P}_0 = -M_0 P_0 + M_1 P_1 + P_0 P_1$$
$$\dot{P}_1 = \quad M_0 P_0 - M_1 P_1 - P_0 P_1 + (M_1 P_0 - M_0 P_1)$$

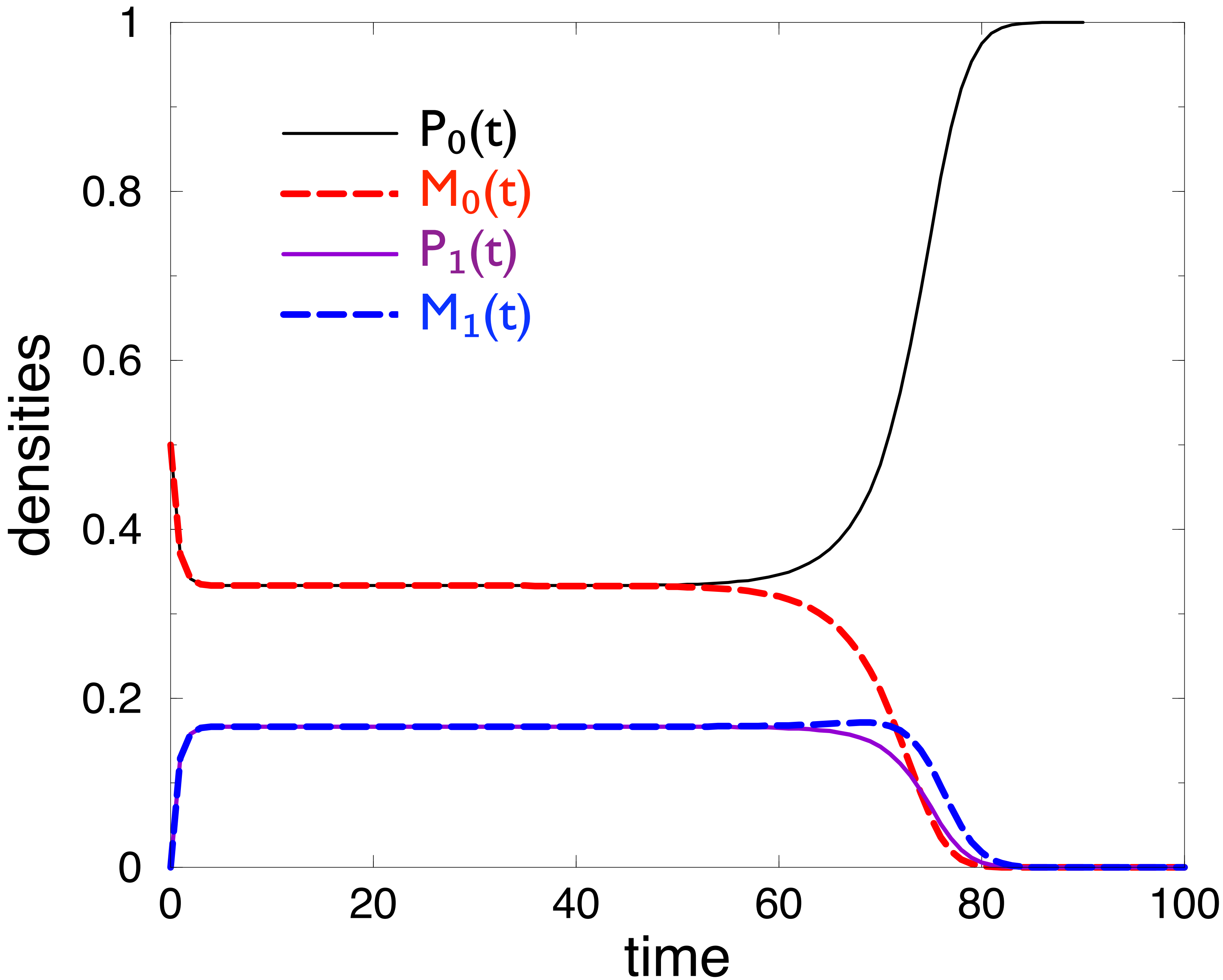
$$\rightarrow \quad \dot{P}_0 = -\dot{P}_1 = P_0^2 + \tfrac{1}{2} P_0 - \tfrac{1}{4}$$
$$= -(P_0 - \lambda_+)(P_0 - \lambda_-)$$

$$\lambda_\pm = \tfrac{1}{4}(-1 \pm \sqrt{5}) \approx 0.309, -0.809$$

solution: $$\frac{P_0(t) - \lambda_+}{P_0(t) - \lambda_-} = \frac{P_0(0) - \lambda_+}{P_0(0) - \lambda_-}\, e^{-(\lambda_+ - \lambda_-)t}$$

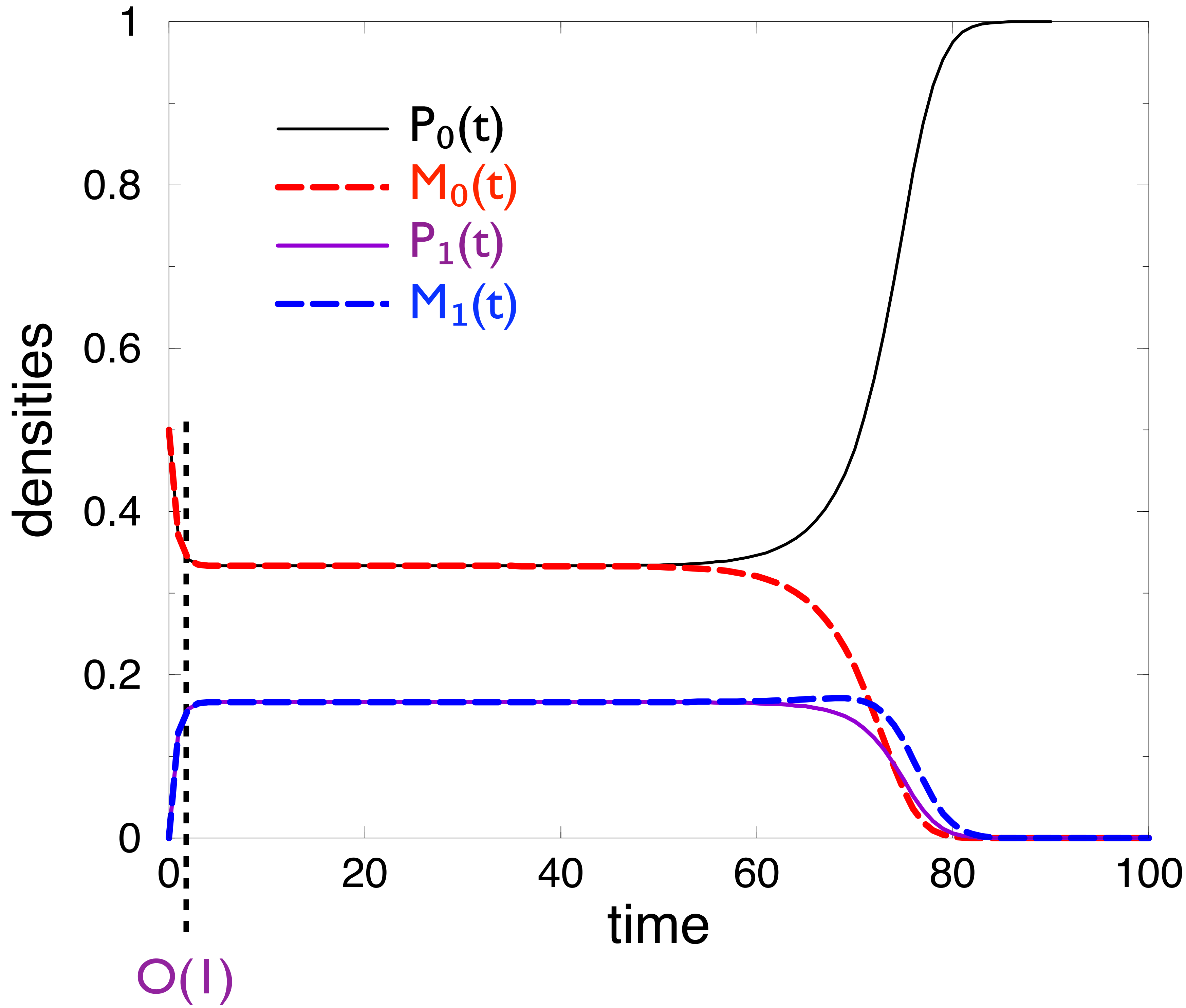
near symmetric limit:

$$P_0 = \frac{1}{2} + 10^{-5},\ M_0 = \frac{1}{2} - 10^{-5},\ P_1 = M_1 = 0$$



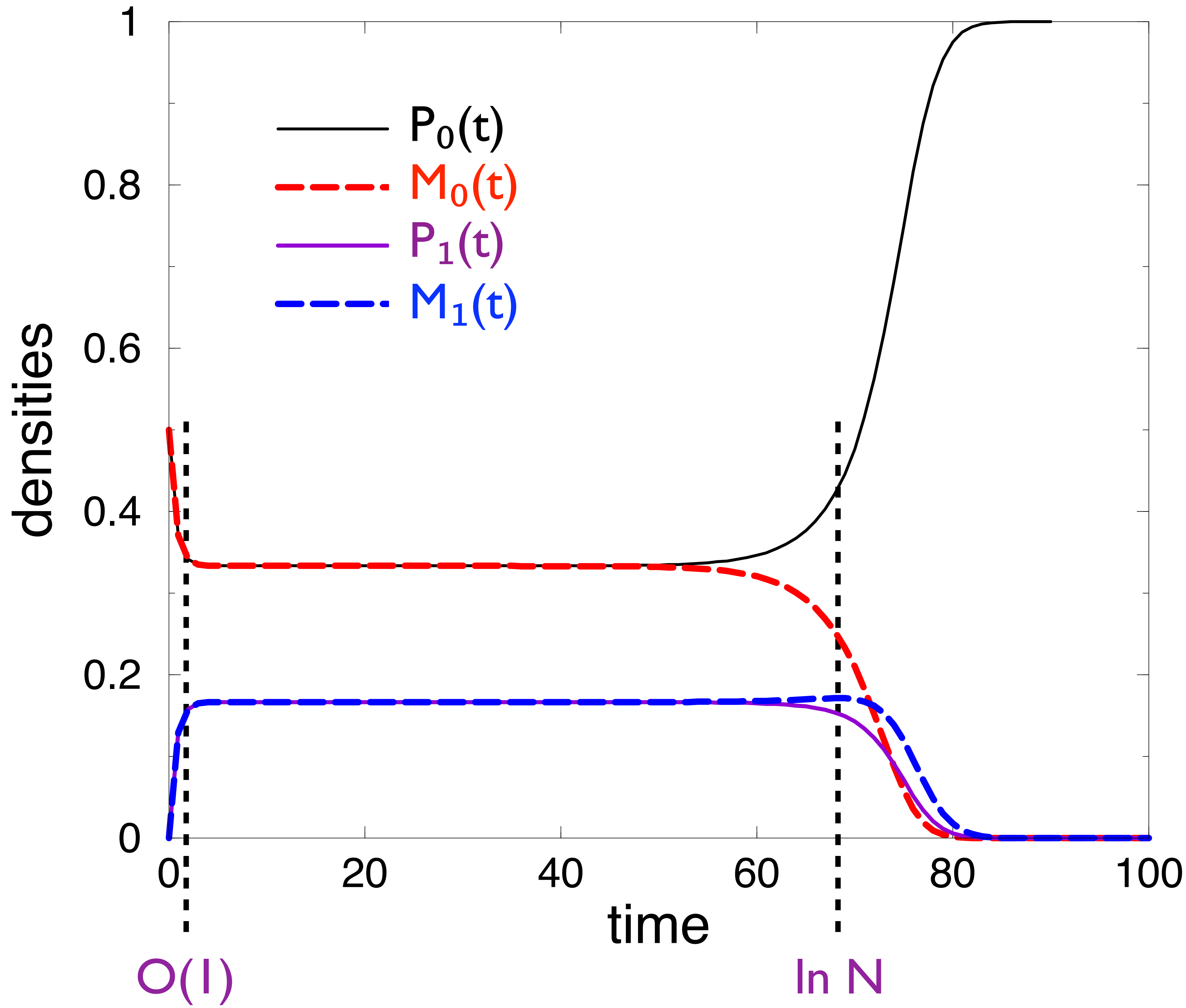
near symmetric limit:

$$P_0 = \frac{1}{2} + 10^{-5},\ M_0 = \frac{1}{2} - 10^{-5},\ P_1 = M_1 = 0$$

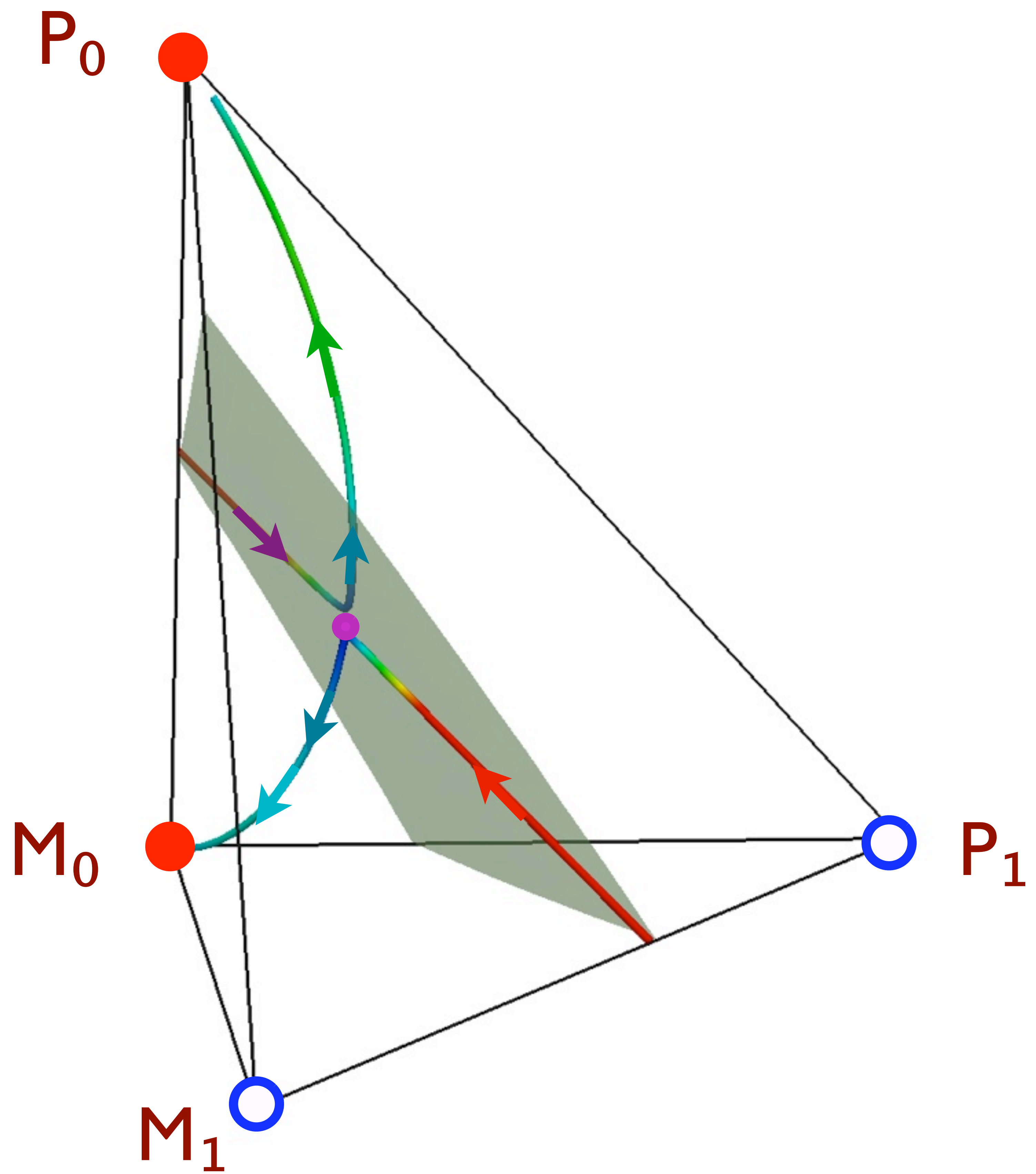


near symmetric limit:

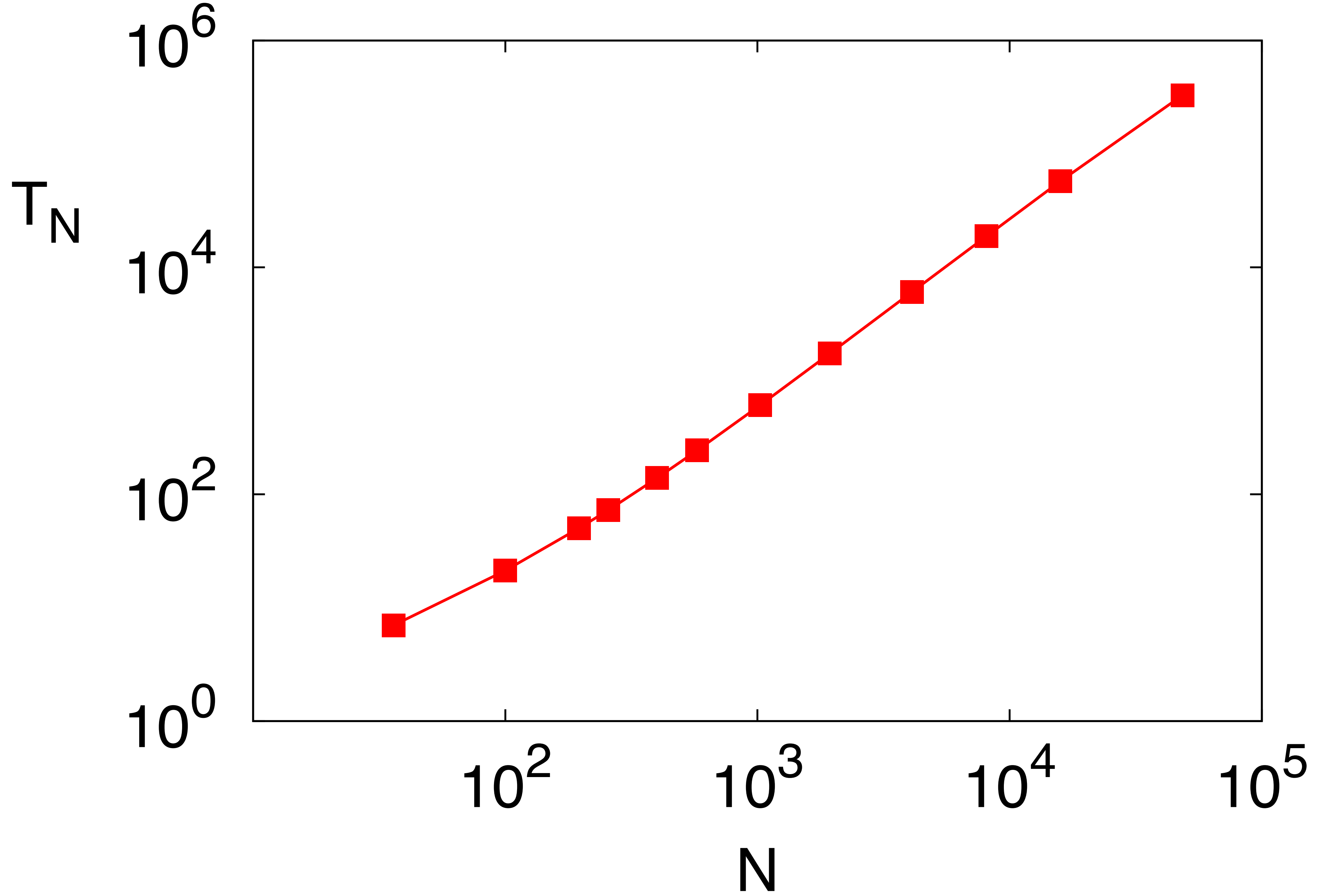
$$P_0 = \frac{1}{2}+10^{-5},\ M_0 = \frac{1}{2}-10^{-5},\ P_1 = M_1 = 0$$



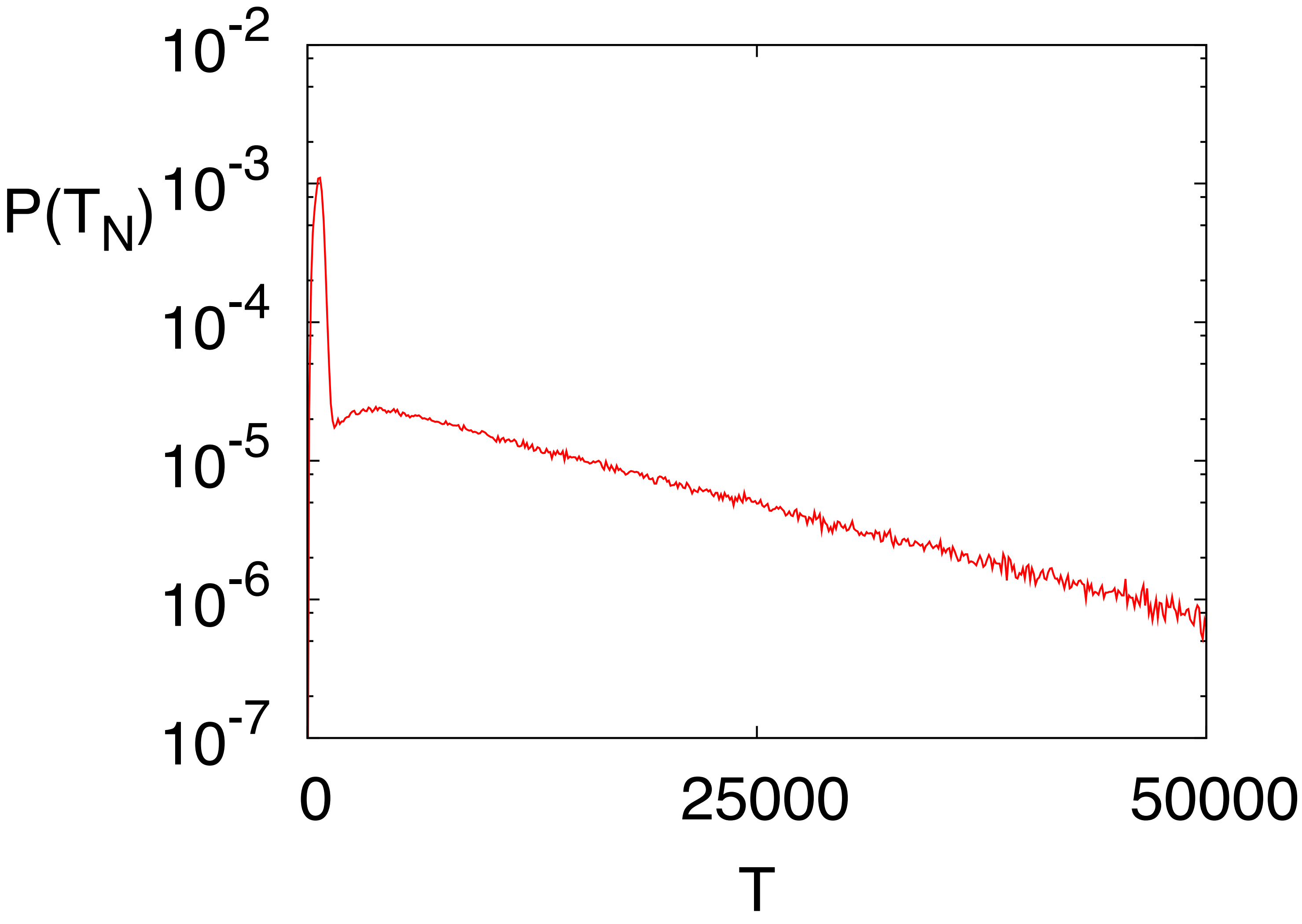
near symmetric limit: composition tetrahedron



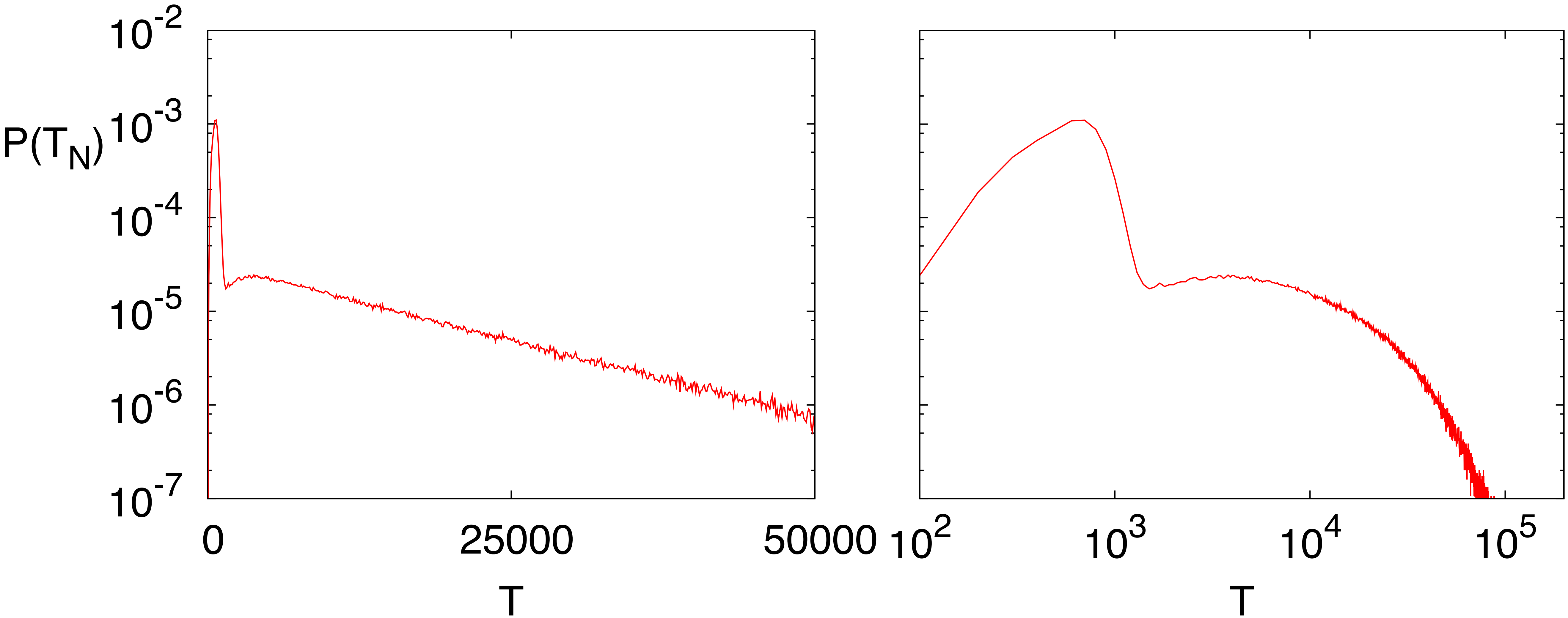
Consensus Time in Two Dimensions



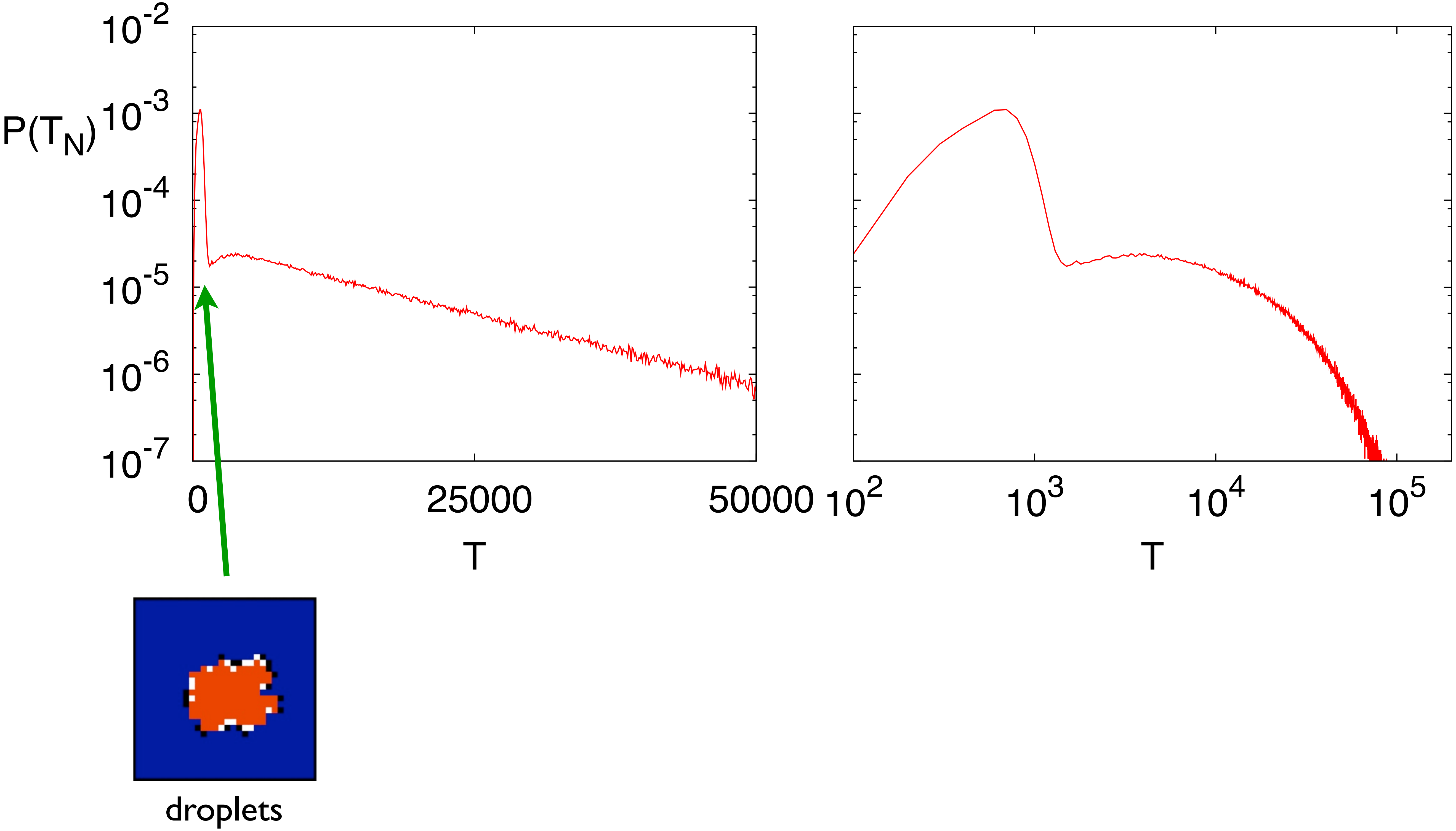
Consensus Time Distribution



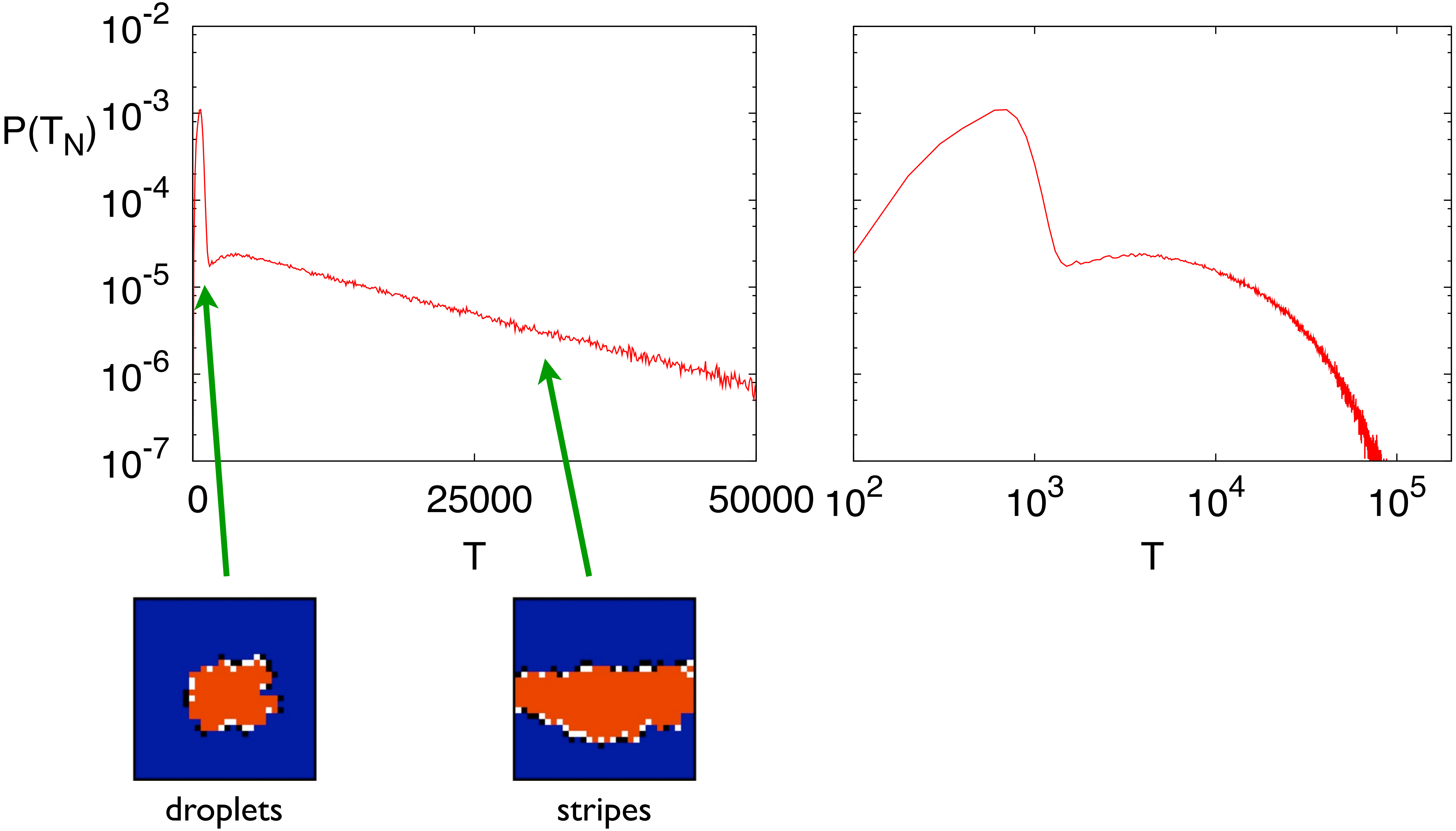
Consensus Time Distribution

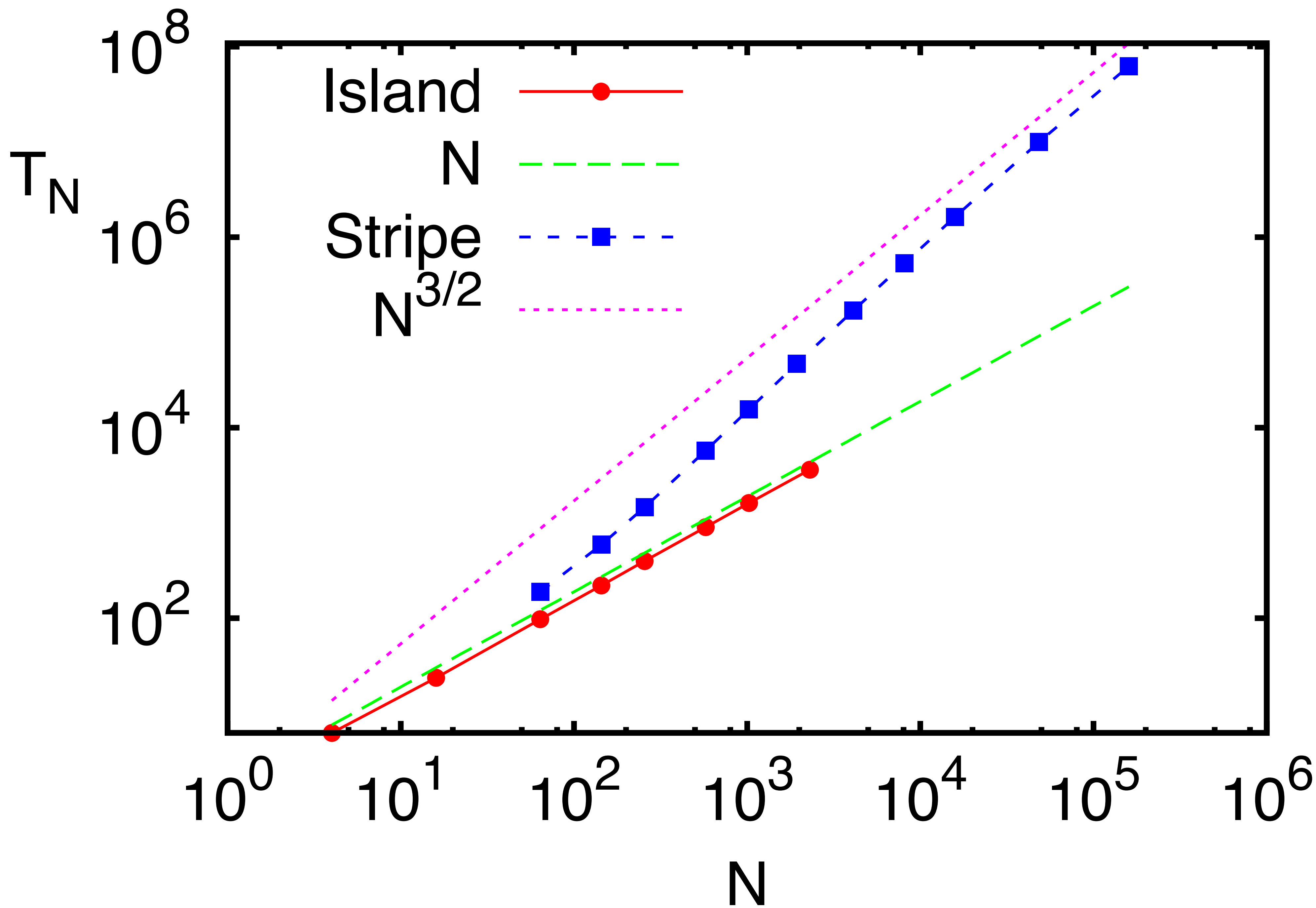


Consensus Time Distribution



Consensus Time Distribution





two time scales control approach to consensus

see also Spirin, Krapivsky, SR (2001), Chen & SR (2005)

Ising model Majority vote model

Summary & Outlook

Summary & Outlook

Voter model:

paradigmatic, soluble, hopelessly naive

Summary & Outlook

Voter model:

paradigmatic, soluble, hopelessly naive

Voter model on complex networks:

new conservation law
route to consensus sensitive to network structure
fast consensus for broad degree distributions

Summary & Outlook

Voter model:

paradigmatic, soluble, hopelessly naive

Voter model on complex networks:

new conservation law
route to consensus sensitive to network structure
fast consensus for broad degree distributions

Confident voting on simple networks:

two time-scale route to consensus
fast consensus ($\ln N$ vs. N) in mean-field limit
slow consensus on lattices & networks

Summary & Outlook

Voter model:

paradigmatic, soluble, hopelessly naive

Voter model on complex networks:

new conservation law
route to consensus sensitive to network structure
fast consensus for broad degree distributions

Confident voting on simple networks:

two time-scale route to consensus
fast consensus (ln N vs. N) in mean-field limit
slow consensus on lattices & networks

Ongoing:

"churn" rather than consensus
heterogeneity of real people
positive and negative social interactions $\rightarrow$ social balance