Dynamic heterogeneities in critical coarsening: 
Exact results for correlation and response 
fluctuations in finite-sized spherical models

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1. **Motivation**
   - Relation between fluctuating correlation and response
   - Comparison with $\chi_4$ in glasses

2. **Spherical Ferromagnet**
   - Model
   - How to coarse-grain the two-time functions
   - Outline of calculation

3. **Results**
   - High temperature
   - Quenches to $T_c$, $d > 4$
   - Quenches to $T_c$, $d < 4$

4. **Summary**
Outline

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4 Summary
Heterogeneities are found in many glassy & aging systems, e.g. spin-glasses and coarsening systems.

Can be detected using two-time local correlation and response functions coarse-grained across mesoscopic regions.

How do these functions fluctuate across dynamical histories?

In spin-glasses these fluctuations are constrained to follow primarily the global FD (fluctuation-dissipation) curve due to a local time-reparametrization symmetry.

In coarsening systems there is no such symmetry.

So how are the fluctuations in correlation and response functions related?

Focus here on exactly solvable spherical model.
Motivation 2: Comparison with $\chi_4$ in glasses

- Variance of coarse-grained correlator is a four-point correlator
- Same as $\chi_4$, used to characterize heterogeneities in glasses
- Maximum in $\chi_4$ identifies a timescale where histories are most diverse, i.e. dynamics is “most heterogeneous”
- Scaling of $\chi_4$ in coarsening?

Below $T_c$

- Time scale of $\chi_4$ set by age since quench, $t_w$
- Amplitude set by correlation volume $\xi^d(t_w)$

At $T_c$: Critical coarsening

- Does $\chi_4$ have a maximum?
- How does position scale with $t_w$?
- How does amplitude scale with $\xi(t_w)$?

- $\xi(t_w) \sim t_w^{1/2}$ is the growing domain size/correlation length
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4 Summary
Spherical ferromagnet

- $N$ spins $S_i$ on cubic lattice in $d$ dimensions
- Hamiltonian: $H = \sum_{(i,j)} (S_i - S_j)^2$
- **Spherical constraint**: $\sum_i S_i^2 = N$
- Produces Lagrange multiplier $z$ in Langevin dynamics
- $z$ fluctuates, but only by $\mathcal{O}(N^{-1/2})$
- Normally neglect fluctuations $\Rightarrow$ linear equations
- Spins then have Gaussian statistics at all times; response does not fluctuate at all
Coarse-grained two-time functions

- Start from **local correlator and susceptibility**
  \[ \hat{C}_{ii}(t, t_w) = S_i(t)S_i(t_w) \quad \hat{\chi}_{ii}(t, t_w) = \frac{\partial S_i(t)}{\partial h_i(t_w)} \]
  
- But \( \hat{C}_{ii} \) fluctuates by \( \mathcal{O}(1) \), \( \hat{\chi}_{ii} \) by \( \mathcal{O}(N^{-1/2}) \)
- Need to **coarse-grain across entire system** to get same order:
  \[ \hat{C}(t, t_w) = \frac{1}{N} \sum_i \hat{C}_{ii}(t, t_w) \quad \hat{\chi}(t, t_w) = \frac{1}{N} \sum_i \hat{\chi}_{ii}(t, t_w) \]

- Averages over dynamics give conventional \( C = \langle \hat{C} \rangle, \chi = \langle \hat{\chi} \rangle \)
- Study **(co)-variances** [scaled with \( N \) and \( T \)]
  \[ V_C(t, t_w) = N \left\langle \left[ \delta C(t, t_w) \right]^2 \right\rangle, \quad \delta C = \hat{C} - C \]
  \[ V_\chi(t, t_w) = NT^2 \left\langle \left[ \delta \chi(t, t_w) \right]^2 \right\rangle, \quad \delta \chi = \hat{\chi} - \chi \]
  \[ V_{C\chi}(t, t_w) = NT \left\langle \delta C(t, t_w) \delta \chi(t, t_w) \right\rangle \]
Connection to $\chi_4$

- Write out correlation variance explicitly: $V_C(t, t_w) =$

$$= \frac{1}{N} \sum_{i,j} \left[ \langle \hat{C}_{ii}(t, t_w)\hat{C}_{jj}(t, t_w) \rangle - \langle \hat{C}_{ii}(t, t_w) \rangle \langle \hat{C}_{jj}(t, t_w) \rangle \right]$$

$$= \frac{1}{N} \sum_{i,j} \left[ \langle S_i(t)S_i(t_w)S_j(t)S_j(t_w) \rangle \right]$$

$$- \langle S_i(t)S_i(t_w) \rangle \langle S_j(t)S_j(t_w) \rangle$$

- This is exactly the four-point correlator $\chi_4$ used to characterize heterogeneities in glassy systems
Alternative ways of coarse-graining?

- Finding the fluctuating response $\hat{\chi}$ requires measuring/calculating $N$ local responses $\hat{\chi}_{ii}$
- Can we simplify using standard trick for mean response?
- Look at response of a single observable $A = \sum_i \epsilon_i S_i$, with $\epsilon_i = \pm 1$ chosen randomly
- Resulting susceptibility $\hat{\chi}_\epsilon = (1/N) \sum_{i,j} \epsilon_i \epsilon_j \hat{\chi}_{ij}$
- Not equivalent: $\hat{\chi}_\epsilon$ has larger variance across histories than $\hat{\chi}$
- Worse: The corresponding $\hat{C}_\epsilon$ has a variance that’s larger by $O(N)$, so no longer comparable to $\hat{\chi}_\epsilon$
We calculate $V_C$, $V_\chi$ and $V_{C\chi}$

Derive correlation coefficient $\gamma = \frac{V_{C\chi}}{(V_C V_\chi)^{1/2}}$: how strongly are fluctuations related?

Graphically: contours of equal probability

\[
\begin{pmatrix}
\delta C \\
\delta \chi
\end{pmatrix}
\begin{pmatrix}
V_C & V_{C\chi} \\
V_{C\chi} & V_\chi
\end{pmatrix}^{-1}
\begin{pmatrix}
\delta C \\
\delta \chi
\end{pmatrix} = \text{const}
\]

Ellipses centered on average values $(C, \chi)$.

Biggest fluctuations occur along principal axis of ellipse

Call the negative slope of this $X_{fl}$

If situation is like in spin glasses, expect $X_{fl} \approx X$

($X =$ standard FD ratio)

Note $\gamma$ and $X_{fl}$ always have opposite sign
Outline of calculation

- Decompose spins as $S_i = s_i + r_i/\sqrt{N}$
- Leading order Gaussian part: $s_i$
- First non-Gaussian correction: $r_i$
- Similar decomposition for response to field
- Expansion in $1/\sqrt{N}$ gives expression for $r_i$:

$$r_i(t) = -\frac{1}{2} \int dt' dt'' \sum_j R_{ij}(t, t') s_j(t') L(t', t'') \frac{\sum_k s_k^2(t'') - N}{\sqrt{N}}$$

- $L(t', t'')$ is inverse of an appropriate two-time kernel
- Eventually end up with averages over the Gaussian $s_i$
- Subtleties with extracting scaling in interesting long-time regime $t \gg 1$
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4 Summary
Cases studied & Overview of results

Quenches to criticality:
- For short time differences, the fluctuations are effectively stationary and one can consider the equilibrium situation.
- For large time differences and $d > 4$ the fluctuations are still as in equilibrium, modulated by weak aging corrections.
- For large time differences and $d < 4$ strong aging effects appear in the fluctuations.

Quenches to above criticality:
- Equilibration process is fast and a genuine equilibrium dynamics results.
- High temperature limit: special case, can be solved explicitly.
Example results for $T = 15$, plotted versus $\tau = T \Delta t$
High temperature: Asymptotics

- For equal time $V_C, V_\chi = 0$ [as it should be]
- For $T \to \infty$ at fixed $\tau = T \Delta t$:
  $V_C, V_\chi = \mathcal{O}(1)$, but $V_C \chi = \mathcal{O}(1/T^2)$
- So fluctuations of correlation and response are weakly correlated for high $T$
- **Short time:**
  
  \[
  V_C = 2T^2 \Delta t^2 \sim \tau^2 \\
  V_\chi = \frac{2}{3} T^3 \Delta t^3 \sim \tau^3 \\
  -V_C \chi \sim T^2 \Delta t^4 \sim \tau^4/T^2
  \]
- **Long time:** $V_C, V_\chi \to \mathcal{O}(1)$, $V_C \chi \to 0$ exponentially
High temperature: Contour plots

- FD plot $\chi(t, t_w)/C(t, t)$ vs $C(t, t_w)/C(t, t)$: straight line
- As time grows, biggest fluctuations line up with $\chi$-axis, rotation sharp but continuous

$T = 10; N = 50$
Quenches to $T_c$, $d > 4$: Schematic plots

- $V_C$ vs $\Delta t$
- $V_\chi$ vs $\Delta t$
- $V_{C\chi}$ vs $\Delta t$
Quenches to $T_c$, $d > 4$: Comments

- **Short time** scalings as for high $T$, with more complicated $T$-dependence for $V_C\chi$
- **Long time**: $V_C, V_\chi \rightarrow O(1)$ but approach is now power law
- Covariance also decays as power law: $-V_C\chi \sim \Delta t^{(4-d)/2}$
- Only significant aging effect: modulation of $V_C\chi$ with a function of $t/t_w$
- Variances approach for long times their equilibrium values
- $V_C \equiv \chi_4$ has maximum but this doesn't scale with age – surprising!?
Quenches to $T_c$, $d > 4$: Contour plots (sketch)

- Aging of means not reflected in fluctuations

- Initial $t - t_w \sim 1$
Quenches to $T_c$, $d < 4$: Scaling functions

(Co)-variances scaled by $t_w^{(4-d)/2}$ against $x - 1 = (t/t_w) - 1$, $d = 3$
Quenches to $T_c$, $d < 4$: Scaling functions

(Co)-variances scaled by $t_w^{(4-d)/2}$ against $x - 1 = (t/t_w) - 1$, $d = 3$
Motivation  Spherical model  Results  Summary

HT  \( d > 4 \)  \( d < 4 \)

Quenches to \( T_c \), \( d < 4 \): Scaling functions

\[(\text{Co})\text{-variances scaled by } t_w^{(4-d)/2} \text{ against } x - 1 = (t/t_w) - 1, \ d = 3\]
Quenches to $T_c$, $d < 4$: Correlation coeff., $X_{fl}$

- For $x \to 1$ (quasi-equilibr., $1 \ll t - t_w \ll t_w$): 
  $\gamma$ and $X_{fl}$ are $O(1)$
- Large $x$: $-\gamma \sim x^{(2-d)/2}$, $X_{fl} \sim x^{d/4}$
- Response has bigger fluct’ns than correlation $\Rightarrow X_{fl}$ diverges
Quenches to $T_c$, $d < 4$
Quasi-equilibrium regime $1 \ll t - t_w \ll t_w$

Variances scale as $V_C \sim t_w^{(4-d)/2} (x - 1)^{(4-d)/2} = \nu_C (t - t_w)^{(4-d)/2}$

Time translation invariant as expected
Quenches to $T_c$, $d < 4$

Quasi-equilibrium regime $1 \ll t - t_w \ll t_w$ (cont.)

$V_C, V_\chi, V_{C\chi}$ have same time-dependence $\Rightarrow$
non-trivial $O(1)$ values for correlation coeff. & fluctuation slope:

For $d \to 4$, $\gamma = -1/\sqrt{3}$ and $X_{fl} = (1 + \sqrt{17})/4$

For $d \to 2$, $\gamma \sim -(d - 2)^{1/2}$ and $X_{fl} \sim d - 2$
Quenches to $T_c$, $d < 4$: Contour plots (sketch)

- **Aging**: $t - t_w \sim t_w$
- **Late quasi-equilibrium**: $1 \ll t - t_w \ll t_w$
- **Initial**: $t - t_w \sim 1$
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4 Summary
Non-Gaussian spin statistics induce fluctuations in $C$ that are of same order in $N$ as the Gaussian term. In the response they are the only source of fluctuations.

- No simple relations between $V_C$ and $V_\chi$, not even at high $T$, though at least covariance $V_{C\chi}$ has expected sign $[X_{fl} > 0]$

- Fluctuations depend on $t - t_w$ only for $d > 4$, and also for $d < 4$ in the quasi-equilibrium regime $x = t/t_w \approx 1$

- Details of $x$ dependence: non-trivial limits for $\gamma$ and $X_{fl}$ in quasi-eq. regime but no obvious constraints or FDT-like relations for fluctuations.
# Conclusion - $\chi_4$ and heterogeneities

## Above critical dimension ($d > 4$)
- Heterogeneities detected via e.g. maximum of $V_C$
- But scaling is surprising...
- Position (timescale) doesn’t grow with $t_w$
- Amplitude doesn’t grow with $\xi(t_w)$

## Below critical dimension ($d < 4$)
- $V_C$, $V_\chi$ and $V_{C\chi}$ are age-dependent, but $V_C$ has no maximum
- Characteristic timescales $\sim t_w$, reasonable
- Amplitudes $\sim t_w^{(4-d)/2}$, unrelated to $\xi^d(t_w) \sim t_w^{d/2}$

## Outlook
- Use field theory to understand scaling exponents?
- Similar studies in genuinely short-ranged models, e.g. $O(n)$