Bragg spectroscopy of an accelerating solitary-wave condensate

with application to a condensate in a time-averaged orbiting potential trap

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Introduction

- Theoretical treatment of Bragg spectroscopy of an accelerating solitary-wave
- Description in translating frame where the condensate is stationary
- Simplified accurate calculations of spectra from two-state momentum-space model
- Illustrate methods using condensate micromotion in a time-averaged orbiting potential (TOP) trap
  - Features of Bragg spectrum experimentally accessible [1]

Solitary-wave solutions

- Particular solutions to the Gross-Pitaevskii equation

$$i\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left(-\nabla^2 + V(\mathbf{r}) + G(\mathbf{r},t) + C|\psi(\mathbf{r},t)|^2\right)\psi(\mathbf{r},t)$$

  - $V(\mathbf{r})$ confines the condensate
  - $G(\mathbf{r},t)$ generates solitary-wave motion
  - $C = 4\pi\hbar aN/m\omega_x x_0^3$ accounts for inter-atomic collisions
- Condensate evolves without changing shape [2, 3]

$$\psi_{\rm SW}(\mathbf{r},t) = \xi(\mathbf{r}-\bar{\mathbf{r}}(t))e^{-i\mu t + iS(\mathbf{r},t)}$$

  - $\mu\xi(\mathbf{r}) = \left(-\nabla^2 + V(\mathbf{r}) + C|\xi(\mathbf{r})|^2\right)\xi(\mathbf{r})$
  - $S(\mathbf{r},t) = \frac{1}{2}\mathbf{r}\cdot\frac{d\bar{\mathbf{r}}(t)}{dt} + \frac{1}{4}\int\left[\bar{\mathbf{r}}^2(t) - \left(\frac{d\bar{\mathbf{r}}(t)}{dt}\right)^2\right]dt$
- Centre of mass motion $\bar{\mathbf{r}}(t)$ governed by

$$\frac{1}{2}\frac{\partial^2\bar{\mathbf{r}}(t)}{\partial t^2} = -\nabla F(\mathbf{r},t)$$

  - $F(\mathbf{r},t) = V(\mathbf{r}) + G(\mathbf{r},t) - V(\mathbf{r}-\bar{\mathbf{r}}(t))$
- Solitary-wave solutions only exist for particular potentials, e.g., for $V(\mathbf{r}) = \frac{1}{4}(x^2+y^2+\lambda^2 z^2)$ we require

$$G(\mathbf{r},t) = \mathbf{g}_1(t)\cdot\mathbf{r} + g_2(t).$$

- Examples of solitary-wave motion
  - dipole oscillations in a harmonic trap
  - micromotion in a TOP trap
  - motion due to experimental noise in the trap position

Translating frame

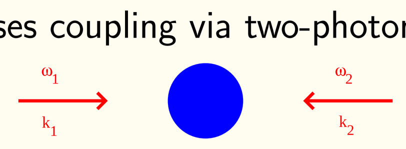
- Transform to frame where solitary-wave is at rest
- Co-ordinates defined by [3]
  $\mathbf{R} = \mathbf{r} - \bar{\mathbf{r}}(t)$ and $\mathbf{P} = \mathbf{p} - \frac{1}{2}\frac{d\bar{\mathbf{r}}(t)}{dt}$
- Applicable to arbitrary centre of mass motion
- Wave function in translating frame obeys

$$i\frac{\partial}{\partial t}\psi^{\rm T}(\mathbf{R},t) = \left(-\nabla^2_{\mathbf{R}} + V(\mathbf{R}) + C|\psi^{\rm T}(\mathbf{R},t)|^2\right)\psi^{\rm T}(\mathbf{R},t)$$

- No explicit time dependence appears

Bragg Spectra

- Bragg pulse causes coupling via two-photon processes



$$\mathbf{q} = \mathbf{k}_1 - \mathbf{k}_2$$
$$\omega = \omega_1 - \omega_2$$

- Optical potential
  $V_{\rm opt}(\mathbf{r},t) = \frac{1}{2}U_0\cos(\mathbf{q}\cdot\mathbf{r} - \omega t)$
- In translating frame

$$V^{\rm t}_{\rm opt}(\mathbf{R},t) = \frac{1}{2}U_0\cos(\mathbf{q}\cdot\mathbf{R} + \mathbf{q}\cdot\bar{\mathbf{r}}(t) - \omega t)$$

- Measure the population $P_{+1}(\omega)$ scattered by momentum $\mathbf{q}$ as a function of ω

Theoretical Treatment

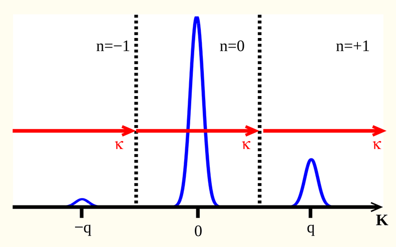
- Need for approximate method
  - Calculating Bragg spectra directly from the Gross-Pitaevskii equation is computationally intensive
  - Seek a physical interpretation

Basis of approximate treatment

- Neglect interactions ($C = 0$)
- Assume the confining potential is turned off ($V(\mathbf{r}) = 0$)
- Bragg pulse applied along the x axis ($\mathbf{q}\cdot\mathbf{R} = qX$)
- System reduces to one dimension with wave function $\psi^{\rm T}(X,t)$

Two-state Model

- Work in momentum space of translating frame
- Partition momentum space into equally sized bins [4]



- Infinite set of equations which couple momentum wave functions of consecutive bins

$$\frac{\partial}{\partial t}\phi^{\rm T}_n(\kappa,t) = \omega_n(\kappa,t)\phi^{\rm T}_n(\kappa,t) + \frac{1}{4}U_0\left[\phi^{\rm T}_{n-1}(\kappa,t) + \phi^{\rm T}_{n+1}(\kappa,t)\right]$$

$$\omega_n(\kappa,t) = (\kappa+nq)^2 - n\omega + nq\frac{d\bar{x}(t)}{dt}$$

- Assume only bins $n=0$ and $n=+1$ are populated [4]

$$i\frac{\partial}{\partial t}\begin{pmatrix}\phi^{\rm T}_0(\kappa,t)\\ \phi^{\rm T}_{+1}(\kappa,t)\end{pmatrix} = \begin{pmatrix}\omega_0(\kappa,t) & \frac{U_0}{4}\\ \frac{U_0}{4} & \omega_0(\kappa,t)+\Delta(\kappa,t)\end{pmatrix}\begin{pmatrix}\phi^{\rm T}_0(\kappa,t)\\ \phi^{\rm T}_{+1}(\kappa,t)\end{pmatrix}$$

$$\Delta(\kappa,t) = 2\kappa q + q^2 - \omega + q\frac{d\bar{x}(t)}{dt}$$

Validity regime

- Resonant coupling only between bins $n=0$ and $n=+1$
  - Neglect other coupling between consecutive bins: $U_0 < 2q(q-\sigma-\delta_p)$
    - $\sigma = 1/e$ condensate momentum width
    - $2\delta_p = \left.\frac{d\bar{x}(t)}{dt}\right|_{\rm max} - \left.\frac{d\bar{x}(t)}{dt}\right|_{\rm min}$
  - Neglect higher order coupling: $t \ll \frac{8\pi q^2}{U_0^2}$

Independent Resonance Approach

- Use Fourier methods to write the optical potential as

$$V^{\rm t}_{\rm opt}(X,t) = \frac{1}{2}\sum_l c_l\cos(qX - (\omega+\omega_l)t + \epsilon_l)$$

- In translating frame multiple resonances possible
- Treat each term independently if resonances are well separated
- Relative phase can cause consecutive terms to act independently,
  e.g., in the limit $\frac{c_l t}{4} \ll 1$,

$$\cos\left[\frac{1}{2}(\omega_{l+1}-\omega_l)t - (\epsilon_{l+1}-\epsilon_l)\right] = 0$$

- Very simple numerically

Example: TOP Trap

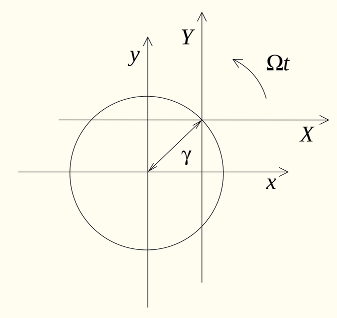
Solitary-wave solutions

- The TOP trap potential can be well approximated by [3, 5]

$$V_{\rm TOP}(\mathbf{r},t) = V_{\rm H}(\mathbf{r}) + r_0(\cos\Omega t, \sin\Omega t, 0)\cdot\mathbf{r}$$

  - r_0 = 'circle of death'
  - Ω = bias field rotation frequency
- Eigenstates of the TOP trap do not exist in the lab frame
- In circularly translating frame eigenstates of the TOP trap exist and are solitary-wave solutions in the lab frame with

$$\bar{\mathbf{r}}(t) = \gamma_{\rm t}(\cos\Omega t, \sin\Omega t, 0)$$
$$\gamma_{\rm t} = \frac{2r_0}{\Omega^2 - 1}$$



- Eigenstates retain their orientation with respect to the lab frame

NOT the rotating frame

- TOP trap has eigenstates in the frame rotating with the bias field
- Gross-Pitaevskii equation in the rotating frame

$$i\frac{\partial}{\partial t}\psi'(\mathbf{r}',t) = \left(-\nabla^2_{\mathbf{r}'} + V'_{\rm TOP}(\mathbf{r}') - \Omega\hat{L}_z(\mathbf{r}') + C|\psi'(\mathbf{r}',t)|^2\right)\psi'(\mathbf{r}',t)$$

- TOP trap eigenstates calculated using the rotating frame are restricted [3], must also satisfy

$$\hat{L}_z(\mathbf{r})\xi(\mathbf{r}) = l_z\xi(\mathbf{r})$$

Bragg spectra in a TOP trap

Bragg pulse

- Optical potential in circularly translating frame is

$$V^{\rm t}_{\rm opt}(X,t) = \frac{1}{2}U_0\cos(qX + q\gamma_{\rm t}\cos\Omega t - \omega t),$$

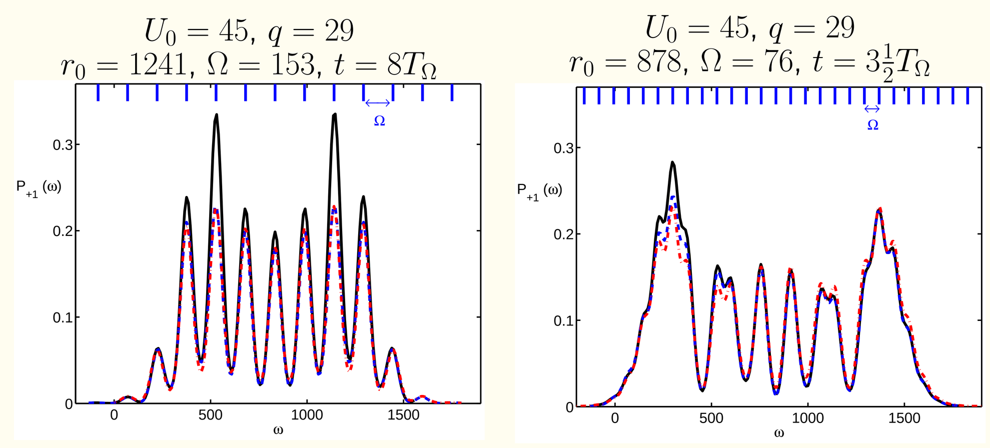
  a frequency modulated potential [1, 5]
- Well known expansion

$$\begin{aligned}V^{\rm t}_{\rm opt}(X,t) = \frac{1}{2}U_0\Big\{&J_0(\gamma q)\cos(qX-\omega t)\\ &-\sum_{l=0}^{\infty}(-1)^l J_{2l+1}(\gamma q)\left[\cos\left(qX-\omega t-(2l+1)\Omega t-\frac{\pi}{2}\right)\right.\\ &\left.+\cos\left(qX-\omega t+(2l+1)\Omega t-\frac{\pi}{2}\right)\right]\\ &+\sum_{l=1}^{\infty}(-1)^l J_{2l}(\gamma q)\left[\cos(qX-\omega t-2l\Omega t)+\cos(qX-\omega t+2l\Omega t)\right]\Big\}\end{aligned}$$

- Resonance frequencies separated by Ω

Bragg spectra

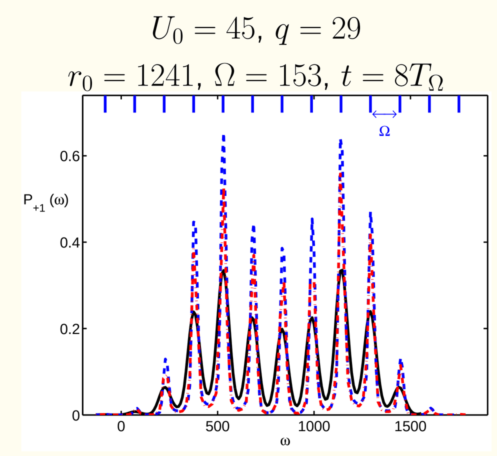
- Two-state model in good agreement with full calculations ($C = 0$)
- Independent resonance approach only accurate when $t = nT_\Omega$



Full numerical simulation $C = 0$ (–), two-state model (- -), and independent resonance approach (-·).

Collisional Effects

- Calculate spectra using full Gross-Pitaevskii equation
- Three primary effects
  - Narrower spectral features
  - Increased peak scattering
  - Resonances shifted up in frequency by $\approx 4\mu/7$ [4]
- Well described in one dimension



Full numerical simulations with $C = 0$ (–), in 1D with $C = 85$, $\mu = 10.08$ (- -), and in 2D with $C = 600$, $\mu = 9.85$ (-·)

Conclusion

- Simple description of Bragg spectroscopy of a condensate accelerating as a solitary-wave
- Developed theoretical models in translating frame
- Methods illustrated using condensate solitary-wave behaviour in a TOP trap
- Excellent agreement between simple theoretical model and calculations of the full Gross Pitaevskii equation

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