Dynamics of Voter Models on Complex & Simple Networks

Sid Redner (physics.bu.edu/~redner)
Mathematical Physics of Complex Networks (MAPCON), MPI Dresden, May 14-18, 2012

T. Antal (BU→Edinburgh), V. Sood (BU→Lausanne), D. Volovik (BU)
NSF DMR0906504
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The classic voter model

3 basic results
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The classic voter model
3 basic results

Voting on complex networks
new conservation law
two time-scale route to consensus
short consensus time

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The classic voter model
3 basic results

Voting on complex networks
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short consensus time

Confident/Reinforced voting
two time-scale dynamics
symmetry breaking clustering
0. Binary voter variable at each site i
0. Binary voter variable at each site $i$

1. Pick a random voter
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1. Pick a random voter
2. Assume state of randomly-selected neighbor
   *individual has no self-confidence & adopts neighbor's state*
Example update:

0. Binary voter variable at each site $i$
1. Pick a random voter
2. Assume state of randomly-selected neighbor

*individual has no self-confidence & adopts neighbor’s state*
Classic Voter Model

Example update:

0. Binary voter variable at each site $i$
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*individual has no self-confidence & adopts neighbor’s state*
Example update:

0. Binary voter variable at each site $i$
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2. Assume state of randomly-selected neighbor

*individual has no self-confidence & adopts neighbor's state*

proportional rule

Clifford & Sudbury (1973)
Holley & Liggett (1975)
0. Binary voter variable at each site i
1. Pick a random voter
2. Assume state of randomly-selected neighbor individual has no self-confidence & adopts neighbor’s state
3. Repeat 1 & 2 until consensus necessarily occurs in a finite system

Example update:

proportional rule
Voter Model Evolution

random initial condition:

Dornic et al. (2001)

t=4  t=16  t=64  t=256

droplet initial condition:
Voter versus Ising Evolution
Voter Model & Cousins
Voter Model & Cousins

Voter Model: Tell me how to vote

lemming
Voter Model & Cousins

Voter Model: Tell me how to vote

Invasion Process: I tell you how to vote
Voter Model & Cousins

Voter Model: Tell me how to vote

Invasion Process: I tell you how to vote

Link Dynamics: Pick two disagreeing agents and change one at random
Voter Model & Cousins

Voter Model: Tell me how to vote

Invasion Process: I tell you how to vote

Link Dynamics: Pick two disagreeing agents and change one at random

*identical* on lattices, *distinct* on degree-heterogeneous graphs

Lattice Voter Model: 3 Basic Properties
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1. Final State (Exit) Probability $\mathcal{E}(\rho_0)$
Lattice Voter Model: 3 Basic Properties

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Evolution of a single active link:
Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability \[ \mathcal{E}(\rho_0) = \rho_0 \]

Evolution of a single active link: average magnetization conserved
Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability \( \mathcal{E}(\rho_0) = \rho_0 \)

Evolution of a single active link:

2. Two-Spin Correlations
Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability \( \mathcal{E}(\rho_0) = \rho_0 \)

Evolution of a single active link:

2. Two-Spin Correlations

\[
\frac{\partial c_2(r, t)}{\partial t} = \nabla^2 c_2(r, t)
\]

\( c_2(r = 0, t) = 1 \)

\( c_2(r > 0, t = 0) = 0 \)
Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability  \( \mathcal{E}(\rho_0) = \rho_0 \)

Evolution of a single active link:

2. Two-Spin Correlations

- For \( d > 2 \):
  \[
  c(r,t) = 1 - (a/r)^{d-2}
  \]

- For \( d \leq 2 \):
  \[
  \frac{\partial c_2(r,t)}{\partial t} = \nabla^2 c_2(r,t)
  \]
  \[
  c_2(r=0,t) = 1 \quad c_2(r > 0, t=0) = 0
  \]
Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability $\mathcal{E}(\rho_0) = \rho_0$

Evolution of a single active link:

2. Two-Spin Correlations

$$\partial c_2(r, t) \over \partial t = \nabla^2 c_2(r, t)$$

- $d > 2$
  - $c_2(r = 0, t) = 1$
  - $c_2(r > 0, t = 0) = 0$

- $d \leq 2$

3. Consensus Time
Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability \( \mathcal{E}(\rho_0) = \rho_0 \)

Evolution of a single active link:

2. Two-Spin Correlations

\[
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3. Consensus Time

<table>
<thead>
<tr>
<th>dimension</th>
<th>consensus time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{2}{N} )</td>
</tr>
<tr>
<td>2</td>
<td>( N \ln N )</td>
</tr>
<tr>
<td>&gt;2</td>
<td>( N )</td>
</tr>
</tbody>
</table>
K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL 69, 228 (2005)
illustrative example:

*complete bipartite graph*

Voter Model on Complex Networks

\[ dN_a = \frac{a}{a + b} \left[ \frac{a - N_a}{a} \frac{N_b}{b} - \frac{N_a}{a} \frac{b - N_b}{b} \right] \]

\[ dN_b = \frac{b}{a + b} \left[ \frac{b - N_b}{b} \frac{N_a}{a} - \frac{N_b}{b} \frac{a - N_a}{a} \right] \]

illustrative example:
complete bipartite graph

K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL 69, 228 (2005)
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\]

pick site on a sublattice
Voter Model on Complex Networks

K. Suckeck, V. M. Eguiluz, M. San Miguel, EPL 69, 228 (2005)

**illustrative example:** complete bipartite graph

differential equations:

\[ dN_a = \frac{a}{a + b} \left[ \frac{a - N_a}{a} \frac{N_b}{b} - \frac{N_a}{a} \frac{b - N_b}{b} \right] \]

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Subgraph densities: \( \rho_a = N_a/a, \rho_b = N_b/b \) \( dt = 1/(a+b) \)

\[ \rho_{a,b}(t) = \frac{1}{2} [\rho_{a,b}(0) - \rho_{b,a}(0)] e^{-2t} + \frac{1}{2} [\rho_a(0) + \rho_b(0)] \]

\[ \rightarrow \frac{1}{2} [\rho_a(0) + \rho_b(0)] \]
Voter Model on Complex Networks

K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL 69, 228 (2005)

Illustrative example: complete bipartite graph

\[
\begin{align*}
\text{pick site on a sublattice} & \quad \text{pick ↓ on a} \quad \text{pick ↑ on b sublattice} \\
\end{align*}
\]

\[
\begin{align*}
dN_a &= \frac{a}{a+b} \left[ \frac{a-N_a}{a} \frac{N_b}{b} - \frac{N_a b - N_b}{a} \right] \\
dN_b &= \frac{b}{a+b} \left[ \frac{b-N_b}{b} \frac{N_a}{a} - \frac{N_b a - N_a}{b} \right]
\end{align*}
\]

Subgraph densities: \( \rho_a = N_a/a, \ \rho_b = N_b/b \)

\[
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\]

\[
\rightarrow \frac{1}{2} [\rho_a(0) + \rho_b(0)] \quad \text{magnetization not conserved}
\]
Voter Model on Complex Networks

mechanism for non-conservation
Voter Model on Complex Networks

mechanism for non-conservation
Voter Model on Complex Networks

mechanism for non-conservation

high degree; few nodes
→ changes rarely
Voter Model on Complex Networks

mechanism for non-conservation

- High degree; few nodes → changes rarely
- Low degree; many nodes → changes often
Voter Model on Complex Networks

mechanism for non-conservation

“flow” from high degree to low degree
New Conservation Law

- low degree → changes often
- high degree → changes rarely
New Conservation Law

low degree → changes often

high degree → changes rarely

to compensate the different rates:

degree-weighted 1st moment:

$$\omega = \frac{1}{\mu_1} \sum_k k n_k \rho_k$$

$$\mu_1 = \text{av. degree}$$

$$n_k = \text{frac. nodes of degree } k$$

$$\rho_k = \text{frac. ↑ on nodes of degree } k$$
New Conservation Law

low degree
→ changes often

high degree
→ changes rarely

to compensate the different rates:

degree-weighted 1st moment:

\[ \omega = \frac{1}{\mu_1} \sum_k k n_k \rho_k \]

\[ \mu_1 = \text{av. degree} \]
\[ n_k = \text{frac. nodes of degree } k \]
\[ \rho_k = \text{frac. ↑ on nodes of degree } k \]

\[ \text{conserved!} \]
Invasion Process on Complex Networks

Castellano (2005)
Antal, Sood, SR (2005, 06, 08)

high degree: changes often
Invasion Process on Complex Networks

Castellano (2005)
Antal, Sood, SR (2005, 06, 08)

high degree: changes often

low degree; changes rarely
Invasion Process on Complex Networks

Castellano (2005)
Antal, Sood, SR (2005, 06, 08)

high degree: changes often

low degree; changes rarely

“flow” from low degree to high degree
Invasion Process on Complex Networks

Castellano (2005)
Antal, Sood, SR (2005, 06, 08)

high degree: **changes often**
low degree; **changes rarely**

“flow” from **low** degree to **high** degree

degree-weighted inverse moment

$$\omega_{-1} = \frac{1}{\mu_1} \sum_k k^{-1} n_k \rho_k$$

**conserved!**
Exit Probability on Complex Graphs

\[ E(\omega) = \omega \]
Exit Probability on Complex Graphs

\[ \mathcal{E}(\omega) = \omega \]

Extreme case: star graph

N nodes: degree 1
1 node: degree N
Exit Probability on Complex Graphs

\[ \mathcal{E}(\omega) = \omega \]

**Extreme case: star graph**

N nodes: degree 1
1 node: degree N

\[ \omega = \frac{1}{\mu_1} \sum_k k n_k \rho_k = \frac{1}{2} \]

Final state: all 1 with prob. 1/2!
Route to Consensus on Complex Graphs
Route to Consensus on Complex Graphs

A complete bipartite graph with $a$ sites of degree $a$ and $b$ sites of degree $b$.

$t \lesssim 1$
Route to Consensus on Complex Graphs

- **Complete Bipartite Graph**
  - Degree a sites
  - Degree b sites

- **Two-Clique Graph**
  - N = 10000, C links/node

- $t \lessapprox 1$

- $c = 100$

- $c = 1$
Consensus Time Evolution Equation
Consensus Time Evolution Equation

warmup: complete graph

\[ T(\rho) \equiv \text{av. consensus time starting with density } \rho \]
Consensus Time Evolution Equation

warmup: complete graph

$T(\rho) \equiv$ av. consensus time starting with density $\rho$

\[
T(\rho) \begin{array}{c} = \\
= \mathcal{R}(\rho) [T(\rho + d\rho) + dt] + \mathcal{L}(\rho) [T(\rho - d\rho) + dt] + [1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)] [T(\rho) + dt]
\end{array}
\]
Consensus Time Evolution Equation

warmup: complete graph

\[ T(\rho) \equiv \text{av. consensus time starting with density } \rho \]

\[
T(\rho) = \mathcal{R}(\rho)[T(\rho + d\rho) + dt] + \mathcal{L}(\rho)[T(\rho - d\rho) + dt] + [1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)][T(\rho) + dt]
\]

\[ \mathcal{R}(\rho) \equiv \text{prob}(\downarrow\uparrow\rightarrow\uparrow\uparrow) = \rho(1 - \rho) \]
Consensus Time Evolution Equation

warmup: complete graph

\( T(\rho) \equiv \text{av. consensus time starting with density } \rho \)

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\( \mathcal{R}(\rho) \equiv \text{prob}(\downarrow \uparrow \rightarrow \uparrow \uparrow) \)

\( \mathcal{L}(\rho) \equiv \text{prob}(\uparrow \downarrow \rightarrow \downarrow \downarrow) \)

\( = \rho(1 - \rho) \)
**Consensus Time Evolution Equation**

**warmup: complete graph**

\[ T(\rho) \equiv \text{av. consensus time starting with density } \rho \]

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\]

\[
\mathcal{R}(\rho) \equiv \text{prob}(\downarrow \uparrow \rightarrow \uparrow \uparrow) \\
\mathcal{L}(\rho) \equiv \text{prob}(\uparrow \downarrow \rightarrow \downarrow \downarrow) \\
\] 

\[ = \rho(1 - \rho) \]
Consensus Time on Complete Graph

\[ T(\rho) = R(\rho)[T(\rho + d\rho) + dt] \]
\[ + L(\rho)[T(\rho - d\rho) + dt] \]
\[ + [1 - R(\rho) - L(\rho)] [T(\rho) + dt] \]

continuum limit:

\[ T'' = - \frac{N}{\rho(1 - \rho)} \]
Consensus Time on Complete Graph

\[ T(\rho) = R(\rho)[T(\rho + d\rho) + dt] \]
\[ + L(\rho)[T(\rho - d\rho) + dt] \]
\[ + [1 - R(\rho) - L(\rho)][T(\rho) + dt] \]

continuum limit: \[ T'' = - \frac{N}{\rho(1 - \rho)} \]
solution: \[ T(\rho) = -N \left[ \rho \ln \rho + (1 - \rho) \ln(1 - \rho) \right] \]
Consensus Time on Heterogeneous Networks

\[ T(\{\rho_k\}) \equiv \text{av. consensus time starting with density } \rho_k \text{ on nodes of degree } k \]

\[ T(\{\rho_k\}) = \sum_k R_k(\{\rho_k\})[T(\{\rho_k^+\}) + dt] \]

\[ + \sum_k L_k(\{\rho_k\})[T(\{\rho_k^-\}) + dt] \]

\[ + \left[ 1 - \sum_k [R_k(\{\rho_k\}) + L_k(\{\rho_k\})] \right] [T(\{\rho_k\}) + dt] \]

\[ R_k(\{\rho_k\}) = \text{prob}(\rho_k \rightarrow \rho_k^+) \quad \text{and} \quad L_k(\{\rho_k\}) = n_k \rho_k (1 - \omega) \]

\[ = \frac{1}{N} \sum_x \sum_x^\prime \frac{1}{k_x} \sum_{y} P(\downarrow, \rightarrow, \uparrow) \]

\[ = n_k \omega (1 - \rho_k) \]
Consensus Time on Heterogeneous Networks

continuum limit:

$$\sum_k \left[ (\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega \rho_k}{2N n_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1$$
Molloy-Reed Configuration Model

\[ n_k \sim k^{-2.5}, \quad \mu_1 = 8 \]
Consensus Time on Heterogeneous Networks

continuum limit:

\[
\sum_k \left[ (\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega \rho_k}{2N n_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1
\]

now use \( \rho_k \to \omega \quad \forall k \)

and \( \frac{\partial}{\partial \rho_k} = \frac{\partial \omega}{\partial \rho_k} \frac{\partial}{\partial \omega} = \frac{kn_k}{\mu_1} \frac{\partial}{\partial \omega} \)
Consensus Time on Heterogeneous Networks

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\]

to give

\[
\frac{\partial^2 T}{\partial \omega^2} = -\frac{N \mu_1^2 / \mu_2}{\omega(1 - \omega)}
\]
Consensus Time on Heterogeneous Networks

continuum limit:

$$\sum_k \left[ (\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega \rho_k}{2Nn_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1$$

now use $$\rho_k \to \omega \quad \forall k$$

and

$$\frac{\partial}{\partial \rho_k} = \frac{\partial \omega}{\partial \rho_k} \frac{\partial}{\partial \omega} = \frac{kn_k}{\mu_1} \frac{\partial}{\partial \omega}$$

to give

$$\frac{\partial^2 T}{\partial \omega^2} = -\frac{N \mu_1^2/\mu_2}{\omega(1 - \omega)}$$

same as

$$T'' = -\frac{N}{\rho(1 - \rho)}$$

with effective size $$N_{\text{eff}} = N \frac{\mu_1^2}{\mu_2}$$
Consensus Time for Power-Law Degree Distribution \( n_k \sim k^{-\nu} \)

Voter model:

\[
T_N \propto N_{\text{eff}} = N \frac{\mu_1^2}{\mu_2} \sim \begin{cases} 
N & \nu > 3 \\
N/ \ln N & \nu = 3 \\
N^2(\nu-2)/(\nu-1) & 2 < \nu < 3 \\
(\ln N)^2 & \nu = 2 \\
\mathcal{O}(1) & \nu < 2
\end{cases}
\]
Consensus Time for Power-Law Degree Distribution \( n_k \sim k^{-\nu} \)

Voter model:

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\end{cases}
\]

fast (<N) consensus
Consensus Time for Power-Law Degree Distribution $n_k \sim k^{-\nu}$

Voter model:

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N^2(\nu-2)/(\nu-1) & 2 < \nu < 3 \\
(\ln N)^2 & \nu = 2 \\
\mathcal{O}(1) & \nu < 2 
\end{cases}$$

Invasion process:

$$T_N \sim \begin{cases} 
N & \nu > 2, \\
N\ln N & \nu = 2, \\
N^{2-\nu} & \nu < 2. 
\end{cases}$$

Fast (\textless N) consensus
Confident/Reinforced Voter Model

motivation: Centola (2010)
related work: Dall’Asta & Castellano (2007)
Confident/Reinforced Voter Model

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Confident/Reinforced Voter Model

motivation: Centola (2010)
related work: Dall’Asta & Castellano (2007)

unsure

confident

extremal
Confident/Reinforced Voter Model

motivation: Centola (2010)
related work: Dall’Asta & Castellano (2007)

unsure

confident

marginal
Simplest case: 2 internal states

densities \( P_0, P_1, M_0, M_1 \),
with \( P_0 + P_1 + M_0 + M_1 = 1 \)
**Simplest case: 2 internal states**

densities $P_0, P_1, M_0, M_1$, with $P_0 + P_1 + M_0 + M_1 = 1$

**basic processes:**

<table>
<thead>
<tr>
<th>Process</th>
<th>$M_1 P_1$</th>
<th>$P_0 P_1$</th>
<th>$M_1 P_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rightarrow$</td>
<td>$P_0 P_1$</td>
<td>$P_0 P_1$</td>
<td>$M_1 P_1$</td>
</tr>
<tr>
<td>or</td>
<td>$M_0 M_1$</td>
<td>$P_0 P_0$</td>
<td>$P_0 P_0$</td>
</tr>
</tbody>
</table>

**rate equations/mean-field limit:**

$$\dot{P}_0 = -M_0 P_0 + M_1 P_1 + P_0 P_1$$

$$\dot{P}_1 = M_0 P_0 - M_1 P_1 - P_0 P_1 + (M_1 P_0 - M_0 P_1)$$

similarly for $M_0, M_1$
special soluble case: symmetric limit

\[ P_0 + P_1 = M_0 + M_1 = \frac{1}{2} \]

\[ \dot{P}_0 = -M_0 P_0 + M_1 P_1 + P_0 P_1 \]

\[ \dot{P}_1 = M_0 P_0 - M_1 P_1 - P_0 P_1 + (M_1 P_0 - M_0 P_1) \]
special soluble case: symmetric limit

\[ P_0 + P_1 = M_0 + M_1 = \frac{1}{2} \]

\[ \dot{P}_0 = -M_0 P_0 + M_1 P_1 + P_0 P_1 \]
\[ \dot{P}_1 = M_0 P_0 - M_1 P_1 - P_0 P_1 + (M_1 P_0 - M_0 P_1) \]

\[ \rightarrow \quad \dot{P}_0 = -\dot{P}_1 = P_0^2 + \frac{1}{2} P_0 - \frac{1}{4} \]
\[ = -(P_0 - \lambda_+)(P_0 - \lambda_-) \]
\[ \lambda_\pm = \frac{1}{4}(-1\pm\sqrt{5}) \approx 0.309, -0.809 \]

solution:

\[ \frac{P_0(t) - \lambda_+}{P_0(t) - \lambda_-} = \frac{P_0(0) - \lambda_+}{P_0(0) - \lambda_-} e^{-(\lambda_+ - \lambda_-)t} \]
near symmetric limit:

\[ P_0 = \frac{1}{2} + 10^{-5}, \quad M_0 = \frac{1}{2} - 10^{-5}, \quad P_1 = M_1 = 0 \]
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near symmetric limit: composition tetrahedron
Consensus Time in Two Dimensions

$T_N$ vs $N$ graph with logarithmic scale.
Consensus Time Distribution

P(T_N) vs T

Y-axis: 10^-2 to 10^-7
X-axis: 0 to 50000

Graph shows the distribution of consensus time as a function of time.
Consensus Time Distribution

$P(T_N)$ vs $T$
Consensus Time Distribution

\[ P(T_N) \]

![Graph showing the distribution of consensus times with droplets in the image.](image-url)
Consensus Time Distribution

\[ P(T_N) \]

- Droplets
- Stripes
two time scales control approach to consensus

see also Spirin, Krapivsky, SR (2001), Chen & SR (2005)

Ising model  Majority vote model
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paradigmatic, soluble, hopelessly naive
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new conservation law
route to consensus sensitive to network structure
fast consensus for broad degree distributions
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Ongoing:
“churn” rather than consensus
heterogeneity of real people
positive and negative social interactions $\rightarrow$ social balance