Pattern Forming Dynamical Instabilities of Bose-Einstein Condensates

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International Workshop on

Mesoscopic Phenomena in Ultracold Matter: From Single Atoms to Coherent Ensembles

October 11 - 15, 2004

Outline

- **The (mean-field theoretic) model and its limitations**

- **Long-wavelength instabilities in BECs**

- **Modulational Instability**
- Time-independent setting (attractive BECs) - **Bright Soliton trains**
- Time-dependent setting: Feshbach Resonance Management (repulsive/attractive BECs) - **Matter Wave Breathers**

- **Transverse ("Snaking") Instability** of dark soliton stripes (repulsive BECs) - **Vortices and vortex arrays**

- **Derivation of relevant instability thresholds**
  (based on length-scale competition arguments)

Some details

- ***The mean-field approximation***
- *Description of the BECs for:* T $\rightarrow$ 0, no fluctuations, low-dimensional (quasi-1D and quasi-2D) systems (relevant to experiments)
- Description of instabilities and patterns (solitons, vortices) with the relevant **GPEs**, for time-scales comparable to the lifetimes: "Signatures" of the predicted phenomena are expected to be observable.

- ***Long-Wavelength instabilities and related patterns***
- *Modulational Instability for quasi-1D attractive BECs:*

  Plane wave solution of the NLS equation becomes unstable, resulting to the formation of a pattern consisting of a "train" of bright solitons.
- *Transverse Instability* for 2D/3D repulsive BECs:

  Dark soliton stripes undergo transverse "snake" deformations, giving rise to vortices and vortex patterns ("vortex necklace").

The mean-field model

- The effective dimensionless GP equation (in quasi-1D)

$$i\psi_t = -\psi_{xx} - g|\psi|^2\psi + V(x)\psi, \quad g = \pm 1$$

$$V(x) = \frac{1}{2}\Omega^2 x^2, \qquad \Omega \equiv \hbar\omega_x / g_{1D} n_0, \qquad g_{1D} = g_{3D}/(4\pi\ell_\perp^2)$$

- x in units of : $\quad \xi \equiv \hbar / \sqrt{2 n_0 g_{1D} m}$
- t in units of : $\quad \xi / c, \qquad c \equiv \sqrt{n_0 g_{1D} / m}$
- ψ in units of : $\quad n_0$

Typically (e.g., for a sodium BEC) : $\omega_x \sim 2\pi \times (10-80)Hz, \quad \omega_\perp \sim 100\omega_x$

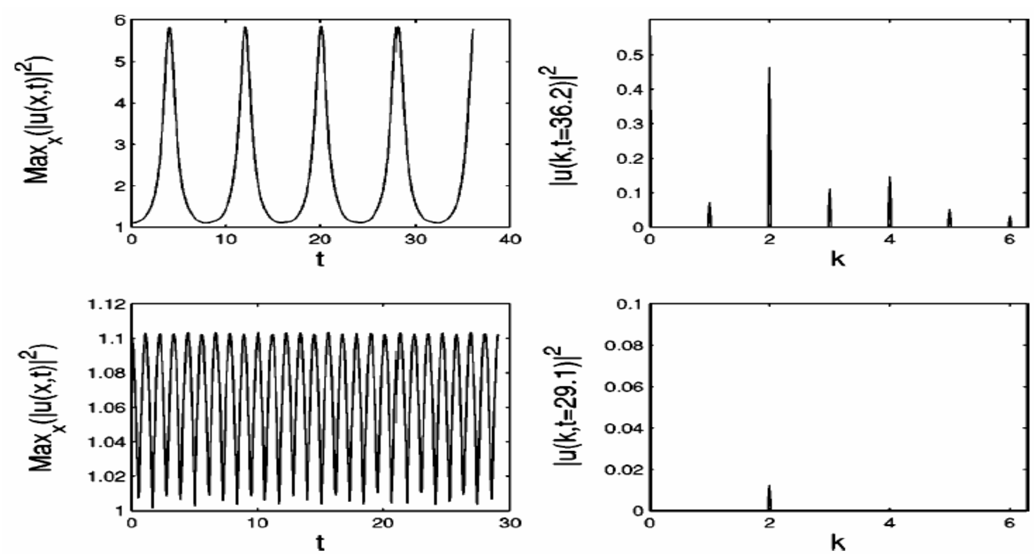
$$\xi \sim 0.6 - 1\,\mu m, \;\; \xi / c \sim 0.1 - 0.3\, ms, \;\; n_0 \sim 10^8 m^{-1}, \;\; N \sim 10^3 - 10^4$$

Modulational Instability in *untrapped* BEC's

- Plane wave solution:

$$\psi(x,t) = \psi_0 \exp[i(kx - \omega t)], \qquad \omega = k^2 + g\psi_0^2$$

- Dispersion Relation: $$(\Omega - 2kQ)^2 = Q^2(Q^2 + 2g\psi_0^2)$$
- Instability Band: $$Q < Q_{cr} \equiv \psi_0\sqrt{-2g} \quad (\text{for } g = -1)$$



Unstable case
(Q=1)

Stable case
(Q=2)

The effect of *linear* or *quadratic* potentials

$$V(x) = \mathcal{E}x$$

$$V(x) = \mathcal{K}x^2$$

- ***Tappert transformation:***

$$\psi(x,t) = v(\eta,t)\exp\left(-i\mathcal{E}xt - \frac{1}{3}i\mathcal{E}^2t^3\right)$$

$$\eta = x + \mathcal{E}t^2$$

brings back to the NLS

$$\boxed{i\upsilon_t + \upsilon_{\eta\eta} + |\upsilon|^2\upsilon = 0}$$

- ***Lens transformation:***

$$\psi(x,t) = \ell^{-1}\exp[if(t)x^2]v(\zeta,\tau)$$

$$\zeta = x/\ell(t) \qquad \tau_t = \frac{1}{\ell^2}$$

$$-f_t = 4f^2 + \mathcal{K}(t) \qquad \ell(t) = \ell(0)\exp\left(4\int_0^t f(s)ds\right)$$

transforms GPE to the NLS

$$\boxed{\begin{gathered} i\upsilon_\tau + \upsilon_{\zeta\zeta} + |\upsilon|^2\upsilon = 2i\lambda\upsilon \\ \lambda \equiv f\ell^2 \end{gathered}}$$

In both cases MI condition ***does not*** change

Phys. Rev. A **67**, 063610 (2003);
Mod. Phys. Lett. B **18**, 173 (2004).

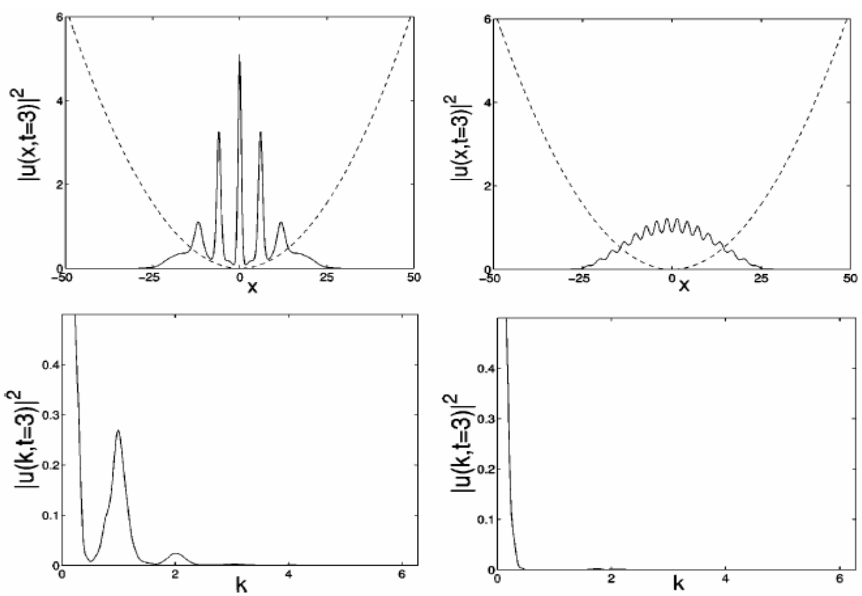
Examples (quadratic potentials)

- Initial condition: $\psi(x,t=0)=\psi_{TF}[1+\epsilon\cos(Qx)]$

(similar to the Texas and Paris experiments)

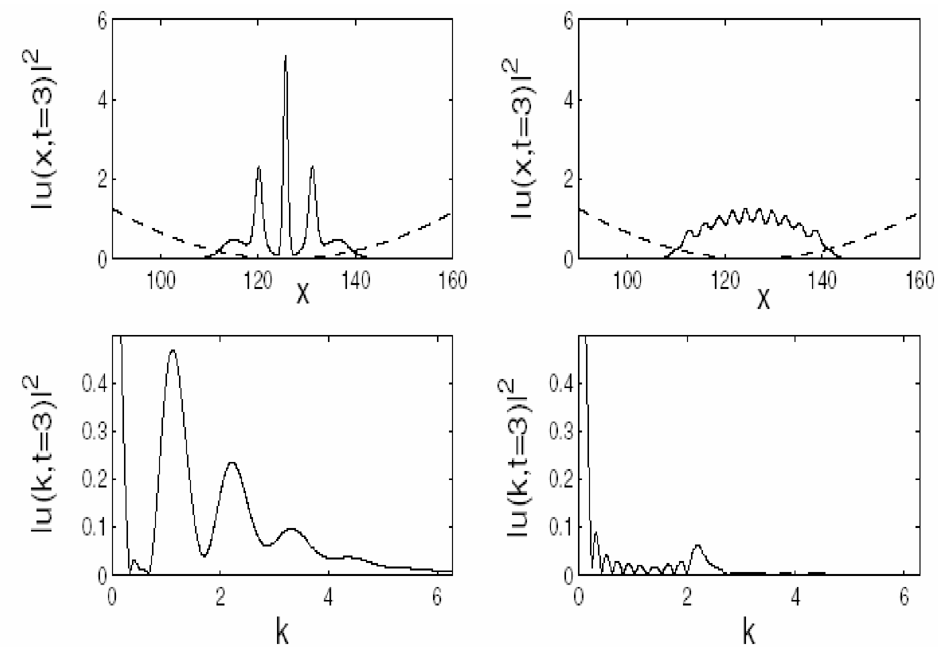
$\mathcal{K}(t)=\text{constant}$

Unstable case (Q=1) **Stable case (Q=2)**



$\mathcal{K}(t) = (1/16)(t+t^*)^{-2}$

Unstable case (Q=1) **Stable case (Q=2)**



Length (time) scale: 0.6 µm (0.1 ms)

Modulational Instability threshold

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + g|\psi|^2\psi + V(x)\psi$$

(we set g = -1)

$$Q < Q_{cr} \equiv \sqrt{2}\psi_0 \longrightarrow \lambda > \lambda_{cr} \equiv \frac{\sqrt{2}\pi}{\psi_0}$$

- In the presence of the magnetic trap, MI is ***avoided*** when the TF diameter

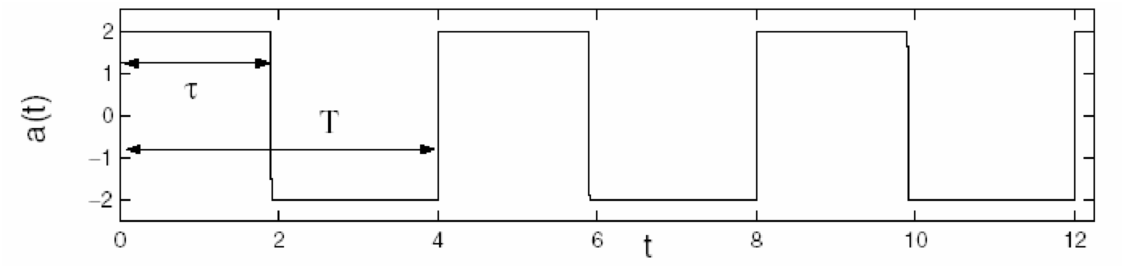
$\lambda_{\mathrm{BEC}} \approx 2\sqrt{2\mu}/\Omega$ is $\lambda_{\mathrm{BEC}} < \lambda_{cr}$

- This occurs when : $\Omega > \Omega_{cr} = \frac{12}{\pi^3}$ or $\boxed{n_0 a \frac{\omega_\perp}{\omega_x} < 1.3}$

Phys. Rev. A 70, 023602 (2004); Mod. Phys. Lett. B 18, 173 (2004).

Time-dependent settings: Feshbach Resonance Management (FRM)

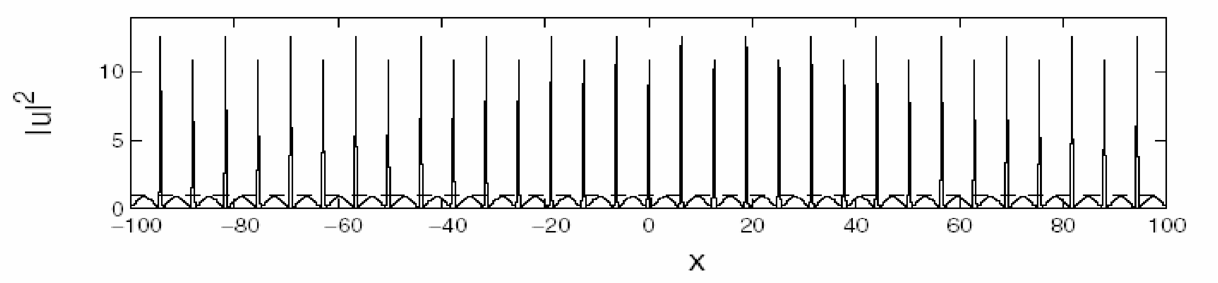
- The model: $$iu_t = -\frac{1}{2}u_{xx} + a(t)|u|^2 u + \frac{1}{2}\Omega^2 x^2 u$$



- Analogy with "***Dispersion Management***" in Nonlinear Fiber Optics

(periodic alternation of fibers with opposite signs of group-velocity dispersion)

- The uniform solution $u_0(t) = A_0 \exp[iA_0^2 \int_0^t a(s)ds]$ is **subject to MI**

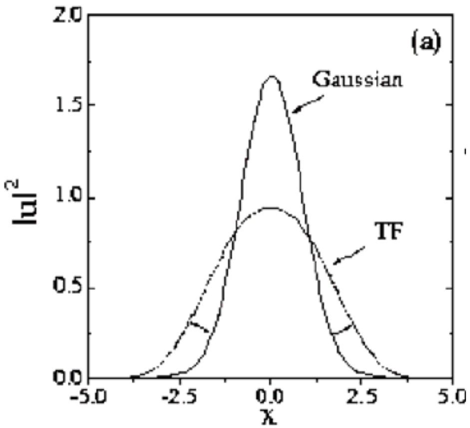


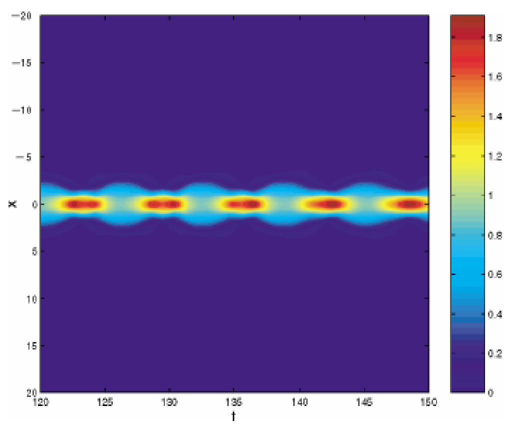
Length (time) scale:
1 μm (0.3 ms)

Phys. Scripta T**107**, 27 (2004).

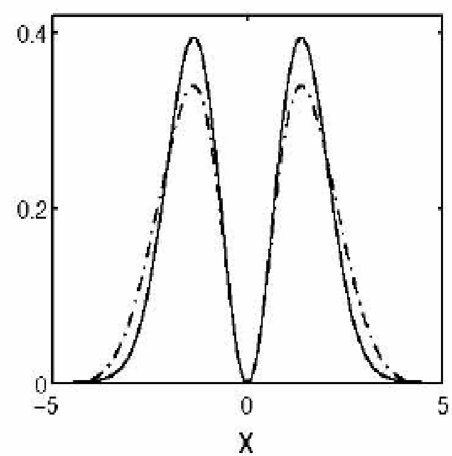
Robust FRM Solitons:
Bright and Dark Matter-Wave Breathers

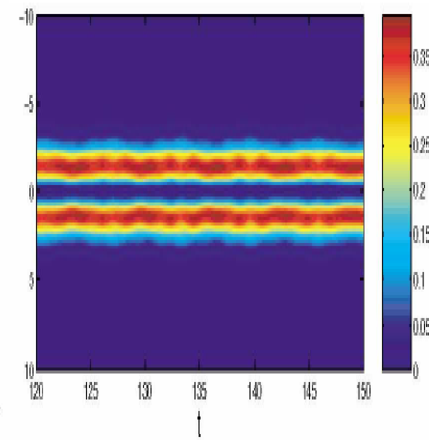
Bright Soliton





Dark Soliton





Length (time) scale: 1 μm (0.3 ms)

Phys. Rev. Lett. **90**, 230401 (2003); Phys. Scripta T**107**, 27 (2004).

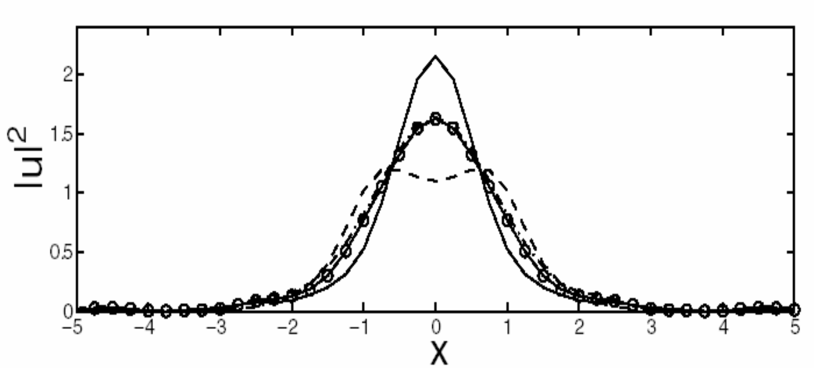
Averaging for Solitons with Nonlinearity Management

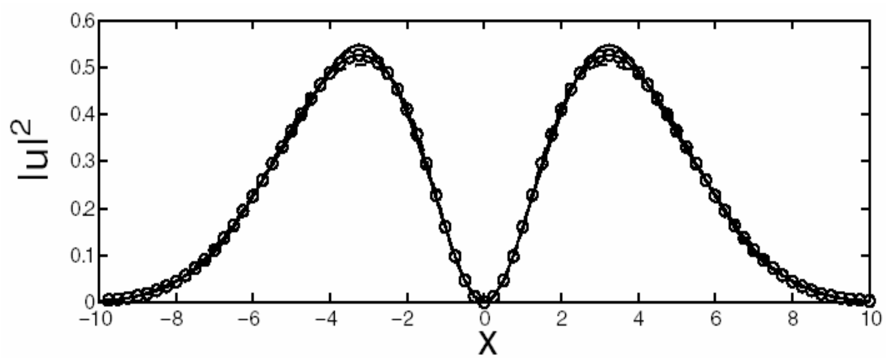
- The model: $$iu_t = -u_{xx} + V(x)u + \left[a_0 + \frac{1}{\varepsilon} a\left(\frac{t}{\varepsilon}\right)\right]|u|^2 u$$
- The transformation:

$$u(x,t) = e^{-i\varphi(x,t)}\upsilon(x,t), \qquad \varphi(x,t) = \varepsilon^{-1}\int_0^t a(\varepsilon^{-1}t')|\upsilon|^2(x,t')dt'$$

- The effective (averaged) NLS equation (with **const.** coefficients):

$$\boxed{iw_t = -w_{xx} + V(x)w + a_0|w|^2 w - \beta\left[\left(|w|_x^2\right)^2 + 2|w|^2|w|_{xx}^2\right]w}$$





Phys. Rev. Lett. 91, 240201 (2003); Phys. Rev. E, in press (2004)

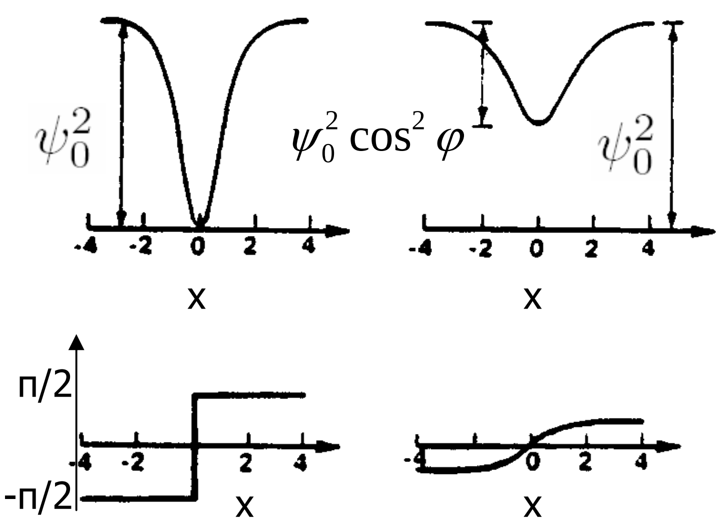
Transverse (Snaking) Instability of Dark Solitons in *quasi-2D* (“*pancake*”) BECs

- The normalized GPE (radially symmetric trap):

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\Delta\psi + |\psi|^2\psi + V(x,y)\psi \qquad V(x,y) = \frac{1}{2}\Omega^2(x^2+y^2)$$

- The dark soliton stripe (untrapped BEC):

Black Soliton   Gray Soliton



$\cos\varphi = 1$ $\cos\varphi < 1$

The general form (both “black” and “gray”)

$$\psi(x,t) = \psi_0(\cos\varphi\tanh\zeta + i\sin\varphi)\exp(-i\mu t)$$

$$\zeta \equiv \psi_0(\cos\varphi)[x - \psi_0(\sin\varphi)t] \qquad \mu \equiv \psi_0^2$$

$\cos\varphi$: *soliton amplitude*

$\sin\varphi$: *soliton velocity* $(\sim \upsilon / c)$

Condition for transverse instability

$$Q < Q_{cr} \equiv [2\sqrt{\sin^4\phi + \cos^2\phi} - (1 + \sin^2\phi)]^{1/2}$$ (for $\mu = 1$)

Instability band can be ***suppressed*** if

- The TF diameter $\lambda_{\mathrm{BEC}} \approx 2\sqrt{2\mu}/\Omega$ is :

  $\lambda_{BEC} < \lambda_{cr} \Rightarrow \Omega > \frac{1}{\pi} \approx 0.31$ or $\boxed{n_0 a \ell_z \frac{\omega_z}{\omega_\perp} < 4}$

  ("***Trap engineering***")

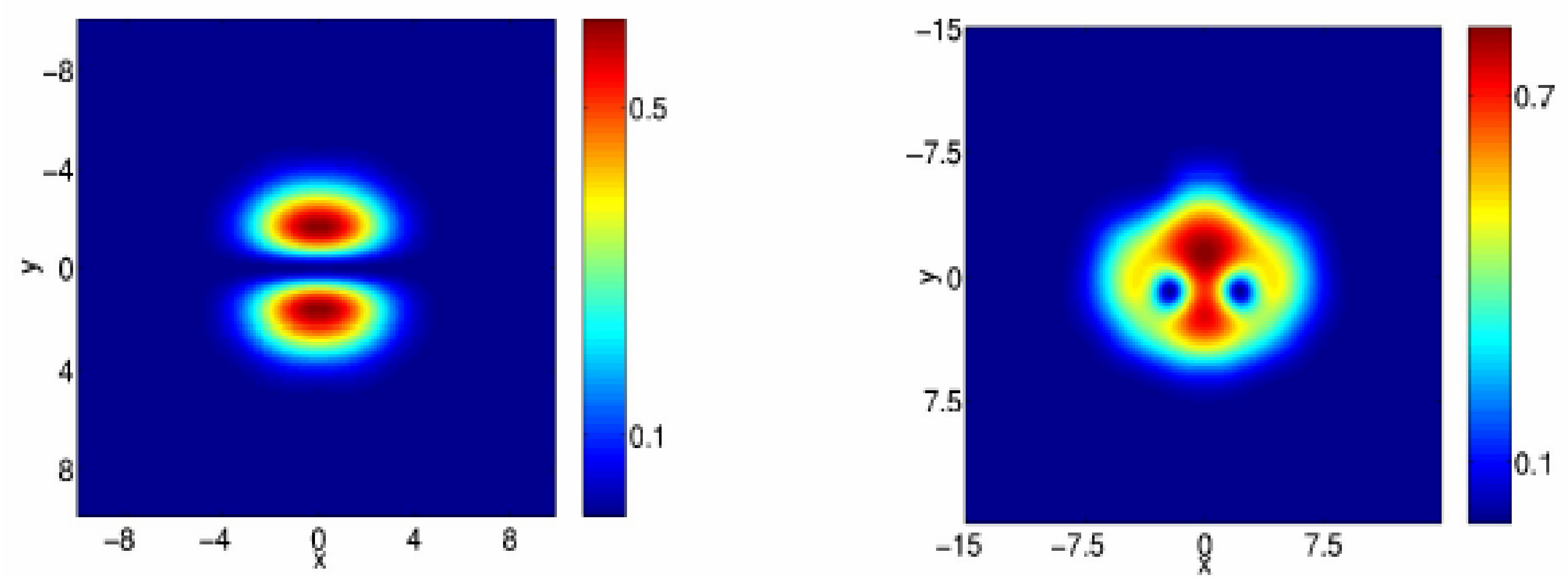
- The dark stripe is bent so as to form a ***ring*** of length

  $L < 2\pi/Q_{cr}$ $\longrightarrow$ *RING DARK SOLITON*

  ("***Soliton engineering***")

"Trap engineering" for suppression / onset of Transverse Instability

- Ω=0.35: TI is suppressed [t = 1000 (or 180 ms)]
- Ω=0.15: TI manifests itself [for t ~150 (or 27 ms)]



Length (time) scale: 0.7 μm (0.18 ms)

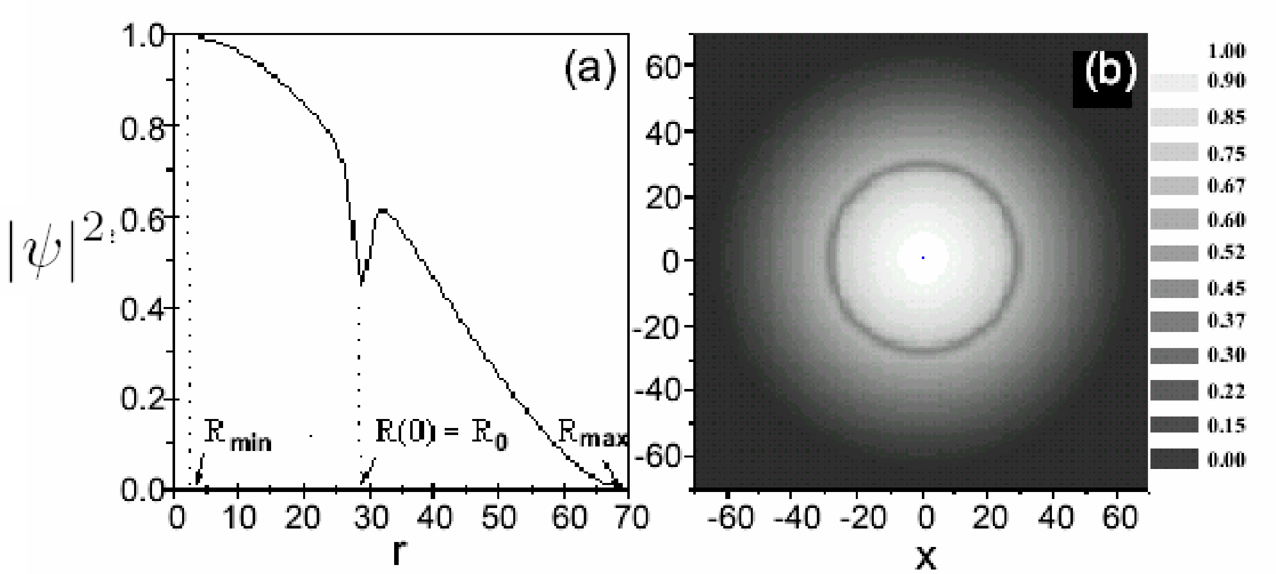
- Phys. Rev. A **70**, 023602 (2004); Mod. Phys. Lett. B **18**, 173 (2004).

"Soliton Engineering" : Ring Dark Solitons

- Approximate solution of the GPE: $\psi = \psi_{\mathrm{TF}} \upsilon$

where: $\upsilon(r,t) = \cos\varphi(t) \cdot \tanh\zeta + i\sin\varphi(t)$

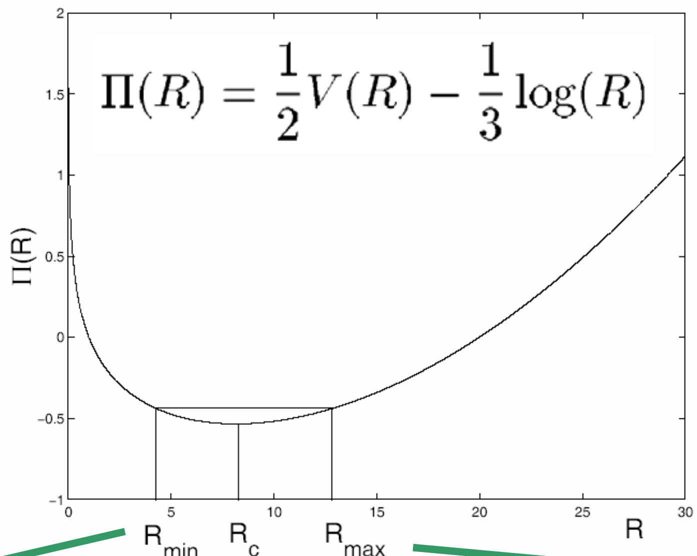
$$\zeta \equiv \cos\varphi(t)\,[r - R(t)]$$



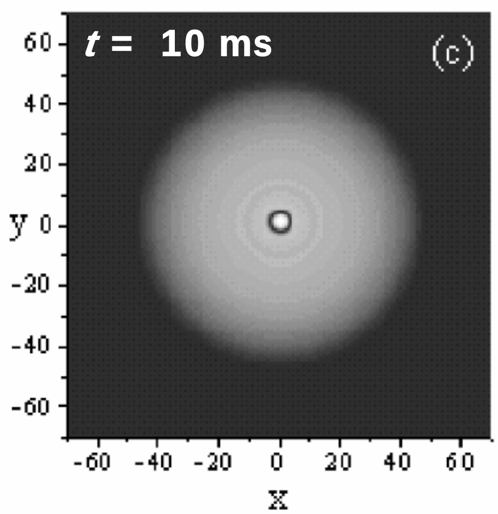
Length scale: 0.4 μm

Phys. Rev. Lett. **90**, 120403 (2003)

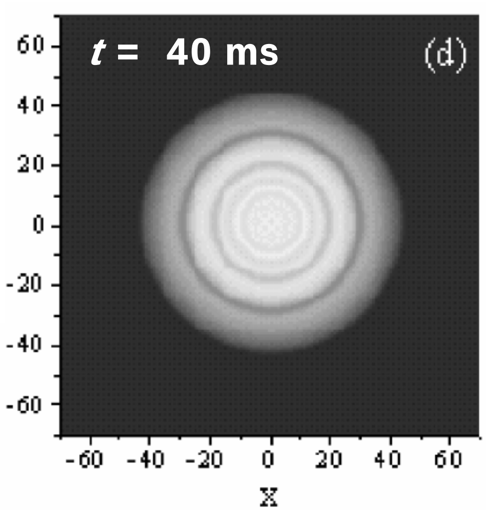
Dynamics of Ring Dark Solitons

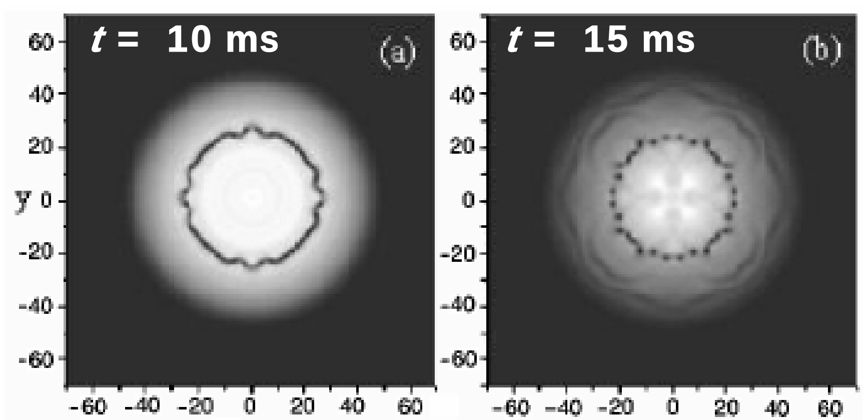


Moving (gray) RDS at R_{min} : *no TI*



Moving (gray) RDS at R_{max} : *no TI*



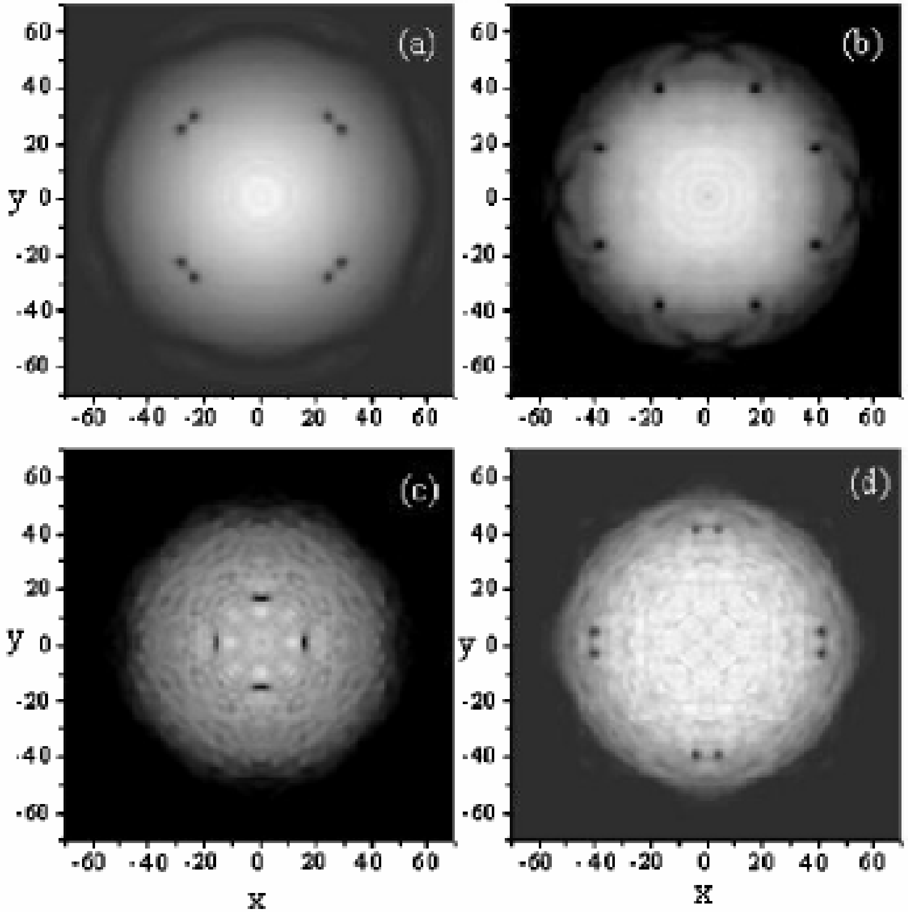


Stationary (black) RDS: *subject to TI*

The “vortex necklace”

t = 30 ms

t = 60 ms



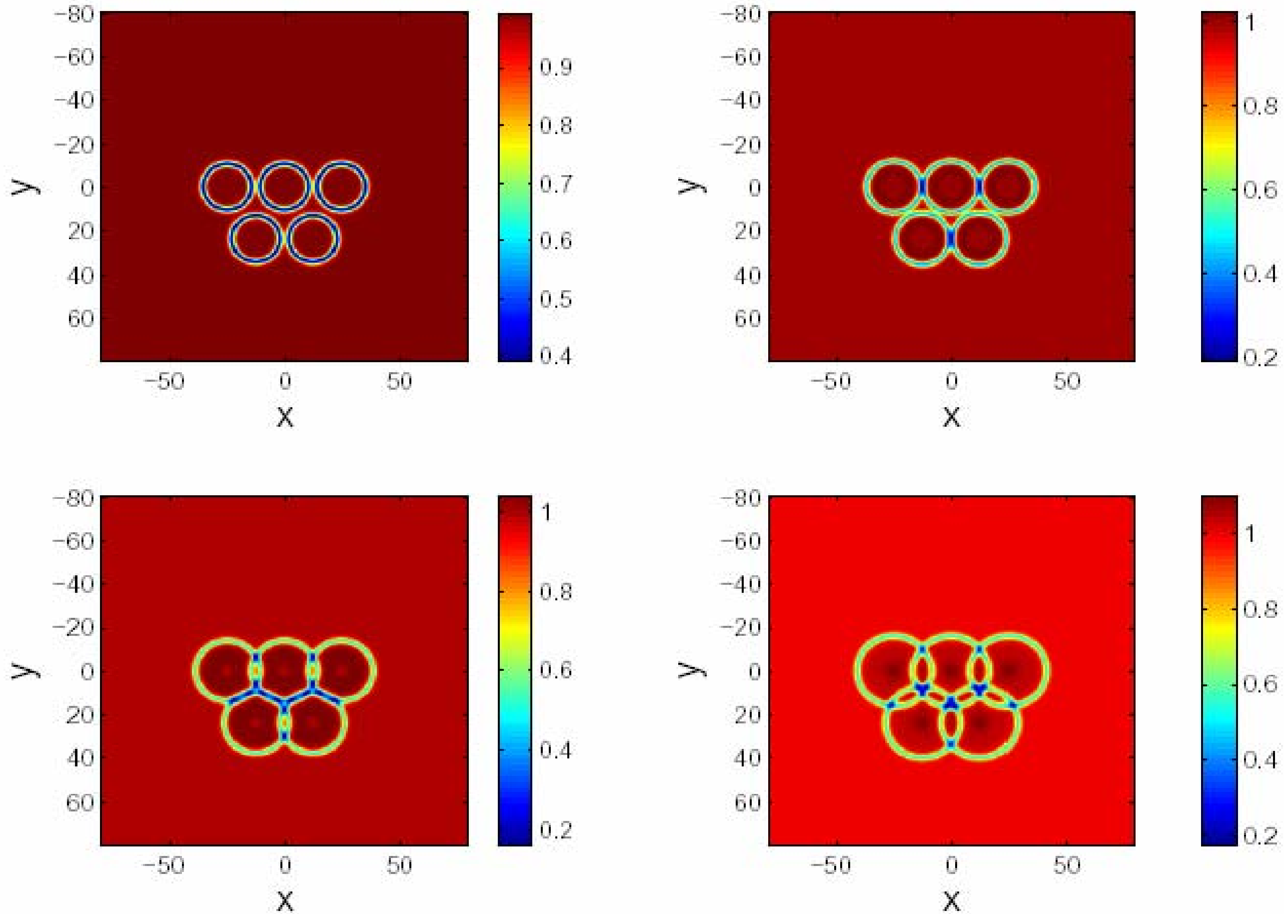
t = 105 ms

t = 135 ms

Length scale: 0.4 μm

Phys. Rev. Lett. 90, 120403 (2003)

The "OLYMPIC" Soliton (Athens 2004)



M. Oberthaler, G. Theocharis

Conclusions and Outlook

- Within the **limitations** of the mean-field theoretical approach :

- **MI** for quasi-1D BECs $\longrightarrow$ **soliton trains, breathers**
- **TI** for repulsive quasi-2D BECs $\longrightarrow$ **vortices, vortex arrays**

- It is of interest to investigate :

- Effects of thermal and quantum fluctuations
- The feasibility of experimental verification of the existence/stability/dynamics of the predicted patterns

In Collaboration with

old

- P. G. Kevrekidis, Z. Rapti (UMass, USA)
- B. A. Malomed (Tel Aviv, Israel)
- Yu. S. Kivshar (Canberra, Australia)
- D. E. Pelinovsky (McMaster, Canada)
- V. V. Konotop (Lisbon, Portugal)
- P. Schmelcher (Heidelberg, Germany)
- A. Trombettoni (Parma, Italy)
- F. Diakonos, H.E. Nistazakis, G. Theocharis (Athens, Greece)

and new friends ...

- N. P. Proukakis (Durham, UK)
- J. Brand (MPI Dresden, Germany)
- M. Oberthaler (Heidelberg, Germany)